Global and Stochastic Analysis
Vol. 4 No. 1, January (2017), 1-10

NON STANDARD FITTED FINITE DIFFERENCE METHOD FOR SINGULAR PERTURBATION PROBLEMS USING CUBIC SPLINE

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ABSTRACT. This paper presents a class of accurate non standard finite difference method based on cubic spline function on uniform mesh for the numerical solution of a second order singularly perturbed two point boundary value problems associate with boundary layer theory. A fitting factor known to be artificial viscosity was introduced in discretization equation of non standard finite difference scheme and its value was obtained from the theory of singular perturbations. We have solved the discretization equation using discrete invariant imbedding. The proposed scheme is analyzed for Convergence. The numerical results for several test examples demonstrate the efficiency of the proposed method.

1. Introduction

Singularly perturbed two point boundary value problems occurs frequently in many areas of applied mathematics and mathematical physics such as fluid mechanics, elasticity, optimal control, chemical-reactor theory, aerodynamics, reaction diffusion process, geophysics and many other related areas. Solutions of this type of equations exhibit boundary layer phenomena; that is, the solution of these problems varies rapidly in some parts and varies slowly in some other parts. The numerical treatment of singularly perturbed differential equations is far from trivial and also gives major computational difficulties due to the presence of boundary and/or interior layers. Standard numerical methods fail to approximate their solutions because of the boundary layer behavior. So to capture the layers, two classical approaches are used. The first approach is to design a special mesh known as Shishkin mesh [15] or graded meshes [3] and the second approach is a fitting factor is introduced in the existing finite difference methods known as fitted operator method [9] to control the behavior in the boundary layer. This paper uses the second approach to solve the singularly perturbed two point boundary value problems.

To describe this method, we consider singularly perturbed two point boundary value problems of the form

$$\varepsilon y''(x) + p(x)y'(x) + q(x)y(x) = r(x), 0 \leq \text{x} \leq 1 \tag{1.1}$$

2000 *Mathematics Subject Classification.* Primary 65L11; Secondary 65L12.
Key words and phrases. Singularly perturbed boundary value problem, Fitting factor, Cubic spline, Tridiagonal system, Invariant imbedding algorithm.

with boundary conditions

$$y(0) = \alpha, y(1) = \beta \tag{1.2}$$

where $0 < \varepsilon << 1$, $p(x), q(x)$ and $r(x)$ are bounded continuous functions in (0, 1), and α, β are finite constants. This type of problem was solved asymptotically by Bellman [1], Bender and Orszag [2], Kevorkian and Cole [6], Nayfeh [10], O'Malley [11] and numerically by Kreiss [7], Miller [9], Kadalbajoo and Kumar [5], Rashidinia etc. [14], Lin and Vancouver [8], Reddy [4, 13].

In this paper, a fitted non standard finite difference scheme using cubic spline functions on a uniform mesh was presented for solving linearly singularly perturbed two-point boundary value problems with boundary layer at one end point. In section 2, cubic spline was discussed briefly. In section 3, numerical method using cubic spline was presented. In section 4, convergence analysis of the method was discussed. To justify the proposed method, test examples are illustrated in section 5. Finally, in section 6, summary and conclusions of the method are given.

2. Description of the method

Discretize the interval [0,1] into N equal subintervals of mesh size $h = \frac{1}{N}$, so that $x_i = ih$, i =0, 1, 2,..., N with $0 = x_0, 1 = x_N$. Let $y(x)$ be the exact solution and y_i be an approximation to $y(x_i)$ obtained by the polynomial cubic spline$S_i(x)$ passing through the points (x_i, y_i) and(x_{i+1}, y_{i+1}). From the theory of the Splines, $S_i(x)$ satisfies not only the interpolatory conditions at x_i andx_{i+1} but also the continuity of first derivative at the common nodes $(x_i, y_i)$are fulfilled. For each i th segment, the cubic polynomial spline function $S_i(x)$has the form

$$S_i(x) = a_i + b_i(x - x_i) + c_i(x - x_i)^2 + d_i(x - x_i)^3,\ i = 0,\ 1,\ 2, ..., N - 1. \tag{2.1}$$

where a_i, b_i, c_i and d_i are constants.

A cubic spline function $S(x)$ of class $C^2[a,\ b]$ interpolates $y(x)$ at the grid pointsx_i for i = 0,1,2.......,N. To develop expressions for the four coefficients of Eq. (2.1) in terms ofy_i,y_{i+1},M_i and M_{i+1}.

We first define $S_i(x_i) = y_i$,$S_i(x_{i+1}) = y_{i+1}$,$S_i''(x_i) = M_i$,$S_i''(x_{i+1}) = M_{i+1}$. From algebraic manipulation, we obtain the following expression:

$$a_i = y_i,\ b_i = \frac{y_{i+1} - y_i}{h} - \frac{M_{i+1} - M_i}{6},\ c_i = \frac{M_i}{2},\ d_i = \frac{M_{i+1} - M_i}{6h}$$

Where i = 0, 1, 2...N-1.

Using the continuity of cubic spline $S(x)$ and its first derivative at(x_i, y_i), that is$S_{i-1}'(x_i) = S_i'(x_i)$, we obtain the following relations for i =1,2,...,N-1.

$$M_{i+1} + 4M_i + M_{i-1} = \frac{6}{h^2}\left(y_{i+1} - 2y_i + y_{i-1}\right) \tag{2.2}$$

3. Numerical Scheme

3.1. Left − end boundary layer. We assume that $q(x) \leq 0$, $p(x) \geq \bar{M} > 0$ throughout the interval [0, 1], where $\bar{M}$ is positive constant. Under these assumptions, Eq. (1.1) has unique solution and display boundary layer at $x = 0$ for small values of ε.

From the theory of singular perturbation it is known that the solution of Eq. (1.1) and Eq. (1.2) is of the form (O'Malley [11])

$$y(x) = y_0(x) + \frac{p(0)}{p(x)}\left(\alpha - y_0(0)\right) e^{-\int_0^x \left(\frac{p(x)}{\varepsilon} - \frac{q(x)}{p(x)}\right)dx} + O(\varepsilon) \tag{3.1}$$

where $y_0(x)$ is the solution of reduced problem $p(x)y_0'(x) + q(x)y_0(x) = r(x)$,with $y_0(1) = \beta$.

Expanding $p(x)$ and $q(x)$ about the point '0' up to first term using Taylor's series, Eq. (3.1) gives

$$y(x) = y_0(x) + \left(\alpha - y_0(0)\right) e^{-\left(\frac{p(0)}{\varepsilon} - \frac{q(0)}{p(0)}\right)x} + O(\varepsilon) \tag{3.2}$$

From Eq. (3.2) we have

$$y(x_i) = y_0(x_i) + \left(\alpha - y_0(0)\right) e^{-\left(\frac{p(0)}{\varepsilon} - \frac{q(0)}{p(0)}\right)x_i} + O(\varepsilon),$$

i.e.,

$$y(ih) = y_0(ih) + \left(\alpha - y_0(0)\right) e^{-\left(\frac{p(0)}{\varepsilon} - \frac{q(0)}{p(0)}\right)ih} + O(\varepsilon),$$

therefore

$$\lim_{h\to 0} y(ih) = y_0(0) + \left(\alpha - y_0(0)\right) e^{-\left(\frac{p^2(0) - \varepsilon q(0)}{p(0)}\right)i\rho} + O(\varepsilon), \text{where} \rho = \frac{h}{\varepsilon} \tag{3.3}$$

At the grid pointx_i, the proposed differential equation Eq. (1.1) may be discretized by

$$\varepsilon\ M_i = r(x_i) - p(x_i)y_i'(x) - q(x_i)y(x_i)$$

The non standard finite differences of y_{i-1}', y_{i+1}' and $y_i'$are

$$y_{i-1}' \approx \frac{-y_{i+1} + 4y_i - 3y_{i-1}}{2h}$$

$$y_{i+1}' \approx \frac{3y_{i+1} - 4y_i + y_{i-1}}{2h}$$

$$y_i' \approx \frac{y_{i+1} - y_{i-1}}{2h}.$$

Substituting the values of M_i, M_{i-1} and M_{i+1} along with above differences in Eq. (2.2), we get

$$\frac{6\varepsilon}{h^2}\left(y_{i+1} - 2y_i + y_{i-1}\right) = \left(\frac{-p_{i+1}}{2h} + \frac{2p_i}{h} - q_{i-1} + \frac{3p_{i-1}}{2h}\right) y_{i-1} + \left(\frac{2p_{i+1}}{h} - 4q_i - \frac{2p_{i-1}}{h}\right) y_i + \left(\frac{-3p_{i+1}}{2h} - \frac{2p_i}{h} - q_{i-1} + \frac{p_{i-1}}{2h}\right) + \left(r_{i+1} + 4r_i + r_{i-1}\right) \tag{3.4}$$

Introducing fitting factor $\sigma(\rho)$ in Eq. (3.4), we get

$$\frac{6\sigma(\rho)\varepsilon}{h^2}\left(y_{i+1} - 2y_i + y_{i-1}\right) = \left(\frac{-p_{i+1}}{2h} + \frac{2p_i}{h} - q_{i-1} + \frac{3p_{i-1}}{2h}\right) y_{i-1} + \left(\frac{2p_{i+1}}{h} - 4q_i - \frac{2p_{i-1}}{h}\right) y_i + \left(\frac{-3p_{i+1}}{2h} - \frac{2p_i}{h} - q_{i-1} + \frac{p_{i-1}}{2h}\right) + \left(r_{i+1} + 4r_i + r_{i-1}\right) \tag{3.5}$$

Multiplying Eq. (3.5) by h and taking limit as $h \to 0$, we get

$$\lim_{h\to 0}\frac{\sigma}{\rho}\left(y(i+1)h-2y(ih)+y(i-1)h\ \right)=\frac{p(0)}{2}\lim_{h\to 0}\left(y(i-1)h-y(i+1)h\right). \tag{3.6}$$

By substituting Eq. (3.3) in to Eq. (3.6), we get

$$\sigma=\frac{\rho}{4}p(0)\coth\left(\left(\frac{p(0)^2-\varepsilon q(0)}{p(0)}\right)\frac{\rho}{2}\right) \tag{3.7}$$

is the required fitting factor in the left end boundary layer. From Eq.(3.5) we get the following tridiagonal system

$$E_i y_{i-1}+F_i\ y_i+G_i y_{i+1}=H_i \text{ for } i=1,2,...,\ N-1 \tag{3.8}$$

where

$$E_i=-\varepsilon\sigma+\frac{h}{4}\ p_{i-1}+\frac{h}{3}p_i-\frac{h}{12}p_{i+1}-\frac{h^2}{6}\ q_{i-1},\quad F_i=2\varepsilon\sigma-\frac{h}{3}p_{i-1}+\frac{h}{3}p_{i+1}-\frac{2h^2}{3}\ q_i$$

$$G_i=-\varepsilon\sigma+\frac{h}{12}\ p_{i-1}-\frac{h}{3}p_i-\frac{h}{4}p_{i+1}-\frac{h^2}{6}\ q_{i+1},\quad H_i=\ -\frac{h^2}{6}\left[r_{i-1}+4\ r_i+r_{i+1}\right]$$

$p(x_i)=p_i$,$q(x_i)=q_i$, $r(x_i)=r_i$, *for* $i=0,\ 1,...,N$.

We solve the tridiagonal system (3.8), using method of invariant imbedding algorithm.

3.2. Right-end boundary layer. Further, if we assume that $q(x)\le 0$, $p(x)\le \bar{M}<0$ throughout the interval [0, 1] where $\bar{M}$ is negative constant. Under these assumptions, (1.1) has unique solution and display boundary layer at x =1 for small values of ε.

From the theory of singular perturbation it is known that the solution of Eq. (1.1) and Eq. (1.2) is of the form

$$y(x)=y_0(x)+\frac{p(1)}{p(x)}\left(\alpha-y_0(1)\right)e^{-\int\limits_0^x\left(\frac{p(x)}{\varepsilon}-\frac{q(x)}{p(x)}\right)dx}+O(\varepsilon) \tag{3.9}$$

where $y_0(x)$ is the solution of $p(x)y_0'(x)+q(x)y_0(x)=r(x)$,with $y_0(0)=\alpha$. Expanding $p(x)$ and $q(x)$ about the point '1' up to first term using Taylor's series, Eq. (3.9) gives

$$y(x)=y_0(x)+\left(\alpha-y_0(1)\right)e^{-\left(\frac{p(1)}{\varepsilon}-\frac{q(1)}{p(1)}\right)(1-x)}+O(\varepsilon) \tag{3.10}$$

$$y(x_i)=y_0(x_i)+\left(\alpha-y_0(1)\right)e^{-\left(\frac{p(1)}{\varepsilon}-\frac{q(1)}{p(1)}\right)(1-x_i)}+O(\varepsilon),$$

i.e., $y(ih)=y_0(ih)+\left(\alpha-y_0(1)\right)e^{-\left(\frac{p(1)}{\varepsilon}-\frac{q(1)}{p(1)}\right)(1-ih)}+O(\varepsilon)$, therefore

$$\lim_{h\to 0}y(ih)=y_0(0)+\left(\alpha-y_0(1)\right)e^{-\left(\frac{p^2(1)-\varepsilon q(1)}{p(1)}\right)\left(\frac{1}{\varepsilon}-i\rho\right)}+O(\varepsilon) \tag{3.11}$$

where $\rho=\frac{h}{\varepsilon}$

Now, we consider cubic spline finite difference scheme and introduce the fitting factor $\sigma(\rho)$ as:

$$\tfrac{6\sigma(\rho)\varepsilon}{h^2}\left(y_{i+1}-2y_i+y_{i-1}\right)=\left(\tfrac{-p_{i+1}}{2h}+\tfrac{2p_i}{h}-q_{i-1}+\tfrac{3p_{i-1}}{2h}\right)y_{i-1}+\left(\tfrac{2p_{i+1}}{h}-4q_i-\tfrac{2p_{i-1}}{h}\right)y_i$$
$$+\left(\tfrac{-3p_{i+1}}{2h}-\tfrac{2p_i}{h}-q_{i-1}+\tfrac{p_{i-1}}{2h}\right)+\left(r_{i+1}+4r_i+r_{i-1}\right) \tag{3.12}$$

Multiplying (3.12) by h and taking limit as $h\to 0$, we get

$$\lim_{h\to 0}\frac{\sigma}{\rho}\left(y(i+1)h-2y(ih)+y(i-1)h\ \right)=\frac{p(0)}{2}\lim_{h\to 0}\left(y(i-1)h-y(i+1)h\right). \tag{3.13}$$

By substituting Eq. (3.11) in to Eq. (3.13) we get

$$\sigma=\frac{\rho}{4}p(0)\coth\left(\left(\frac{p(1)^2-\varepsilon q(1)}{p(1)}\right)\frac{\rho}{2}\right) \tag{3.14}$$

is the fitting factor in the right end boundary layer.

From Eq.(3.12) we get the following tridiagonal system

$$E_iy_{i-1}+F_i\ y_i+G_iy_{i+1}=H_i \text{ for } i=1,2,...,\ N-1 \tag{3.15}$$

where

$$E_i=-\varepsilon\sigma+\frac{h}{4}\ p_{i-1}+\frac{h}{3}p_i-\frac{h}{12}p_{i+1}-\frac{h^2}{6}\ q_{i-1},\quad F_i=2\varepsilon\sigma-\frac{h}{3}p_{i-1}+\frac{h}{3}p_{i+1}-\frac{2h^2}{3}\ q_i$$

$$G_i=-\varepsilon\sigma+\frac{h}{12}\ p_{i-1}-\frac{h}{3}p_i-\frac{h}{4}p_{i+1}-\frac{h^2}{6}\ q_{i+1},\quad H_i=\ \text{-}\frac{h^2}{6}\left[r_{i-1}+4\ r_i+r_{i+1}\right]$$

$p(x_i)=p_i\ ,q(x_i)=q_i\ ,\ r(x_i)=r_i,\ for\ \ i=0,\ 1,...,N$

To solve the above system (3.15), we used method of invariant imbedding algorithm.

4. Convergence Analysis

Incorporating the given boundary conditions we obtain the matrix equation as

$$(D+P)Y+Q+T(h)=0 \tag{4.1}$$

where $D=[-\varepsilon\sigma,\ 2\varepsilon\sigma,\ -\varepsilon\sigma]=\begin{bmatrix} 2\varepsilon\sigma & -\varepsilon\sigma & 0 & 0 & .... & 0\\ -\varepsilon\sigma & 2\varepsilon\sigma & -\varepsilon\sigma & 0 & .... & 0\\ 0 & -\varepsilon\sigma & 2\varepsilon\sigma & -\varepsilon\sigma & .... & 0\\ .. & .. & .. & .. & .. & ..\\ .. & .. & .. & .. & .. & ..\\ 0 & .. & .. & 0 & -\varepsilon\sigma & 2\varepsilon\sigma \end{bmatrix}$

and

$$P=[z_i,v_i,w_i]=\begin{bmatrix} v_1 & w_1 & 0 & 0 & .... & 0\\ z_2 & v_2 & w_2 & 0 & .... & 0\\ 0 & z_3 & v_3 & w_3 & .... & 0\\ .. & .. & .. & .. & .. & ..\\ .. & .. & .. & .. & .. & ..\\ 0 & .. & .. & 0 & z_{N-1} & v_{N-1} \end{bmatrix}$$

where

$$z_i = \frac{h}{4}\ p_{i-1} + \frac{h}{3}p_i - \frac{h}{12}p_{i+1} - \frac{h^2}{6}\ q_{i-1},\ v_i = -\frac{h}{3}p_{i-1} + \frac{h}{3}p_{i+1} - \frac{2h^2}{3}\ q$$
$$w_i = \frac{h}{12}\ p_{i-1} - \frac{h}{3}p_i - \frac{h}{4}p_{i+1} - \frac{h^2}{6}\ q_{i+1} fori = 1, 2, 3, 4........N-1$$

and $Q = [q_1 + (-\varepsilon\sigma + z_1)\,\alpha, q_2, q_3, ..., q_{N-2,}q_{N-1} + (-\varepsilon\sigma + w_{N-1})\beta]^T$

where $q_i = \frac{h^2}{6}\ \ (r_{i-1} + 4r_i + r_{i+1})\ \ i = 1, 2, 3, 4, ..., N-1$
$T(h) = 0(h^4)$ and $Y = [Y_1, Y_2, Y_3, ..., Y_{N-1}]^T,\ T(h) = [T_1, T_2, ..., T_{N-1}]^T,$
$O = [0, 0, ..., 0]^T$ are associated vectors of equation (4.1).

Let $y = [y_1, y_2, ..., y_{N-1}]^T \cong Y$ which satisfies the equation

$$(D + P)\,y + Q = 0 \tag{4.2}$$

Let$e_i = y_i - Y_i,\ i = 1(1)N-1$be the discretization error so that

$$E = [e_1, e_2, ..., e_{N-1}]^T = y - Y.$$

Subtracting (4.1) from (4.2), we obtain the error equation

$$(D + P)\,E = T(h) \tag{4.3}$$

Let $|p(x)| \le C_1$ and $|q(x)| \le C_2$ where C_1, C_2 are positive constants. If $P_{i,j}$ be the $(i,j)^{th}$element of the matrix $(D+P)$, then

$$|P_{i,i+1}| \le \varepsilon\sigma + \frac{2h}{3}C_1 + \frac{h^2}{6}C_2, i = 1, 2, ..., N-2$$
$$|P_{i,i-1}| \le \varepsilon\sigma + \frac{2h}{3}C_1 + \frac{h^2}{6}C_2 i = 2, 3, ..., N-1$$

Thus for sufficiently small h_i(i.e. as$h \to 0$) we have

$$\begin{aligned} |P_{i,i+1}| &< \varepsilon\sigma,\ i = 1, 2, ..., .N-2 \\ |P_{i,i-1}| &< \varepsilon\sigma,\ i = 2, 3, ..., N-1 \end{aligned} \tag{4.4}$$

Hence $(D + P)$ is irreducible (see Ref. [16]).

Let $S_i$be the sum of the elements of the ith row of the matrix $(D+P)$, then we have

$$S_i = \varepsilon\sigma - \ \frac{h}{4}p_{i-1} - \frac{h}{3}p_i + \frac{h}{12}p_{i+1} \text{-} \frac{h^2}{6}\,(4q_i + q_{i+1})\ \text{ for } i = 1$$
$$S_i = -\frac{h^2}{6}\ \ \left(q_{\text{i-1}} + 4q_i + q_{i+1}\right)\ \text{ for } i = 2, 3, ..., N-2$$
$$S_i = \varepsilon\sigma - \frac{h}{12}p_{i-1} + \frac{h}{3}p_i + \frac{h}{4}p_{i+1} - \frac{h^2}{6}\,(4q_i + q_{i-1})\,\text{for } i = N-1$$

Let $C_{1^*} = \min\limits_{1 \le i \le N} |p(x)|$ and $C_1^* = \max\limits_{1 \le i \le N} |p(x)|,\ C_{2^*} = \min\limits_{1 \le i \le N} |q(x)|$ and $C_2^* = \max\limits_{1 \le i \le N} |q(x)|$.

Since $0 < \varepsilon << 1$ and $\varepsilon \propto O(h)$,it is verify that for sufficiently small $h, (D+P)$ is monotone [16, 17]. Hence $(D+P)^{-1}$ exists and $(D+P)^{-1} \ge 0$.

Thus from Eq.(4.3), we have

$$\|E\| \le \left\|(D+P)^{-1}\right\|\ \ \|T\| \tag{4.5}$$

For sufficiently small h, we have

$$S_i > h^2 C_2 \ for \ i = 1$$
$$S_i > h^2 C_2 \ for \ i = n-1 \ and$$
$$S_i > h^2 C_2 \ for \ i = 2, 3, ..., n-2.$$

Let $(D+P)^{-1}_{i,k}$ be the $(i,k)^{th}$ element of $(D+P)^{-1}$ and we define

$$\left\|(D+P)^{-1}\right\| = \max_{1 \le i \le N-1} \sum_{k=1}^{N-1} (D+P)^{-1}_{i,k} \text{ and } \|T(h)\| = \max_{1 \le i \le N-1} |T(h)|, \quad (4.6)$$

Since $(D+P)^{-1}_{i,k} \ge 0$ and $\sum_{k=1}^{N-1} (D+P)^{-1}_{i,k} .S_k = 1 \ for \ i = 1, 2, 3, ..., N-1.$

Hence

$$(D+P)^{-1}_{i,k} \le \frac{1}{S_i} < \frac{1}{h^2 C_2}, \ i = 1 \quad (4.7)$$

$$(D+P)^{-1}_{i,k} \le \frac{1}{S_i} < \frac{1}{h^2 C_2}, \ i = N-1 \quad (4.8)$$

Furthermore,

$$\sum_{k=1}^{N-1} (D+P)^{-1}_{i,k} \le \frac{1}{\min_{2 \le i \le N-2} S_i} \ < \ \frac{1}{h^2 C_2} \text{ for i = 1.2.3............N-1} \quad (4.9)$$

By the help of Eqs. (4.6) - (4.9), from (4.5), we obtain

$$\|E\| \le O(h^2). \quad (4.10)$$

Hence the method is second order convergence.

5. Numerical examples

This section presents test examples to demonstrate the efficiency of the method computationally. We consider two numerical examples with left-end boundary layer and one problem with right-end boundary layer of the underlying interval were considered. These problems were chosen because they have been widely discussed in the literature and exact solutions were available for comparison. The maximum absolute errors with and without fitting factor were presented to support the given method.

Example 1. Consider the boundary value problem from Bender and Orszag [2],

$$\varepsilon y''(x) + y'(x) - y(x) = 0; \quad x \in [0, 1]$$

with $y(0)$=1 and $y(1)$=1. Clearly this problem has a boundary layer at x = 1. The exact solution is given by

$$y(x) = \frac{[(e^{m_2} - 1) e^{m_1 x} + (1 - e^{m_1}) e^{m_2 x}]}{(e^{m_2} - e^{m_1})}$$

where $m_1 = (-1 + \sqrt{1 + 4\varepsilon})/2\varepsilon$ and $m_2 = (-1 - \sqrt{1 + 4\varepsilon})/2\varepsilon$.

Table 1 gives the maximum absolute errors for different values of h and ε with and without fitting factor.

Example 2. Consider the following non-homogeneous problem from fluid dynamics for fluid of small viscosity

$$\varepsilon y''(x) + y'(x) = 1 + 2x; \quad x \in [0, 1]$$

with $y(0) = 0$ and $y(1) = 1$. The exact solution is given by

$$y(x) = x(x + 1 - 2\varepsilon) + \frac{(2\varepsilon - 1)(1 - e^{-x/\varepsilon})}{(1 - e^{-1/\varepsilon})}$$

. Table 2 gives the maximum absolute errors for different values of h and ε with and without fitting factor.

Example 3. Consider the boundary problem

$$\varepsilon y''(x) - y'(x) - (1 + \varepsilon)y(x) = 0; \quad x \in [0, 1]$$

with $y(0) = 1 + \exp(-(1 + \varepsilon)/\varepsilon)$ and $y(1) = 1 + 1/e$ Clearly, this problem has a boundary layer at $x = 1$. The exact solution is given by

$$y(x) = e^{(1+\varepsilon)(x-1)/\varepsilon} + e^{-x}$$

Table 3 gives the maximum absolute errors for different values of h and ε with and without fitting factor.

6. Summary and conclusions

Non standard fitted finite difference method for singular perturbation problems using cubic splines was developed. We have introduced a fitting factor which is called artificial viscosity in the spline difference scheme to control the rapid changes in the boundary layer. Convergence analysis of the method is discussed and it shows that our method is second order convergent. We have presented maximum absolute errors for the standard test examples chosen from literature with the proposed method with and without fitting factor. From the numerical results, it shows the importance of the fitting factor introduced in the spline difference scheme.

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TABLE 1. Maximum absolute errors in the solution of Example 1

$\delta\backslash h$	2^4	2^5	2^6	2^7	2^8
With fitting factor					
10^{-3}	1.06(−2)	5.30(−3)	2.50(−3)	1.10(−3)	3.78(−4)
10^{-4}	1.09(−2)	5.60(−3)	2.80(−3)	1.40(−3)	6.79(−4)
10^{-6}	1.09(−2)	5.60(−3)	2.80(−3)	1.40(−3)	7.16(−4)
10^{-8}	1.09(−2)	5.60(−3)	2.80(−3)	1.40(−3)	7.16(−4)
Without fitting factor					
10^{-3}	7.11(−1)	5.55(−1)	4.85(−1)	3.73(−1)	2.16(−1)
10^{-4}	1.05(+0)	8.87(−1)	6.80(−1)	5.99(−1)	5.69(−1)
10^{-6}	1.20(+0)	1.21(+0)	1.19(+0)	1.12(+0)	9.52(−1)
10^{-8}	1.20(+0)	1.22(+0)	1.22(+0)	1.23(+0)	1.22(+0)

TABLE 2. Maximum absolute errors in the solution of Example 2

$\delta\backslash h$	2^4	2^5	2^6	2^7	2^8
With fitting factor					
10^{-3}	5.67(−2)	2.83(−2)	1.34(−2)	5.77(−3)	2.05(−3)
10^{-4}	5.84(−2)	3.01(−2)	1.52(−2)	7.55(−3)	3.69(−3)
10^{-6}	5.86(−2)	3.03(−2)	1.54(−2)	7.75(−3)	3.89(−3)
10^{-8}	5.86(−2)	3.03(−2)	1.54(−2)	7.75(−3)	3.89(−3)
Without fitting factor					
10^{-3}	2.02(+0)	9.09(−1)	7.71(−1)	5.91(−1)	3.42(−1)
10^{-4}	19.5(+0)	4.91(+0)	1.45(+0)	9.52(−1)	9.02(−1)
10^{-6}	1.95(+3)	4.88(+2)	1.22(+2)	30.52(+0)	7.66(+0)
10^{-8}	1.95(+5)	4.88(+4)	1.22(+4)	3.05(+3)	7.62(+2)

TABLE 3. Maximum absolute errors in the solution of Example 3

$\delta\backslash h$	2^4	2^5	2^6	2^7	2^8
With fitting factor					
10^{-3}	1.06(−2)	5.27(−3)	2.48(−3)	1.06(−3)	3.78(−4)
10^{-4}	1.09(−2)	5.59(−3)	2.81(−3)	1.39(−3)	6.80(−4)
10^{-6}	1.09(−2)	5.62(−3)	2.84(−3)	1.43(−3)	7.16(−4)
10^{-8}	1.09(−2)	5.62(−3)	2.84(−3)	1.43(−3)	7.16(−4)
Without fitting factor					
10^{-3}	1.13(+0)	8.79(−1)	7.69(−1)	5.91(−1)	3.42(−1)
10^{-4}	1.66(+0)	1.40(+0)	1.07(+0)	9.48(−1)	9.01(−1)
10^{-6}	1.90(+0)	1.91(+0)	1.89(+0)	1.77(+0)	1.51(+0)
10^{-8}	1.91(+0)	1.92(+0)	1.93(+0)	1.94(+0)	1.94(+0)

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