Global and Stochastic Analysis
Vol. 3, No. 1, June (2016), 1-10

LOOSING STABILITY AND EXHIBITING LIMIT CYCLE AS CONSEQUENCE OF TIME DELAY INTRODUCTION INTO A COMPUTER VIRUS INFECTION MODEL

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ABSTRACT. This paper deals with analytical investigations on the possibility to model nonlinear dynamics emerging by the introduction of time delays into mathematical models based on differential equations. Specifically a delayed three-dimension ODE-based model is considered and an analytical sensitivity analysis on the time delays is performed. The results show that the magnitude of the time delay, considered as a parameter of the model, modifies the stability of the equilibrium points and induces the onset of a Poincaré-Andronov-Hopf bifurcation. Applications and future research directions are also discussed within the paper.

1. Introduction

The development and analysis of delayed mathematical models is an important topic considering the recent applications in biology [4, 5, 6], economics [7, 8, 9] and social sciences [10]. The introduction of the time delay in differential equations (models) has been shown to be an efficient method for the modeling of nonlinear dynamics appearing in many complex phenomena of the applied sciences [11]. Consequently the development of mathematical methods for the nonlinear dynamics analysis of such a model are required and the applicability of previous methods is a fundamental issue towards the definition of a robust and generalized stability theory. Specifically nonlinear dynamics includes the emergence of chaos, the onset of Hopf bifurcations, and fluctuations as the magnitude of the parameters of the model is varied, see, among others, paper [12, 13, 14], the review paper [15] and the book [16]. It is worth pointing out that the introduction of time delays takes into account that most of the emerging phenomena in the applied sciences at a certain time are strictly related to the state of the system at a previous time.

The pertinent literature comprises many contributions in the case of the introduction of only one time delay and for ordinary differential equation (ODE) models consisting of a system of at most two ODEs. The literature appears very limited in the case of delayed models that are defined on more than one time delay (see papers [17, 18, 19] and therein references) and, in particular, for ODE-based models defined by more than two differential equations. The analytical stability analysis of the latter mentioned delayed models is difficult considering the problem to manage the relation among a large number of parameters of the model.

2000 *Mathematics Subject Classification.* Primary 34C23; Secondary 58Z05.
Key words and phrases. Nonlinearity, Time delay, Asymptotic analysis.

The development of new methods is thus required in order to construct a general theory and this step can be pursued by considering specific models.

The present paper is devoted to analytical investigations on the possibility to control the nonlinear dynamics emerging into a delayed three-dimension ODE-based model. Specifically two time delays are introduced into a mathematical model proposed in [9] and an analytical sensitivity analysis on the time delays is performed. The results show that the magnitude of the time delay, considered as a parameter of the model, modifies the stability of the equilibrium points and induces the onset of a Poincaré-Andronov-Hopf bifurcation.

It is worth stressing that, to the best of our knowledge, this is the first time that time delays are introduced into the differential model [9] and, more in general, for a three-dimensional ordinary differential equation model.

The present paper is organized as follows: After this introduction, Section 2 outlines the delayed ODE-based model and the related equilibrium points. Section 3 is devoted to the sensitivity analysis on the time delays taken as parameters of the model and specifically to the local stability analysis of the equilibrium states and the conditions under which a Poincaré-Andronov-Hopf bifurcation occurs. Finally Section 4 deals with a critical analysis on the results and the definition of further research directions.

2. The Delayed Three-dimensional Model

Recently in [9] the following model has been proposed:

$$\begin{cases} \dot{S}(t) &= -\delta S(t) - \eta S(t-\tau)I(t-\tau) - \gamma_1 S(t) + \alpha_2 I(t) + \gamma_2 E(t), \\ \dot{I}(t) &= \eta S(t-\tau)I(t-\tau) - \delta I(t) - \alpha_1 I(t) - \alpha_2 I(t), \\ \dot{E}(t) &= -\delta E(t) - \gamma_2 E(t) + \gamma_1 S(t) + \alpha_1 I(t) + \varepsilon, \end{cases}$$

where S, I and E denote the numbers of the susceptible computers, the infected computers and external computers, respectively; δ is the rate at which each computer dies out; ε is the birth rate of external computers; α_1, α_2, γ_1, γ_2 and η are the states transmission rates.

Bearing in mind how the spread of virus computers occurs, the introduction of two time delays for S and I is proposed in this paper as follows:

$$\begin{cases} \dot{S}(t) &= -\delta S(t) - \eta S(t-\tau_1)I(t-\tau_2) - \gamma_1 S(t) + \alpha_2 I(t) + \gamma_2 E(t), \\ \dot{I}(t) &= \eta S(t-\tau_1)I(t-\tau_2) - (\delta + \alpha_1 + \alpha_2)\, I(t), \\ \dot{E}(t) &= -(\delta + \gamma_2)\, E(t) + \gamma_1 S(t) + \alpha_1 I(t) + \varepsilon. \end{cases} \tag{2.1}$$

The delayed model (2.1) shares the same equilibrium points of [9]. Accordingly, the system (2.1) has a unique positive equilibrium (S_*, I_*, E_*), where

$$S_* = \frac{\alpha_1 + \alpha_2 + \delta}{\eta}, \qquad I_* = \frac{\varepsilon\gamma_2\eta - \delta\,(\gamma_1 + \gamma_2 + \delta)\,(\alpha_1 + \alpha_2 + \delta)}{\eta\delta\,(\alpha_1 + \gamma_2 + \delta)},$$

$$E_* = \frac{\varepsilon}{\delta + \gamma_2} + \frac{\gamma_1\,(\alpha_1 + \alpha_2 + \delta)}{\eta\,(\delta + \gamma_2)} + \frac{\alpha_1 I_*}{\delta + \gamma_2}.$$

The above equilibrium point has been proved, under suitable technical assumptions that are not reported here, to be locally asymptotically stable for the non-delayed model [9]. The next section is devoted to show how the stability of this equilibrium changes as the magnitude of the time delays τ_1 and τ_2 is let varied.

3. Local Stability and Onset of Poincaré-Andronov-Hopf Bifurcation

The characteristic equation of the linearized system of (2.1) at the equilibrium reads $\det \mathbf{A} = 0$ where $\mathbf{A}$ writes:

$$\begin{pmatrix} -\delta-\gamma_1-\lambda-\eta I_* e^{-\lambda\tau_1} & \alpha_2-\eta S_* e^{-\lambda\tau_2} & \gamma_2 \\ \eta I_* e^{-\lambda\tau_1} & -(\delta+\alpha_1+\alpha_2)-\lambda+\eta S_* e^{-\lambda\tau_2} & 0 \\ \gamma_1 & \alpha_1 & -(\delta+\gamma_2)-\lambda \end{pmatrix},$$

namely

$$\lambda^3+a_2\lambda^2+a_1\lambda+a_0+\left(b_2\lambda^2+b_1\lambda+b_0\right)e^{-\lambda\tau_1}+\left(c_2\lambda^2+c_1\lambda+c_0\right)e^{-\lambda\tau_2}=0, \quad (3.1)$$

where

$$a_2 = 3\delta+\alpha_1+\alpha_2+\gamma_1+\gamma_2, \qquad a_1 = \delta\left[3\delta+2\left(\alpha_1+\alpha_2\right)\right]+\left(\gamma_1+\gamma_2\right)\left(2\delta+\alpha_1+\alpha_2\right),$$

$$a_0 = \delta\left(\delta+\gamma_1+\gamma_2\right)\left(\delta+\alpha_1+\alpha_2\right),$$

$$b_2 = \eta I_*, \qquad b_1 = \left(2\delta+\alpha_1+\gamma_2\right)\eta I_*, \qquad b_0 = \left(\delta+\alpha_1+\gamma_2\right)\delta\eta I_*,$$

$$c_2 = -\eta S_*, \qquad c_1 = -\left(2\delta+\gamma_1+\gamma_2\right)\eta S_*, \qquad c_0 = -\left(\delta+\gamma_1+\gamma_2\right)\delta\eta S_*.$$

The equilibrium point (S_*, I_*, E_*) of (2.1) is locally asymptotically stable if each of eigenvalues in (3.1) has negative real parts. Moreover, the boundary of the stability region is determined by the equations $\lambda = 0$ and $\lambda = i\omega$ ($\omega > 0$). If $\lambda = 0$ in (3.1) yields $\left(\delta+\alpha_1+\gamma_2\right)\delta\eta I_* = 0$, which is a contradiction. Hence, in what follows, the case $\lambda = i\omega$, with ω positive is taken into account.

3.1. Case $\tau_1 = 0, \tau_2 > 0$. The characteristic equation (3.1) rewrites:

$$\lambda^3+\left(a_2+b_2\right)\lambda^2+\left(a_1+b_1\right)\lambda+a_0+b_0+\left(c_2\lambda^2+c_1\lambda+c_0\right)e^{-\lambda\tau_2}=0. \quad (3.2)$$

Let $\lambda = i\omega$, $\omega > 0$, be a root of Eq. (3.2). Then, after separating the real and the imaginary parts, the following holds:

$$\left(a_2+b_2\right)\omega^2-\left(a_0+b_0\right) = c_1\omega\sin\omega\tau_2+\left(c_0-c_2\omega^2\right)\cos\omega\tau_2, \quad (3.3)$$

$$\omega^3-\left(a_1+b_1\right)\omega = c_1\omega\cos\omega\tau_2-\left(c_0-c_2\omega^2\right)\sin\omega\tau_2. \quad (3.4)$$

Squaring and adding these two equations, it follows:

$$\omega^6+p\omega^4+q\omega^2+r=0, \quad (3.5)$$

where

$$p = \left(a_2+b_2\right)^2-c_2^2-2\left(a_1+b_1\right),$$

$$q = \left(a_1+b_1\right)^2-c_1^2-2\left(a_0+b_0\right)\left(a_2+b_2\right)+2c_0c_2,$$

$$r = \left(a_0+b_0\right)^2-c_0^2.$$

It is worth to noting that a direct calculation shows that $r > 0$. Let $z = \omega^2$, then Eq. (3.5) rewrites as follows:

$$f(z) = z^3 + pz^2 + qz + r = 0. \tag{3.6}$$

The following Lemma holds true.

Lemma 3.1. *Let* $f(z) = z^3 + pz^2 + qz + r = 0$ *and*

$$z_* = \frac{-p + \sqrt{p^2 - 3q}}{3}. \tag{3.7}$$

Then

1) *If* $p \geq 0$ *and* $q \geq 0$ *or if* $p < 0, q > 0$ *and* $p^2 - 3q < 0$, *then Eq.* (3.6) *has no positive roots.*
2) *If* $p \geq 0$ *and* $q < 0$ *or if* $p < 0$ *and* $q \leq 0$ *or if* $p < 0, q > 0$ *and* $p^2 - 3q \geq 0$, *then Eq.* (3.6) *has no positive roots if* $f(z_*) > 0$; *it has one positive root* $z_0 = z_*$ *if* $f(z_*) = 0$ *and one has* $f'(z_0) = 0$; *it has two positive roots* z_- *and* z_+, $z_- < z_+$, *if* $f(z_*) < 0$, *and it is* $f'(z_-) < 0$ *and* $f'(z_+) > 0$.

Proof. The statement 1) is immediate since $r > 0$. Next, notice $f(0) = r > 0$, $f(+\infty) = +\infty$, $f'(z) = 3z^2 + 2pz + q$, $f''(z) = 6z + 2p$; $f(z)$ has a minimum in z_* when $p \geq 0$ and $q < 0$, $p < 0$ and $q \leq 0$, $p < 0$ and $q > 0$ with $p^2 - 3q \geq 0$. The statement 2) follows from Descartes' rule of signs that says that the number of positive roots of (3.6) is equal to the number of sign changes in the sequence formed of the polynomial's coefficients or less than the sign changes by a multiple of 2. $\square$

Bearing all the above in mind, the Eq. (3.2) has purely imaginary roots at certain values of τ_2, which can be determined by (3.3) and (3.4). Indeed, we can derive

$$\sin \omega\tau_2 = \frac{\omega_k B(\omega_k)}{c_1^2\omega_k^2 + (c_0 - c_2\omega_k^2)^2}, \qquad \cos \omega\tau_2 = \frac{A(\omega_k)}{c_1^2\omega_k^2 + (c_0 - c_2\omega_k^2)^2},$$

with

$$\begin{aligned} A(\omega_k) &= [c_1 - (a_2 + b_2) c_2] \omega_k^4 + [(a_0 + b_0) c_2 + (a_2 + b_2) c_0 - (a_1 + b_1) c_1] \omega_k^2 \\ &\quad - (a_0 + b_0) c_0, \end{aligned}$$

and

$$B(\omega_k) = -c_2\omega_k^4 + [(a_1 + b_1) c_2 + (a_2 + b_2) c_1 + c_0] \omega_k^3 - [(a_0 + b_0) c_1 + (a_1 + b_1) c_0].$$

Therefore,

$$\tau_{2,j}^{(k)} = \begin{cases} \dfrac{1}{\omega_k} \cos^{-1} \left\{ \dfrac{A(\omega_k)}{c_1^2\omega_k^2 + (c_0 - c_2\omega_k^2)^2} \right\} + \dfrac{2j\pi}{\omega_k}, \text{ if } B(\omega_k) \geq 0, \\[2ex] \dfrac{2(j+1)\pi}{\omega_k} - \dfrac{1}{\omega_k} \cos^{-1} \left\{ \dfrac{A(\omega_k)}{c_1^2\omega_k^2 + (c_0 - c_2\omega_k^2)^2} \right\}, \text{ if } B(\omega_k) < 0, \end{cases} \tag{3.8}$$

where $j = 0, 1, 2, ...,$ $\omega_k = \sqrt{z_k}$, $k \in \{0, \pm\}$, $\omega_- < \omega_+$. In conclusion, $\lambda = \pm i\omega_k$ is a pair of purely imaginary roots of (3.2) with $\tau_2 = \tau_{2,j}^{(k)}$. According to the Hopf bifurcation Theorem, we need to verify the transversality condition.

Proposition 3.2. *Let $\lambda(\tau_2)$ be the root of* (3.2) *such that*

$$Re(\tau_{2,j}^{(k)}) = 0, \quad Im(\tau_{2,j}^{(k)}) = \omega_k.$$

Then $\lambda = \pm i\omega_k$ are simple roots of (3.2) *at $\tau_2 = \tau_{2,j}^{(k)}$ and*

$$\left[\frac{dRe(\lambda)}{d\tau_2}\right]_{\tau_2=\tau_{2,j}^{(0)},\omega=\omega_0} = 0, \qquad \left[\frac{dRe(\lambda)}{d\tau_2}\right]_{\tau_2=\tau_{2,j}^{(+)},\omega=\omega_+} > 0,$$

and

$$\left[\frac{dRe(\lambda)}{d\tau_2}\right]_{\tau_2=\tau_{2,j}^{(-)},\omega=\omega_-} < 0.$$

Proof. Substituting $\lambda = \lambda(\tau_2)$ into (3.2) and taking the derivative with respect to τ_2, we have

$$\begin{aligned}\big\{3\lambda^2 + 2(a_2+b_2)\lambda + (a_1+b_1) + (2c_2\lambda + c_1)e^{-\lambda\tau_2} \\ -\left(c_2\lambda^2 + c_1\lambda + c_0\right)\tau_2 e^{-\lambda\tau_2}\big\}\frac{d\lambda}{d\tau_2} = \left(c_2\lambda^2 + c_1\lambda + c_0\right)\lambda e^{-\lambda\tau_2},\end{aligned} \quad (3.9)$$

namely

$$\left(\frac{d\lambda}{d\tau_2}\right)^{-1} = \frac{3\lambda^2 + 2(a_2+b_2)\lambda + a_1 + b_1}{\lambda(c_2\lambda^2 + c_1\lambda + c_0)e^{-\lambda\tau_2}} + \frac{2c_2\lambda + c_1}{\lambda(c_2\lambda^2 + c_1\lambda + c_0)} - \frac{\tau_2}{\lambda},$$

which, together with (3.2), yields

$$\begin{aligned}\left(\frac{d\lambda}{d\tau_2}\right)^{-1} &= -\frac{3\lambda^2 + 2(a_2+b_2)\lambda + a_1 + b_1}{\lambda(\lambda^3 + (a_2+b_2)\lambda^2 + (a_1+b_1)\lambda + (a_0+b_0))} \\ &\quad + \frac{2c_2\lambda + c_1}{\lambda(c_2\lambda^2 + c_1\lambda + c_0)} - \frac{\tau_2}{\lambda}.\end{aligned} \quad (3.10)$$

Plugging $\lambda = i\omega_k$ and $\tau_2 = \tau_{2,j}^{(k)}$ into (3.10), and using (3.6), we derive that

$$\begin{aligned}sign\left\{\left.\frac{d(Re\lambda)}{d\tau_2}\right|_{\tau_2=\tau_{2,j}^{(k)}}\right\} &= sign\left\{Re\left(\frac{d\lambda}{d\tau_2}\right)^{-1}_{\tau_2=\tau_{2,j}^{(k)}}\right\} \\ &= sign\left\{\frac{C(\omega_k)}{D(\omega_k)}\right\} = sign\left\{f'(z_k)\right\},\end{aligned} \quad (3.11)$$

, where

$$\begin{aligned}C(\omega_k) &= 3\omega_k^4 + 2\left[(a_2+b_2)^2 - c_2^2 - 2(a_1+b_1)\right]\omega_k^2 + (a_1+b_1)^2 - c_1^2 \\ &\quad -2(a_0+b_0)(a_2+b_2) + 2c_0c_2,\end{aligned} \quad (3.12)$$

and

$$D(\omega_k) = \omega_k^6 + \Big[(a_2+b_2)^2 - 2(a_1+b_1)\Big]\omega_k^4 + \Big[(a_1+b_1)^2 - 2(a_0+b_0)(a_2+b_2)\Big]\omega_k^2 + (a_0+b_0)^2. \quad (3.13)$$

The conclusion is now straightforward from the previous Lemma. It remains to prove the simplicity of the root $\lambda = i\omega_k$. Assuming that this root is a repeated root of (3.2), then (3.9) implies $\left(-c_2\omega_k^2 + c_1 i\omega_k + c_0\right) i\omega_k e^{-i\omega_k \tau_{2,j}^{(k)}} = 0$, and then a contradiction $\omega_k = 0$. This concludes the proof. □

Therefore a pair of pure imaginary roots $\lambda = \pm i\omega_k$ cross the imaginary axis from left to right if $dRe(\lambda(\tau_{2,j}^{(k)}))/d\tau_2 > 0$. On the contrary, if that derivative is negative the crossing of imaginary axis is from right to left.

Remark 3.3. The transversality condition does not hold for $\lambda = i\omega_0$. For simplicity, the analysis of this case will not be considered here.

Bearing all the above in mind, the following theorem holds true.

Theorem 3.4. *Assume that condition* (H_1) *in* [9] *holds true and let* $f(z), z_*$ *and* $\tau_{2,j}^{(k)}$ $(j = 0, 1, 2, ...)$ *be defined as in* (3.6), (3.7) *and* (3.8), *respectively.*

1) *If* $p \geq 0$ *and* $q \geq 0$ *or if* $p < 0, q > 0$ *and* $p^2 - 3q < 0$ *or if* $p \geq 0, q < 0$ *and* $f(z_*) > 0$ *or if* $p < 0, q \leq 0$ *and* $f(z_*) > 0$ *or if* $p < 0, q > 0, p^2 - 3q \geq$ *and* $f(z_*) > 0$, *then the equilibrium* (S_*, I_*, E_*) *of system* (3.1) *is locally asymptotically stable for all* $\tau_2 \geq 0$.
2) *If* $p \geq 0$ *and* $q < 0$ *or if* $p < 0$ *and* $q \leq 0$ *or if* $p < 0, q > 0$ *and* $p^2 - 3q \geq 0$, *and we also have* $f(z_*) < 0$, *then the stability of the equilibrium point* (S_*, I_*, E_*) *of system* (2.1) *can change a finite of times, at most, as* τ_2 *is increased and eventually it becomes unstable. System* (2.1) *undergoes a Poincaré-Andronov-Hopf bifurcation at* (S_*, I_*, E_*) *for those values of* $\tau_2 = \tau_{2,j}^{(k)}$ $(j = 0, 1, 2, ...)$ *for which a stability switch occurs.*

3.2. Case $\tau_1 > 0$, τ_2 fixed in its stable interval. In order to investigate the effects of multiple delays on the local stability of equilibrium point, in this section we consider Eq. (3.1) with $\tau_1 > 0$ regarded as a parameter and τ_2 fixed in its stable interval. Let $\lambda = i\omega$ $(\omega > 0)$ be a root of (3.1). Then, we obtain

$$b_1\omega \sin\omega\tau_1 + \left(b_0 - b_2\omega^2\right)\cos\omega\tau_1 = a_2\omega^2 - a_0 + \left(c_2\omega^2 - c_0\right)\cos\omega\tau_2 - c_1\omega\sin\omega\tau_2, \quad (3.14)$$

and

$$b_1\omega\cos\omega\tau_1 - \left(b_0 - b_2\omega^2\right)\sin\omega\tau_1 = \omega^3 - a_1\omega - \left(c_2\omega^2 - c_0\right)\sin\omega\tau_2 - c_1\omega\cos\omega\tau_2, \quad (3.15)$$

Taking squares of (3.14) and (3.15), and then adding them together, we get

$$g(\omega) = 0, \quad (3.16)$$

where

$$
\begin{aligned}
g(\omega) = {} & \omega^6 + \left(a_2^2 - b_2^2 + c_2^2 - 2a_1\right)\omega^4 + \left(a_1^2 - b_1^2 + c_1^2 - 2a_0a_2 + 2b_0b_2 - 2c_0c_2\right)\omega^2 \\
& + \left[2\left(a_2c_2 - c_1\right)\omega^4 + 2\left(a_1c_1 - a_0c_2 - a_2c_0\right)\omega^2 + 2a_0c_0\right]\cos\omega\tau_2 \\
& + \left[-2c_2\omega^5 + 2\left(a_1c_2 + c_0 - a_2c_1\right)\omega^3 + 2\left(a_0c_1 - a_1c_0\right)\omega\right]\sin\omega\tau_2 \\
& + a_0^2 + b_0^2 + c_0^2.
\end{aligned}
$$

In what follows, Eq. (3.16) is assumed to have at least a positive real root. Let ω_l, $l = 1, 2, ..., N$, be the positive real roots of (3.16). Then, for every fixed ω_l, there is a sequence of critical values given by

$$
\tau_{1,l}^{(j)} = \begin{cases} \dfrac{1}{\omega_l}\cos^{-1}\left\{\dfrac{E(\omega_l)}{\left(b_1\omega_l\right)^2 + \left(b_0 - b_2\omega_l^2\right)^2}\right\} + \dfrac{2j\pi}{\omega_l}, & \text{if } F(\omega_l) \geq 0, \\ \dfrac{2(j+1)\pi}{\omega_l} - \dfrac{1}{\omega_l}\cos^{-1}\left\{\dfrac{E(\omega_l)}{\left(b_1\omega_l\right)^2 + \left(b_0 - b_2\omega_l^2\right)^2}\right\}, & \text{if } F(\omega_l) < 0, \end{cases} \tag{3.17}
$$

where

$$
\begin{aligned}
E(\omega_l) = {} & b_1\omega_l\left[\omega_l^3 - a_1\omega_l - \left(c_2\omega_l^2 - c_0\right)\sin\omega_l\tau_2 - c_1\omega_l\cos\omega_l\tau_2\right] \\
& + \left(b_0 - b_2\omega_l^2\right)a_2\omega_l^2 - a_0 + \left(c_2\omega_l^2 - c_0\right)\cos\omega_l\tau_2 - c_1\omega_l\sin\omega_l\tau_2,
\end{aligned}
$$

and

$$
\begin{aligned}
F(\omega) = {} & \left(b_0 - b_2\omega_l\right)^2\left[a_2\omega_l^2 - a_0 + \left(c_2\omega_l^2 - c_0\right)\cos\omega_l\tau_2 - c_1\omega_l\sin\omega_l\tau_2\right] \\
& - b_1\omega_l\left(b_0 - b_2\omega_l^2\right)\left[\omega_l^3 - a_1\omega_l - \left(c_2\omega_l^2 - c_0\right)\sin\omega_l\tau_2 - c_1\omega_l\cos\omega_l\tau_2\right].
\end{aligned}
$$

Let

$$
\tau_1^c = \min\left\{\tau_{1,l}^{(j)}, l = 1, 2, ..., N, j = 0, 1, 2, ...\right\}. \tag{3.18}
$$

When $\tau_1 = \tau_1^c$, Eq. (3.1) has a pair of purely imaginary roots $\lambda = \pm i\omega_c$. We need now to verify that $\lambda = \pm i\omega_c$ is a simple root and the transversality condition of Hopf bifurcation holds. Differentiating λ with respect to τ_1 in (3.1), we can get

$$
\begin{aligned}
& \left\{3\lambda^2 + 2a_2\lambda + a_1 + \left(2b_2\lambda + b_1\right)e^{-\lambda\tau_1}\right. \\
& \qquad - \left(b_2\lambda^2 + b_1\lambda + b_0\right)\tau_1 e^{-\lambda\tau_1} + \left(2c_2\lambda + c_1\right)e^{-\lambda\tau_2} \\
& \qquad \left. - \left(c_2\lambda^2 + c_1\lambda + c_0\right)\tau_2 e^{-\lambda\tau_2}\right\}\frac{d\lambda}{d\tau_1} = \left(b_2\lambda^2 + b_1\lambda + b_0\right)\lambda e^{-\lambda\tau_1}
\end{aligned}
$$

We prove $\lambda = \pm i\omega_c$ is a simple root. If it is not simple, then

$$
\left(-b_2^2\omega_c + b_1 i\omega_c + b_0\right)i\omega_c e^{-i\omega_c\tau_1^c} = 0,
$$

which yields the contradiction $\omega_c = 0$. Next, we derive

$$\left(\frac{d\lambda}{d\tau_1}\right)^{-1} = \frac{3\lambda^2 + 2a_2\lambda + a_1 + (2b_2\lambda + b_1)\, e^{-\lambda\tau_1} + (2c_2\lambda + c_1)\, e^{-\lambda\tau_2}}{(b_2\lambda^2 + b_1\lambda + b_0)\,\lambda e^{-\lambda\tau_1}} - \frac{\left(c_2\lambda^2 + c_1\lambda + c_0\right)\tau_2 e^{-\lambda\tau_2}}{(b_2\lambda^2 + b_1\lambda + b_0)\,\lambda e^{-\lambda\tau_1}} - \frac{\tau_1}{\lambda},$$

and using (3.1), we find

$$sign\left[\frac{dRe(\lambda)}{d\tau_1}\right]_{\tau_1=\tau_1^c} = sign\left[Re\left(\frac{d\lambda}{d\tau_1}\right)^{-1}\right]_{\tau_1=\tau_1^c} = sign\,(P_1Q_1 - P_2Q_2)\,. \quad (3.19)$$

where

$$P_1 = 2b_2\omega_c \sin\omega_c\tau_1^c + b_1\cos\omega_c\tau_1^c + \tau_2\left(c_2\omega_c^2 - c_0\right)\cos\omega_c\tau_2 + (2c_2 - \tau_2c_1)\sin\omega_c\tau_2 + a_1 - 3\omega_c^2,$$

$$P_2 = 2b_2\omega_c\cos\omega_c\tau_1^c - b_1\sin\omega_c\tau_1^c + (2c_2 - \tau_2c_1)\,\omega_c\cos\omega_c\tau_2 - \tau_2\left(c_2\omega_c^2 - c_0\right)\sin\omega_c\tau_2 + 2a_2\omega_c,$$

$$Q_1 = (-b_2\omega_c^3 + b_0\omega_c)\sin\omega_c\tau_1^c - b_1\omega_c^2\cos\omega_c\tau_1^c,$$

and

$$Q_2 = (-b_2\omega_c^3 + b_0\omega_c)\cos\omega_c\tau_1^c + b_1\omega_c^2\sin\omega_c\tau_1^c.$$

If $sign\,(P_1Q_1 - P_2Q_2) > 0$ (resp. < 0), then each crossing at τ_1^c is from left to right (resp. right to left). Bearing all the above in mind, the following result holds true.

Theorem 3.5. *Let $g(\omega)$ and τ_1^c be defined as in* (3.16) *and* (3.18), *respectively.*

1) *If $g(\omega)$ has no positive zero, then the equilibrium (S_*, I_*, E_*) of system* (2.1) *is locally asymptotically stable for $\tau_1 \geq 0$.*
2) *If $g(\omega)$ has at least a positive zero, then there exists $\tau_1^c > 0$ such that the equilibrium (S_*, I_*, E_*) of system* (2.1) *is locally asymptotically stable for $\tau_1 \in [0, \tau_1^c)$ and the system* (2.1) *undergoes a Poincaré-Andronov-Hopf bifurcation at (S_*, I_*, E_*) when $\tau_1 = \tau_1^c$ if $sign\,(P_1Q_1 - P_2Q_2) > 0$.*
3) *If $g(\omega)$ has at least a positive zero, then the equilibrium (S_*, I_*, E_*) of system* (2.1) *remains stable when τ_1 crosses τ_1^c if $sign\,(P_1Q_1 - P_2Q_2) < 0$. On the other hand, it becomes unstable when τ_1 crosses a value $\tau_1 = \tau_{1,l}^{(j)}$ such that the corresponding value $sign\,(P_1Q_1 - P_2Q_2)$ is positive. In this case, a Poincaré-Andronov-Hopf bifurcation occurs at this $\tau_1 = \tau_{1,l}^{(j)}$. Moreover there may exist a lot of stability switches for $\tau_1 > \tau_{1,l}^{(j)}$.*

4. Critical Analysis and Research Perspectives

The mathematical analysis developed in the present paper has been addressed to the emergence of nonlinear dynamics in a delayed three-dimensional mathematical model. The analysis shows how the magnitude of a time delay, considered as a parameter, can be responsible of the onset of a Poincaré-Andronov-Hopf bifurcation. The results have been obtained employing the methods of the stability theory and the main problem is to be able to manage the parameters. In particular the parameters have to satisfy some technical assumptions whose validity is essential for gaining our analysis. This is an important step that cannot be relaxed considering the all mathematical models are based on the definition of parameters which have a specific meaning. The whole analysis has been performed where the transversality condition holds true, then a first research direction should be investigate also in the case $\lambda = i\omega_0$. According to our results, the model proposed in [9] with two independent time delays shows a more complicated dynamics. That is why it seems to be more realistic.

It is worth stressing that the introduction of the two delays has been motivated by the applications. Time delay can be introduced also in other terms of the differential system proposed in [9]. In this case the whole analysis should be revised in order to take into account the role of the other parameters. However we are confident that our method well fit also in the other cases.

An important research perspective is the (analytical) derivation of explicit formulas for determining the properties of the Poincaré-Andronov-Hopf bifurcation, namely the nature of the bifurcation (supercritical or subcritical), to determine the stability of the bifurcating periodic solutions, and to determine the period of the bifurcating periodic solutions. This analysis can be performed by employing the normal form method and the center manifold theory [20].

The Poincaré-Andronov-Hopf bifurcation analysis developed in the present paper cannot be straightforward applied to mathematical models that are based on partial differential equation. Much effort is required in this direction in particular for what concerns the stability analysis of the stationary states. In this context our analysis assumes an important role in particular for the mathematical models based on differential equations that presents a term of control and optimization [21, 22] or are based on equations of fractional order [23].

References

1. Gross, L.: Abstract Wiener spaces, in: *Proc. 5th Berkeley Symp. Math. Stat. and Probab.* **2**, part 1 (1965) 31–42, University of California Press, Berkeley.
2. Itô, K.: Stochastic integral, *Proc. Imp. Acad. Tokyo* **20** (1944) 519–524.
3. McKean, H. P.: *Stochastic Integrals*, Academic Press, New York, 1969.
4. Song, Y., Wei, J. and Han M.: Local and global Hopf bifurcation in a delayed hematopoiesis model, *International Journal of Bifurcation and Chaos in Applied Sciences and Engineering* **14** (2004) 3909-3919.
5. Piotrowska, M.J.: Hopf bifurcation in a solid avascular tumour growth model with two discrete delays, *Mathematical and Computer Modelling* **47** (2008) 597-603.
6. Ma, Y.: Global Hopf bifurcation in the Leslie-Gower predator-prey model with two delays, *Nonlinear Analysis: Real World Applications* **13** (2012) 370-375.

7. Chen, T., Li, X. and He, J.: Complex dynamics of credit risk contagion with time-delay and correlated noises, *Abstract and Applied Analysis* Volume 2014, Article ID 456764, 10 pages http://dx.doi.org/10.1155/2014/456764.
8. Bianca, C., Ferrara, M. and Guerrini L.: Hopf bifurcations in a delayed-energy-based model of capital accumulation, *Applied Mathematics & Information Sciences*, **7** (2013) 139-143.
9. Zhang, Z., Bi, D.: Bifurcation analysis in a delayed computer virus model with the effect of external computers, *Advances in Difference Equations* **317** (2015).
10. Erneux, T: *Applied Delay Differential Equations*, Springer, New York, NY, USA, 2009.
11. Bar-Yam, Y.: *Dynamics of Complex Systems, Studies in Nonlinearity*, Westview Press, 2003.
12. Byrne H.M.: The effect of time delays on the dynamics of avascular tumor growth, *Mathematical Biosciences* **144** (1997) 83-117.
13. Wei, J.: Bifurcation analysis in a scalar delay differential equation, *Nonlinearity* **20** (2007) 2483-2498.
14. Bianca, C., Ferrara, M., Guerrini, L.: Qualitative analysis of a retarded mathematical framework with applications to living systems, *Abstract and Applied Analysis* **2013** (2013) 736058.
15. Baker, C.T.H., Bocharov, G.A. and Paul, C.A.H.: Mathematical modelling of the interleukin-2 T-cell system: a comparative study of approaches based on ordinary and delay differential equations, *Journal of Theoretical Medicine* **2** (1997) 117-128.
16. Kuang, Y.: *Delay Differential Equations with Applications in Population Dynamics*, vol. 191 of Mathematics in Science and Engineering, Academic Press, Boston, Mass, USA, 1993.
17. Li, W., Ruan, S. and Wei, J.: Stability and bifurcation in delay differential equations with two delays, *Journal ofMathematical Analysis and Applications* **236** (1999) 254-280.
18. Piotrowska, M.J.: A remark on the ODE with two discrete delays, *Journal of Mathematical Analysis and Applications* **329** (2007) 664-676.
19. Blair, J., Mackey, M.C. and Mahay, J.M.: Age-structured and two delay models for erythropoiesis, *Mathematical Biosciences* **128** (1995) 317-346.
20. Hassard, B., Kazarino D. and Wan Y.: *Theory and Application of Hopf Bifurcation*, Cambridge, UK, 1981.
21. Isidori, A.: *Nonlinear Control Systems*, New York: Springer-Verlag, 1996.
22. Sivasundaram, S., Uvah, J.: Controllability of impulsive hybrid integro-differential systems, *Nonlinear Analysis: Hybrid Systems* **2** (2008) 1003-1009.
23. Li, C.G., Kostic, M., Li, M., Piskarev, S.: On a class of time-fractional differential equations, *Fractional Calculus and Applied Analysis* **15** (2012) 639-668.

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