Global and Stochastic Analysis
Vol. 4 No. 1, January (2017), 49-61

AN APPLICATION OF MULTIVARIATE MARKOV CHAIN MODEL FOR FUEL PRICE CHANGES IN INDIA

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ABSTRACT. Markov chain is an essential tool for modeling a lot of practical systems such as queuing systems, manufacturing systems and categorical data sequences. Multiple categorical data sequences occur in many applications such as inventory control, data mining and finance market. In many practical situations, we would like to consider multiple categorical data sequences in the same time period. The reason is that the data sequences can be correlated and therefore by exploring their relationships, we can develop better models. In this paper, the monthly changes of the Petrol and Diesel prices and the monthly changes of the crude oil price are taken into consideration as two categorical data sequences and it is analyzed that with multivariate Markov chain model to what extent these categorical data sequences may be affected each other. In addition that, a degree of relationship through correlation analysis and a simple linear regression analysis were done for these categorical data sequence.

1. INTRODUCTION

Fuel is the world's most useful commodity. The fuel data is multidimensional and nonlinear Markov chain model is one of the statistical techniques that play an important role to analyze such type of data. In this paper, before applying multivariate Markov chain model, we have to check whether the independent and dependent variables are highly correlated. So correlation analysis is done. The multivariate Markov chain model is an advance level of Markov chain.

An elementary form of dependence between values of X_n in successive transition was introduced by Russian mathematician A.A. Markov and is known as Markov dependence. Markov dependence is a form of dependence which states that X_{n+1} when it is known and is independent of X_{n-1},X_{n-2},...,X_n.This implies that the future of the process depends only on the present, irrespective of the past. This property is known as Markov property.

2000 *Mathematics Subject Classification.* Primary 65C40, 62H20; Secondary 62M10.

Key words and phrases. Markov Chain, Categorical Data Sequences, Multivariate Markov Chain, Stock Market.

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In general, the Markov property can be defined as:

$$P[X_{n+1} = i_{n+1}|X_n = i_n, X_{n+1} = i_{n-1}, ..., X_1 = i_1, X_0 = i_0]$$

$$= P[X_{n+1} = i_{n+1}|X_n = i_n] \quad (1.1)$$

For all states $i_0, i_1, ..., i_{n+1}$ and for all n. This is called Markov dependence of the first order.

The square matrix P consisting of the elements $p_{ij}^{(1)}$ for all possible states i and j is called one-step transition probability matrix of the chain. Therefore

$$P_{ij}^{(1)} = \begin{bmatrix} P_{11} & P_{12} & P_{13} & ... \\ P_{21} & P_{22} & P_{23} & ... \\ ... & ... & ... & ... \\ ... & ... & ... & ... \end{bmatrix} \quad (1.2)$$

The transition probabilities P_{ij} should satisfy $P_{ij}^{(m)} \geq 0$ and $\sum_j P_{ij}^{(m)} = 1$

Vasek Chavatal [1983][14] has discussed about formulation of linear programming problem(LPP) formulated from having optimization problem and solved by simplex method. Raftery [1985][8] has introduced higher order Markov chain model which combines realism and parsimony. Raftery et al., [1994][9] have introduced the mixture transition distribution model for higher order Markov chain and also proposed a computational algorithm for maximum likelihood estimation which is based on a way of reducing the large number of constrains. Ching et al., [2002][1] have proposed a multivariate Markov chain model for modeling multiple categorical data sequence and also developed efficient estimation method for the model parameters. Ching et al., [2003][2] have proposed a new higher-order Markov model whose number of states is linear and also developed a new parameter estimation method based on linear programming.

Ching et al., [2008][3] has proposed an nth order multivariate Markov chain model for modeling multiple categorical data sequences. Zhilong et al., [2009][15] have presented a higher-order multivariate Markov chain model combined with particle swarm optimization algorithm, capable of searching for the optimal parameter values for η level characteristics value to obtain a high accuracy model for forecasting of multidimensional time series. Particle swarm optimization algorithm is employed to optimize the coefficient of level characteristics value. Tie Liu [2010][12] has adopted Markov chains model to analyze and predict the time series. Some series can be expressed by a first-order discrete-time Markov chain model. Ersoy [2011][5] has discussed the application of monthly changes of the US Dollar selling rates and the monthly changes of the Euro selling rates. The two changes are taken into consideration as two categorical data sequences and it is revealed with multivariate Markov chain model.

2. METHODOLOGY

In this section, the methodology of data collection and analysis of the data using appropriate statistical techniques are discussed.

2.1. Correlation Analysis. The degree of the linear relationship between the two variables is measured by the correlation coefficient that is shown in equation (2.1) and denoted by "ρ". The correlation coefficient is the measure

$$\gamma_{xy} = \frac{\sum(x_i - \bar{x})^2(y_i - \bar{y})^2}{n\sigma_x\sigma_y} \tag{2.1}$$

where $\bar{x}$, $\bar{y}$ are the arithmetic means and σ_x, σ_y are the standard deviations of the variables x and y respectively.

2.2. Regression Analysis. Simple linear regression is expressed as showing the real linear relationship that is assumed to have been between the variables by a linear equation which includes only on independent variable. It is used when two or more variables are thought to be systematically connected by a linear relationship. This is shown in equation (2.2), where Y denotes the dependent variable, X denotes the independent variable and α denotes the constant coefficient that shows the intersection point and β denotes the coefficient that shows the slope ε denotes the error.

$$Y_i = \alpha + \beta X_i + \varepsilon \tag{2.2}$$

For calculating the parameters of α and β in the model, the normal equations, shown in equation (2.3) and (2.4), obtained by the least square method is used.

$$\sum Y = n\alpha + \beta \sum X \tag{2.3}$$

$$\sum XY = n \sum X + \beta \sum X^2 \tag{2.4}$$

2.3. Multivariate Markov Chain Model. Time series data occur frequently in many real world applications. One of the major important steps in analysing a time series data is the selection of appropriate statistical model for the data. Because it helps in prediction, hypothesis testing and rule discovery. Higher order Markov chain model is developed [Raftery, 1985 [8] and Ching, W.K et al., 2003 [2]] for modelling categorical data sequence. Here it is assumed that there is S categorical data sequence and each has m possible states.

It is proposed that a multivariate Markov chain model represents the behavior of multiple categorical sequences generated by similar sources or the same source. Here it is assumed that there are S categorical sequences and each has m possible state. Let $X_n^{(k)}$ be the state vector of the k^{th} sequence at time n. If the k^{th} sequence is in state j at time n then $X_n^{(k)} = j$. The multivariate Markov chain model, it is assumed that there is the following relationship:

$$X_{n+1}^{(k)} = \sum_{k=j}^{s} \lambda_{jk} P^{(jk)} X_n^{(k)}, forj = 1, 2, ..., S \tag{2.5}$$

where

$$\lambda_{jk} \geq 1,\ 1 \leq j,\ k \leq s \sum_{k=1}^{s} \lambda_{jk} = 1 \text{ for } j = 1,\ 2,\ \ldots,\ S$$

The state probability distribution of the k^{th} sequence at the $(n+1)^{th}$ step depends on the weighted average of $p^{(jk)} X_n^{(k)}$. Here $p^{(jk)}$ is a transition probability matrix from the states in the k^{th} sequence to the states in the jth sequence, and $X_n^{(k)}$ is the state probability distribution of the k^{th} sequences at the n^{th} step. In the matrix form, it is written as follows:

$$X_{n+1} \equiv \begin{bmatrix} X_{n+1}^1 \\ X_{n+1}^2 \\ . \\ . \\ . \\ X_{n+1}^s \end{bmatrix} = \begin{bmatrix} \lambda_{11} P^{(11)} & \lambda_{12} P^{(12)} & ... & \lambda_{1s} P^{(1s)} \\ \lambda_{21} P^{(21)} & \lambda_{22} P^{(22)} & ... & \lambda_{2s} P^{(2s)} \\ . & . & & . \\ . & . & & . \\ . & . & & . \\ \lambda_{s1} P^{(s1)} & \lambda_{s2} P^{(s2)} & ... & \lambda_{ss} P^{(ss)} \end{bmatrix} \begin{bmatrix} X_n^1 \\ X_n^2 \\ . \\ . \\ . \\ X_n^s \end{bmatrix} \equiv Q X_n \tag{2.6}$$

$$\text{(or) } X_{(n+1)} = Q X_n$$

From the above equation the column sum of $Q \neq 1$ (the column sum of $P^{(jk)} = 1$)

Proposition 2.1. *If $\lambda_{jk} > 0$ for $1 \leq j, k \leq s$, then the matrix Q has an eigenvalue equal to one and the eigenvalues of Q have modulus less than or equal to 1.*

Proposition 2.2. *Suppose that $P^{(jk)} (1 \leq j, k \leq s)$ are irreducible and $\lambda_{jk} > 0$ for $1 \leq j, k \leq S$. Then there is vector $X = \left[X^{(1)}, X^{(2)}, ..., X^{(s)}\right]^T$ such that X=QX and $\sum_{i=1}^{m} \left[X^{(j)}\right]_i = 1,\ 1 \leq j \leq S$.*

2.4. Estimations of Model Parameters. It is proposed that there are some methods for the estimations of λ_{jk}. For each data sequence, the transition probability matrix is estimated by the following method. Given the data sequence, the transition frequency from the states in the kth sequence to the states in the jth sequence is counted. Hence, the transition frequency matrix for the data sequence can be constructed. After the normalization, the estimates of the transition probability matrices can also be obtained. More precisely, the transition frequency $f_{i_j i_k}^{(jk)}$ from the state in the sequence i_k in the sequence $\left\{x_n^{(k)}\right\}$ to the state $i_j$in the sequence $\left\{x_n^{(j)}\right\}$ is counted and therefore the transition frequency matrix for the sequence is constructed as follows:

$$\hat{P}^{(jk)} = \begin{bmatrix} \hat{p}_{11}^{(jk)} & \hat{p}_{21}^{(jk)} & \cdots & \hat{p}_{m1}^{(jk)} \\ \hat{p}_{12}^{(jk)} & \hat{p}_{22}^{(jk)} & \cdots & \hat{p}_{m2}^{(jk)} \\ \vdots & \vdots & \vdots & \vdots \\ \hat{p}_{1m}^{(jk)} & \hat{p}_{2m}^{(jk)} & \cdots & \hat{p}_{mm}^{(jk)} \end{bmatrix} \tag{2.7}$$

$$\hat{p}^{jk}_{i_j i_k} = \begin{cases} \dfrac{f^{jk}_{i_j i_k}}{\sum_{i_k=1}^{m} f^{jk}_{i_j i_k}} & if \quad \sum_{i_k=1}^{m} f^{jk}_{i_j i_k} \neq 0 \\ 0 & otherwise \end{cases} \tag{2.8}$$

In order to construct the transition probability matrix for the above observed Markov chain (categorical sequence), we consider the following simple procedures. Besides the estimates of $\hat{P}^{(jk)}$, the parameters that the multivariate Markov chain has a stationary vector x. The vector x can be estimated from the sequences by computing the proportion of the occurrence of each state in each of the sequences, and it is denoted by

$$\hat{x} = \left(\hat{x}^{(1)}, \hat{x}^{(2)}, ..., \hat{x}^{(s)}\right)^T \tag{2.9}$$

we expect

$$\begin{bmatrix} \lambda_{11}P^{(11)} & \lambda_{12}P^{(12)} & \dots & \lambda_{1s}P^{(1s)} \\ \lambda_{21}P^{(21)} & \lambda_{22}P^{(22)} & \dots & \lambda_{2s}P^{(2s)} \\ \vdots & \vdots & \vdots & \vdots \\ \lambda_{s1}P^{(s1)} & \lambda_{s2}P^{(s2)} & \dots & \lambda_{ss}P^{(ss)} \end{bmatrix} \hat{x} \approx \hat{x} \tag{2.10}$$

From (2.10), we suggest one possible way to estimate the parameters $\lambda = \{\lambda_{jk}\}$. we may consider solving the following optimization problem:

$$\min_{\lambda} \max_{i} \left| \left[\sum_{k=1}^{m} \lambda_{jk} \hat{P}^{(jk)} \hat{x}^{(k)} \ - \hat{x}^{(j)} \right]_i \right| \tag{2.11}$$

Subject to

$$\sum_{k=1}^{s} \lambda_{jk} = 1, \, and \, \lambda_{jk} \geq 0, \ \forall k. \tag{2.12}$$

Problem (2.11) & (2.12) can be formulated as s LPP for each j :

min w_j

Subject to:

$$\begin{cases} \begin{pmatrix} w_j \\ w_j \\ \vdots \\ w_j \end{pmatrix} \geq -\hat{x}^{(j)} + B \begin{pmatrix} \lambda_{j1} \\ \lambda_{j2} \\ \vdots \\ \lambda_{js} \end{pmatrix}, \begin{pmatrix} w_j \\ w_j \\ \vdots \\ w_j \end{pmatrix} \geq -\hat{x}^{(j)} + B \begin{pmatrix} \lambda_{j1} \\ \lambda_{j2} \\ \vdots \\ \lambda_{js} \end{pmatrix}, \\ w_j \geq 0, \\ \sum_{k=1}^{s} \lambda_{jk} = 1, \ \lambda_{jk} \geq 0, \forall k, \end{cases} \tag{2.13}$$

where

$$B = \left[\hat{P}^{(j1)} \hat{x}^{(1)} \ | P^{(j2)} \ \hat{x}^{(2)} \right| \dots \left| \hat{P}^{(js)} \hat{x}^{(s)} \right] \tag{2.14}$$

3. MATERIALS AND METHODS

3.1. Data Source. The monthly price changes of Crude oil, Petrol and Diesel price data for the period of October 2005 to September 2016 were obtained from the yahoofinance.com. The basic statistics for all the data are presented in Table 1.

TABLE 1. **Descriptive statistics for the data**

Statistics	**Variables**		
	Crude Oil Price	**Petrol Price**	**Diesel Price**
Mean	4100.5	59.6048	43.1417
S.D	1215.7	10.4489	8.98487
Maximum	7246.00	78.5100	62.9200
Minimum	1996.00	44.2400	32.8200
Skewness	0.39012	0.155553	0.501874
Kurtosis	-0.85708	-1.40706	-1.04780

3.2. Correlation Analysis. Our objective is to study the system behavior of the Petrol and Diesel prices and also predict the future behavior of the Petrol and Diesel prices.

TABLE 2. **Correlation and determination for Crude oil and Petrol**

Model Summary

Model	R	R Square	Adjusted R Square	S.E of Estimate	Change Statistics					Durbin-Watson
					R Square Change	F Change	df1	df2	Sig. F Change	
1	.745	.554	.551	7.00261	.554	161.669	1	130	.000	.177

a. Predictors:(Constant), Crude
b. Dependent Variable: Petroll

From the Table 2, there is a 74.5% positive relationship between the Crude oil and Petrol price and the changes in the Petrol price is 55.4% dependent on the Crude oil price. Also it can be shown that there is correlation between residual terms since the Durbin Watson test statistic is 0.177.

TABLE 3. **Correlation and determination for Crude oil and Diesel**

Model Summary

Model	R	R Square	Adjusted R Square	S.E of Estimate	Change Statistics					Durbin-Watson
					R Square Change	F Change	df1	df2	Sig. F Change	
1	.686	.544	.539	7.30654	.544	148.094	1	130	.000	.170

a. Predictors:(Constant), Crude
b. Dependent Variable: Diesel

From the Table 3, there is a 68.6% Crude oil and Diesel prices are positively related and the changes in the Diesel price is 54.4% dependent on the Crude oil price. Also it can be shown that there is correlation between residual terms since the Durbin Watson test statistic is 0.170.

4. Multivariate Markov Chain Analysis

The mean value of increase and the mean value of decrease are calculated for the values of Crude oil, Petrol and Diesel price are determined. The values of increase over the mean value of increase are denoted by D_1, the values of increase below the mean value of increase are denoted by D_2, the values of decrease over the mean value of decrease are denoted by D_3, the values of decrease below the mean value of decrease are denoted by D_4 and the no change values are denoted by D_5. According to the values of increase and decrease, the data sequences are denoted by S_1, S_2 and S_3 for Crude oil, Petrol price and Diesel price respectively.

According to these data sequences there are five states of multivariate Markov chain model. To construct the transition probability matrix for the above observed Markov chain (categorical sequence), we consider the following simple procedures. By counting the transition frequency from State k to state j in the sequence, we can construct the transition frequency matrix $F^{(jk)}$ (then the transition probability matrix $P^{(jk)}$) for the sequence as follows:

$$S_1 = \left\{ \begin{array}{cccccccccccc}
D_2 & D_3 & D_3 & D_2 & D_2 & D_3 & D_3 & D_2 & D_2 & D_3 & D_2 & D_1 \\
D_3 & D_2 & D_3 & D_1 & D_5 & D_1 & D_1 & D_1 & D_4 & D_3 & D_4 & D_4 \\
D_4 & D_4 & D_2 & D_2 & D_2 & D_2 & D_1 & D_2 & D_3 & D_1 & D_3 & D_1 \\
D_3 & D_2 & D_4 & D_1 & D_2 & D_2 & D_4 & D_2 & D_2 & D_3 & D_2 & D_2 \\
D_1 & D_2 & D_2 & D_2 & D_1 & D_1 & D_4 & D_4 & D_3 & D_3 & D_3 & D_1 \\
D_1 & D_2 & D_4 & D_1 & D_2 & D_2 & D_4 & D_3 & D_2 & D_1 & D_4 & D_3 \\
D_2 & D_2 & D_2 & D_3 & D_1 & D_4 & D_2 & D_1 & D_1 & D_1 & D_4 & D_4 \\
D_3 & D_1 & D_5 & D_2 & D_3 & D_3 & D_2 & D_2 & D_3 & D_3 & D_3 & D_4 \\
D_4 & D_4 & D_4 & D_2 & D_3 & D_1 & D_2 & D_3 & D_4 & D_2 & D_3 & D_2 \\
D_3 & D_3 & D_3 & D_2 & D_2 & D_1 & D_1 & D_3 & D_4 & D_2 & D_5 &
\end{array} \right\}$$

$$S_2 = \left\{ \begin{array}{cccccccccccc}
D_4 & D_5 & D_5 & D_4 & D_3 & D_5 & D_3 & D_5 & D_5 & D_5 & D_5 & D_5 \\
D_5 & D_5 & D_5 & D_2 & D_5 & D_5 & D_2 & D_1 & D_5 & D_5 & D_5 & D_5 \\
D_5 & D_4 & D_4 & D_5 & D_5 & D_5 & D_5 & D_5 & D_1 & D_5 & D_5 & D_5 \\
D_5 & D_5 & D_1 & D_1 & D_5 & D_2 & D_5 & D_1 & D_5 & D_5 & D_2 & D_2 \\
D_2 & D_1 & D_5 & D_5 & D_2 & D_5 & D_1 & D_5 & D_2 & D_5 & D_1 & D_5 \\
D_2 & D_4 & D_5 & D_5 & D_5 & D_5 & D_1 & D_4 & D_4 & D_3 & D_5 & D_3 \\
D_3 & D_5 & D_3 & D_2 & D_2 & D_4 & D_4 & D_1 & D_1 & D_2 & D_1 & D_4 \\
D_3 & D_2 & D_2 & D_5 & D_2 & D_3 & D_3 & D_2 & D_2 & D_4 & D_3 & D_4 \\
D_4 & D_4 & D_4 & D_4 & D_1 & D_3 & D_1 & D_2 & D_3 & D_4 & D_4 & D_4 \\
D_2 & D_3 & D_3 & D_3 & D_4 & D_1 & D_2 & D_1 & D_4 & D_4 & D_3 &
\end{array} \right\}$$

$$S_3 = \left\{ \begin{array}{cccccccccccc} D_2 & D_5 & D_5 & D_3 & D_3 & D_5 & D_3 & D_5 & D_5 & D_5 & D_5 & D_5 \\ D_5 & D_5 & D_5 & D_2 & D_5 & D_5 & D_2 & D_1 & D_3 & D_5 & D_5 & D_5 \\ D_5 & D_4 & D_4 & D_5 & D_5 & D_5 & D_5 & D_5 & D_1 & D_5 & D_5 & D_5 \\ D_5 & D_5 & D_5 & D_1 & D_5 & D_2 & D_5 & D_1 & D_5 & D_5 & D_2 & D_5 \\ D_5 & D_5 & D_5 & D_5 & D_5 & D_5 & D_5 & D_1 & D_2 & D_5 & D_5 & D_5 \\ D_5 & D_5 & D_5 & D_5 & D_5 & D_5 & D_5 & D_5 & D_3 & D_2 & D_1 & D_2 \\ D_5 & D_5 & D_3 & D_2 & D_2 & D_5 & D_2 & D_2 & D_2 & D_2 & D_2 & D_2 \\ D_2 & D_2 & D_2 & D_2 & D_2 & D_5 & D_1 & D_2 & D_2 & D_2 & D_2 & D_4 \\ D_4 & D_4 & D_4 & D_4 & D_1 & D_4 & D_1 & D_2 & D_4 & D_4 & D_3 & D_2 \\ D_2 & D_3 & D_4 & D_3 & D_1 & D_1 & D_1 & D_1 & D_3 & D_4 & D_2 & \end{array} \right\}$$

According to the above data sequences there are five states of multivariate Markov chain model are constructed. The below transition frequency matrixes, $F^{(11)}$, $F^{(22)}$ and $F^{(33)}$ are shows the intra-transition frequencies derived for the states of S_1, S_2 and S_3 respectively. $F^{(12)}$ shows the transition frequencies from the states of S_1 data sequence to the state of S_2 data sequence. Similarly $F^{(21)}$, $F^{(13)}$ and $F^{(31)}$ are constructed. The transition frequency matrices are calculated on the basis of these data sequences as follows:

$$F^{(11)} = \begin{array}{c} D_1 \\ D_2 \\ D_3 \\ D_4 \\ D_5 \end{array} \overset{\begin{array}{ccccc} D_1 & D_2 & D_3 & D_4 & D_5 \end{array}}{\begin{bmatrix} 7 & 6 & 4 & 5 & 2 \\ 7 & 15 & 13 & 4 & 1 \\ 7 & 11 & 9 & 4 & 0 \\ 2 & 6 & 5 & 8 & 0 \\ 1 & 1 & 0 & 0 & 0 \end{bmatrix}}$$

$$F^{(22)} = \begin{array}{c} D_1 \\ D_2 \\ D_3 \\ D_4 \\ D_5 \end{array} \overset{\begin{array}{ccccc} D_1 & D_2 & D_3 & D_4 & D_5 \end{array}}{\begin{bmatrix} 1 & 3 & 1 & 3 & 7 \\ 4 & 5 & 3 & 3 & 5 \\ 1 & 3 & 4 & 3 & 4 \\ 3 & 1 & 5 & 10 & 3 \\ 6 & 8 & 3 & 2 & 27 \end{bmatrix}}$$

$$F^{(33)} = \begin{array}{c} D_1 \\ D_2 \\ D_3 \\ D_4 \\ D_5 \end{array} \overset{\begin{array}{ccccc} D_1 & D_2 & D_3 & D_4 & D_5 \end{array}}{\begin{bmatrix} 3 & 4 & 2 & 1 & 3 \\ 2 & 15 & 1 & 2 & 7 \\ 1 & 3 & 1 & 2 & 3 \\ 2 & 1 & 2 & 6 & 1 \\ 5 & 6 & 4 & 1 & 40 \end{bmatrix}}$$

$$F^{(12)} = \begin{array}{c} D_1 \\ D_2 \\ D_3 \\ D_4 \\ D_5 \end{array} \overset{\begin{array}{ccccc} D_1 & D_2 & D_3 & D_4 & D_5 \end{array}}{\begin{bmatrix} 7 & 5 & 0 & 2 & 10 \\ 5 & 10 & 4 & 2 & 19 \\ 1 & 4 & 10 & 6 & 10 \\ 2 & 1 & 2 & 11 & 5 \\ 0 & 0 & 0 & 0 & 2 \end{bmatrix}}$$

$$F^{(13)} = \begin{array}{c} \\ D_1 \\ D_2 \\ D_3 \\ D_4 \\ D_5 \end{array} \begin{array}{c} \begin{array}{ccccc} D_1 & D_2 & D_3 & D_4 & D_5 \end{array} \\ \left[\begin{array}{ccccc} 5 & 5 & 1 & 0 & 13 \\ 4 & 10 & 3 & 0 & 24 \\ 1 & 7 & 6 & 4 & 12 \\ 3 & 5 & 0 & 8 & 5 \\ 0 & 1 & 0 & 0 & 1 \end{array} \right] \end{array}$$

$$F^{(21)} = \begin{array}{c} \\ D_1 \\ D_2 \\ D_3 \\ D_4 \\ D_5 \end{array} \begin{array}{c} \begin{array}{ccccc} D_1 & D_2 & D_3 & D_4 & D_5 \end{array} \\ \left[\begin{array}{ccccc} 6 & 3 & 2 & 4 & 0 \\ 4 & 4 & 6 & 4 & 2 \\ 1 & 9 & 3 & 2 & 0 \\ 3 & 8 & 6 & 4 & 1 \\ 10 & 15 & 14 & 7 & 0 \end{array} \right] \end{array}$$

$$F^{(31)} = \begin{array}{c} \\ D_1 \\ D_2 \\ D_3 \\ D_4 \\ D_5 \end{array} \begin{array}{c} \begin{array}{ccccc} D_1 & D_2 & D_3 & D_4 & D_5 \end{array} \\ \left[\begin{array}{ccccc} 4 & 3 & 4 & 2 & 0 \\ 6 & 3 & 10 & 7 & 2 \\ 1 & 5 & 4 & 0 & 0 \\ 0 & 6 & 2 & 3 & 1 \\ 13 & 22 & 11 & 9 & 0 \end{array} \right] \end{array}$$

The transition probability matrices are calculated by normalization of the transition frequency matrixes as follows:

$$\hat{p}^{(11)} = \begin{array}{c} \\ D_1 \\ D_2 \\ D_3 \\ D_4 \\ D_5 \end{array} \begin{array}{c} \begin{array}{ccccc} D_1 & D_2 & D_3 & D_4 & D_5 \end{array} \\ \left[\begin{array}{ccccc} 0.292 & 0.250 & 0.167 & 0.208 & 0.083 \\ 0.175 & 0.375 & 0.325 & 0.100 & 0.025 \\ 0.226 & 0.355 & 0.290 & 0.129 & 0.000 \\ 0.095 & 0.286 & 0.238 & 0.381 & 0.000 \\ 0.500 & 0.500 & 0.000 & 0.000 & 0.000 \end{array} \right] \end{array}$$

$$\widehat{p}^{(22)} = \begin{array}{c} \\ D_1 \\ D_2 \\ D_3 \\ D_4 \\ D_5 \end{array} \begin{array}{c} \begin{array}{ccccc} D_1 & D_2 & D_3 & D_4 & D_5 \end{array} \\ \left[\begin{array}{ccccc} 0.067 & 0.200 & 0.067 & 0.200 & 0.467 \\ 0.200 & 0.250 & 0.150 & 0.150 & 0.250 \\ 0.067 & 0.200 & 0.267 & 0.200 & 0.267 \\ 0.136 & 0.045 & 0.227 & 0.455 & 0.136 \\ 0.130 & 0.174 & 0.065 & 0.043 & 0.587 \end{array} \right] \end{array}$$

$$\widehat{p}^{(33)} = \begin{array}{c} \\ D_1 \\ D_2 \\ D_3 \\ D_4 \\ D_5 \end{array} \begin{array}{c} \begin{array}{ccccc} D_1 & D_2 & D_3 & D_4 & D_5 \end{array} \\ \left[\begin{array}{ccccc} 0.231 & 0.308 & 0.154 & 0.077 & 0.231 \\ 0.074 & 0.556 & 0.037 & 0.074 & 0.259 \\ 0.100 & 0.300 & 0.100 & 0.200 & 0.300 \\ 0.167 & 0.083 & 0.167 & 0.500 & 0.083 \\ 0.089 & 0.107 & 0.071 & 0.018 & 0.714 \end{array} \right] \end{array}$$

$$\hat{p}^{(12)} = \begin{array}{c} \\ D_1 \\ D_2 \\ D_3 \\ D_4 \\ D_5 \end{array} \begin{array}{c} \begin{array}{ccccc} D_1 & D_2 & D_3 & D_4 & D_5 \end{array} \\ \left[\begin{array}{ccccc} 0.292 & 0.208 & 0.000 & 0.083 & 0.417 \\ 0.125 & 0.250 & 0.100 & 0.050 & 0.475 \\ 0.032 & 0.129 & 0.323 & 0.194 & 0.323 \\ 0.095 & 0.048 & 0.095 & 0.524 & 0.238 \\ 0.000 & 0.000 & 0.000 & 0.000 & 1.000 \end{array} \right] \end{array}$$

$$\hat{p}^{(21)} = \begin{array}{c} D_1 \\ D_2 \\ D_3 \\ D_4 \\ D_5 \end{array} \overset{\begin{array}{ccccc} D_1 & D_2 & D_3 & D_4 & D_5 \end{array}}{\begin{bmatrix} 0.400 & 0.200 & 0.133 & 0.267 & 0.000 \\ 0.200 & 0.200 & 0.300 & 0.200 & 0.100 \\ 0.067 & 0.600 & 0.200 & 0.133 & 0.000 \\ 0.136 & 0.364 & 0.273 & 0.182 & 0.045 \\ 0.217 & 0.326 & 0.304 & 0.152 & 0.000 \end{bmatrix}}$$

$$\hat{p}^{(13)} = \begin{array}{c} D_1 \\ D_2 \\ D_3 \\ D_4 \\ D_5 \end{array} \overset{\begin{array}{ccccc} D_1 & D_2 & D_3 & D_4 & D_5 \end{array}}{\begin{bmatrix} 0.208 & 0.208 & 0.042 & 0.000 & 0.542 \\ 0.098 & 0.244 & 0.073 & 0.000 & 0.585 \\ 0.033 & 0.233 & 0.200 & 0.133 & 0.400 \\ 0.143 & 0.238 & 0.000 & 0.381 & 0.238 \\ 0.000 & 0.500 & 0.000 & 0.000 & 0.500 \end{bmatrix}}$$

$$\hat{p}^{(31)} = \begin{array}{c} D_1 \\ D_2 \\ D_3 \\ D_4 \\ D_5 \end{array} \overset{\begin{array}{ccccc} D_1 & D_2 & D_3 & D_4 & D_5 \end{array}}{\begin{bmatrix} 0.308 & 0.231 & 0.308 & 0.154 & 0.000 \\ 0.214 & 0.107 & 0.357 & 0.250 & 0.071 \\ 0.100 & 0.500 & 0.400 & 0.000 & 0.000 \\ 0.000 & 0.500 & 0.167 & 0.250 & 0.083 \\ 0.236 & 0.400 & 0.200 & 0.164 & 0.000 \end{bmatrix}}$$

The vector x below are derived by computing the proportion of occurrence of each state in S_1, S_2 and S_3 sequences.

$$\hat{x}^{(1)} = \begin{bmatrix} \frac{24}{119} & \frac{40}{119} & \frac{31}{119} & \frac{21}{119} & \frac{3}{119} \end{bmatrix}^T$$

$$\hat{x}^{(2)} = \begin{bmatrix} \frac{15}{119} & \frac{20}{119} & \frac{16}{119} & \frac{22}{119} & \frac{46}{119} \end{bmatrix}^T$$

$$\hat{x}^{(3)} = \begin{bmatrix} \frac{13}{119} & \frac{29}{119} & \frac{10}{119} & \frac{12}{119} & \frac{55}{119} \end{bmatrix}^T$$

In theoretical explanations variable m shows the number of the states, therefore m=*5* and s variable shows the number of the categorical sequences, therefore s=*3*. In this case, separate computations should be made for j=*1* and j=*2*.

The below calculations are made in order to solve the LPP which was represented in (2.13) for the above Crude oil and Petrol price data.

For j=1:
Objective function: min w_1
Subject to:

$$\begin{pmatrix} w_1 \\ w_1 \\ w_1 \\ w_1 \\ w_1 \end{pmatrix} \geq \hat{x}^{(1)} - B \begin{pmatrix} \lambda_{11} \\ \lambda_{12} \end{pmatrix}, \begin{pmatrix} w_1 \\ w_1 \\ w_1 \\ w_1 \\ w_1 \end{pmatrix} \geq -\hat{x}^{(1)} + B \begin{pmatrix} \lambda_{11} \\ \lambda_{12} \end{pmatrix}$$

$$w_1 \geq 0$$

$$\sum_{k=1}^{2} \lambda_{1k} = 1 \Rightarrow \lambda_{11} + \lambda_{12} = 1, \;\; \lambda_{11}, \, \lambda_{12} \geq 0,$$

where

$$B = \left[\hat{p}^{(11)} \;\; \hat{x}^{(1)} \middle| \; \hat{p}^{(12)} \hat{x}^{(2)} \right] = \begin{bmatrix} 0.2251 & 0.2482 \\ 0.2643 & 0.2641 \\ 0.2632 & 0.2296 \\ 0.2445 & 0.2217 \\ 0.2689 & 0.3866 \end{bmatrix}$$

For j=2:
Objective function: min w_2
Subject to:

$$\begin{pmatrix} w_2 \\ w_2 \\ w_2 \\ w_2 \\ w_2 \end{pmatrix} \geq \hat{x}^{(2)} - B \begin{pmatrix} \lambda_{21} \\ \lambda_{22} \end{pmatrix}, \begin{pmatrix} w_2 \\ w_2 \\ w_2 \\ w_2 \\ w_2 \end{pmatrix} \geq -\hat{x}^{(2)} + B \begin{pmatrix} \lambda_{21} \\ \lambda_{22} \end{pmatrix}$$

$$w_2 \geq 0$$

$$\sum_{k=1}^{2} \lambda_{2k} = 1 \Rightarrow \lambda_{21} + \lambda_{22} = 1, \;\; \lambda_{21}, \, \lambda_{22} \geq 0,$$

where

$$B = \left[\hat{p}^{(21)} \;\; \hat{x}^{(1)} \middle| \; \hat{p}^{(22)} \hat{x}^{(2)} \right] == \begin{bmatrix} 0.2297 & 0.2683 \\ 0.2235 & 0.2118 \\ 0.2908 & 0.2179 \\ 0.2540 & 0.1921 \\ 0.2596 & 0.2894 \end{bmatrix}$$

λ values are obtained by computations of the LPP for equation (2.13) with j=1 and j=2.

$$\lambda = \begin{bmatrix} \lambda_{11} & \lambda_{12} \\ \lambda_{21} & \lambda_{22} \end{bmatrix} = \begin{bmatrix} 0.4917 & 0.5083 \\ 0.4620 & 0.5380 \end{bmatrix}$$

The multivariate Markov chain model for the Crude oil and Petrol price categorical data sequence $S_1$and S_2 is given below by putting λ matrix values into equation (2.6),

$$x_{n+1}^{(1)} = 0.4917 \hat{p}^{(11)} x_n^{(1)} + 0.5083 \hat{p}^{(12)} x_n^{(2)} \tag{4.1}$$

$$x_{n+1}^{(2)} = 0.4620 \hat{p}^{(21)} x_n^{(1)} + 0.5380 \hat{p}^{(22)} x_n^{(2)} \tag{4.2}$$

The above same procedures are made in order to solve the LPP which was represented in (2.6) for the Crude oil and Diesel price data.

For j=2:

$$B = \left[\hat{p}^{(31)}\ \hat{x}^{(1)} \middle| \hat{p}^{(33)}\hat{x}^{(3)}\right] = \begin{bmatrix} 0.2469 & 0.2275 \\ 0.2182 & 0.2739 \\ 0.2924 & 0.2513 \\ 0.2577 & 0.1415 \\ 0.2631 & 0.1415 \end{bmatrix}$$

λ values are obtained by computations of the LPP for equation (2.13) with $j = 1$ and $j = 2$.

$$\lambda = \begin{bmatrix} \lambda_{11} & \lambda_{12} \\ \lambda_{21} & \lambda_{22} \end{bmatrix} = \begin{bmatrix} 0.4281 & 0.5719 \\ 0.4732 & 0.5268 \end{bmatrix}$$

The multivariate Markov chain model for the Crude oil and Petrol price categorical data sequence $S_1$and S_3 is given below by putting λ matrix values into equation (2.6),

$$x_{n+1}^{(1)} = 0.4281\hat{p}^{(11)}x_n^{(1)} + 0.5719\hat{p}^{(13)}x_n^{(3)} \tag{4.3}$$

$$x_{n+1}^{(2)} = 0.4732\hat{p}^{(31)}x_n^{(1)} + 0.5268\hat{p}^{(33)}x_n^{(3)} \tag{4.4}$$

5. CONCLUSIONS

In this paper, monthly changes of Crude oil, Petrol and Diesel price are taken as three categorical data sequences. In multivariate Markov chain model, data sequences should have correlation among themselves are used. In this paper the multivariate Markov chain model has been discussed from the theoretical perspective clearly to the extent that is possible and within an application of Petrol and Diesel price prediction. Using the equation (4.2) and (4.4), we can predict the Petrol and Diesel price respectively. From the equation (4.2), substitute $\hat{p}^{(21)}$, $x_n^{(1)}$, $\hat{p}^{(22)}$ and $x_n^{(2)}$values, we can get the $x_{n+1}^{(2)}$ value. Similarly from the equation (4.4), substitute$\hat{p}^{(31)}$, $x_n^{(1)}$, $\hat{p}^{(33)}$and $x_n^{(2)}$values, we can get the $x_{n+1}^{(3)}$ value. From these values we can predict probability values of moving from one state to another states are predicted for Petrol and Diesel prices.

Acknowledgment. The authors acknowledge University Grants Commission for providing the infrastructure facility under the scheme of SAP (DRS-I). The second author acknowledges UGC for financial support to carry out this research work under Basic Science Research Fellowship.

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