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CONE PROPERTIES OF LINEAR FUZZY NUMBERS

FELBIN C KENNEDY AND DHANALAKSHMI V

Abstract. In this paper, we introduce a new class of fuzzy numbers called the linear fuzzy numbers, which is a generalization of the classical triangular, trapezoidal fuzzy numbers and some recent numbers like octagonal, hexagonal, pentagonal fuzzy numbers. Due to this representation, it is convenient to the study the common properties of these piecewise linear fuzzy numbers. The various cone properties of this class is studied. We also discuss the projection of any element from the base space $\mathbb{R}^n$ to the class of linear fuzzy numbers. The interest in the subject of projection arises in several situations, having a wide range of applications in pure and applied mathematics such as Convex Analysis, Numerical Linear Algebra, Statistics, Computer Graphics and so on.

1. Introduction

The concept of fuzzy numbers was introduced by Chang and Zadeh in 1972 [5]. Henceforth the research on fuzzy numbers has received considerable attention, and many theoretical[6], [12], [21] and practical achievements[4], [7], [13], [15], [15], [20], [19] have emerged. Fuzzy numbers have found useful in many research fields, such as programming problems, control systems, neural networks, system analysis, signal processing, expert system, regression analysis, decision making and so on. Triangular and trapezoidal fuzzy numbers are two important kinds of fuzzy numbers that have been thoroughly studied by researchers in interval fuzzy analysis. An extensive survey and bibliography on fuzzy intervals can be found in [8]. Recently many scholars have tried to define new fuzzy numbers as basic tools to deal with their respective problems with vagueness and uncertainty. Most of the new classes have piecewise linear membership function. Thus all these fuzzy numbers with piecewise linear membership function can be considered under one class of linear fuzzy numbers and the properties common to all these numbers can be dealt together.

In this paper, we deal linear fuzzy numbers as a subset of $\mathbb{R}^n$ with a closed, convex cone structure. In section 2, we discuss some of the familiar fuzzy numbers in literature and we see that $n = 3$ and $n = 4$ respectively denotes the familiar triangular and trapezoidal fuzzy numbers. In section 3, we prove that addition and scalar multiplication on linear fuzzy numbers is same as that performed in $\mathbb{R}^n$. In section 4, basic concepts and results on covex cones are discussed. Also we introduce a new class of positive linear fuzzy numbers, denoted LFN and discuss the various cone properties that this class satisfies. In section 5, we show that

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the linear complementarity problem on the cone LFN reduces to a classical linear complementarity problem. In section 6, we discuss the projection of any element of $\mathbb{R}^n$ to the class of positive linear fuzzy number.

2. Preliminaries

Definition 2.1 (Fuzzy Sets)**.** The characteristic function χ_A of a crisp set $A \subseteq X$ assigns a value either 0 or 1 to each member in X. This function can be generalized to a function $\mu_{\tilde{A}}$ such that the value assigned to the element of the universal set X fall within a specified range i.e. $\mu_{\tilde{A}} : X \to [0,1]$. The assigned value indicates the membership grade of the element in the set $\tilde{A}$. The function $\mu_{\tilde{A}}$ is called the membership function and the set $\tilde{A} = \{(x, \mu_{\tilde{A}}(x)) : x \in X\}$ is called a fuzzy set.

Definition 2.2 (Fuzzy Numbers)**.** A fuzzy set $\tilde{A}$, defined on the universal set of real numbers $\mathbb{R}$, is said to be a fuzzy number if its membership function has the following characteristics:
i. $\tilde{A}$ is convex i.e. $\mu_{\tilde{A}}(\lambda x_1 + (1-\lambda)x_2) \geq \min(\mu_{\tilde{A}}(x_1), \mu_{\tilde{A}}(x_2))\ \forall\ x_1, x_2 \in \mathbb{R},\ \forall\ \lambda \in [0,1]$
ii. $\tilde{A}$ is normal i.e. $\exists x_0 \in \mathbb{R}$ such that $\mu_{\tilde{A}}(x_0) = 1$
iii. $\mu_{\tilde{A}}$ is piecewise continuous

Definition 2.3 (Linear Fuzzy Number)**.** A fuzzy number is said to be linear fuzzy number if its membership function is piece-wise linear.

Remark 2.4. Some of the familiar and most applied fuzzy numbers, like triangular, trapezoidal fuzzy numbers are linear.

Definition 2.5 (Triangular Fuzzy Number)**.** A linear fuzzy number $\widetilde{A}$ is said to be a triangular fuzzy number denoted by $\widetilde{A} = (a_1, a_2, a_3)$ where $a_1 \leq a_2 \leq a_3$ are real numbers and its membership function $\mu_{\tilde{A}}$is given by

$$\mu_{\tilde{A}}(x) = \begin{cases} 0 & x \leq a_1 \\ m_1(x) & a_1 \leq x \leq a_2 \\ w & x = a_2 \\ m_3(x) & a_2 \leq x \leq a_3 \\ 0 & x \geq a_3 \end{cases}$$

where $w = height(\tilde{A})$, $y = m_1(x)$ is the line joining the points $(a_1, 0)$ and (a_2, w), $y = m_3(x)$ is that of (a_2, w) and $(a_3, 0)$.

Definition 2.6 (Trapezoidal Fuzzy Number)**.** A fuzzy number $\widetilde{A}$ is said to be a trapezoidal fuzzy number denoted by $\widetilde{A} = (a_1, a_2, a_3, a_4)$ where $a_1 \leq a_2 \leq a_3 \leq a_4$ are real numbers and its membership function $\mu_{\tilde{A}}$is given by

$$\mu_{\tilde{A}}(x) = \begin{cases} 0 & x \leq a_1 \\ m_1(x) & a_1 \leq x \leq a_2 \\ w & a_2 \leq x \leq a_3 \\ m_3(x) & a_3 \leq x \leq a_4 \\ 0 & x \geq a_4 \end{cases}$$

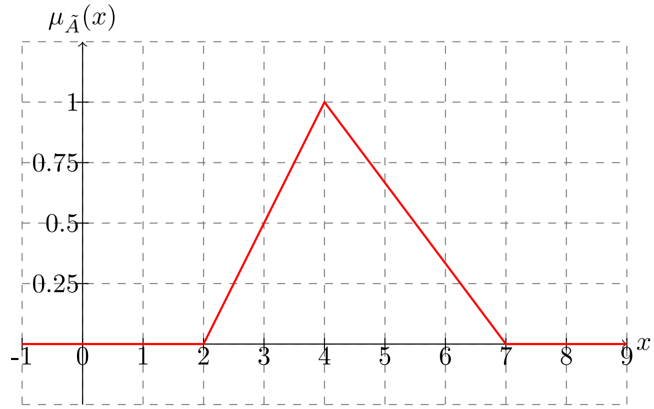


FIGURE 1. Graphical representation of triangular fuzzy number $\tilde{A} = (2, 4, 7)$

where $w = height(\tilde{A})$, $y = m_1(x)$ is the line joining the points $(a_1, 0)$ and (a_2, w), $y = m_3(x)$ is that of (a_2, w), and $(a_3, 0)$.

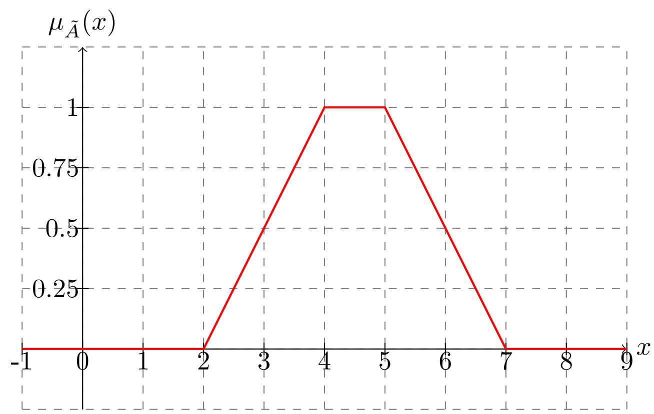


FIGURE 2. Graphical representation of trapezoidal fuzzy number $\tilde{A} = (2, 4, 5, 7)$

Definition 2.7 (Octagonal Fuzzy Number)**.** [14] A fuzzy number $\widetilde{A}$ is said to be an octagonal fuzzy number denoted by $\widetilde{A} = (a_1, a_2, a_3, a_4, a_5, a_6, a_7, a_8; k, w)$ where $a_1 \leq a_2 \leq a_3 \leq a_4 \leq a_5 \leq a_6 \leq a_7 \leq a_8$are real numbers and its membership function $\mu_{\tilde{A}}$is given by

$$\mu_{\tilde{A}}(x) = \begin{cases} 0 & x \leq a_1 \\ m_1(x) & a_1 \leq x \leq a_2 \\ k & a_2 \leq x \leq a_3 \\ m_3(x) & a_3 \leq x \leq a_4 \\ w & a_4 \leq x \leq a_5 \\ m_5(x) & a_5 \leq x \leq a_6 \\ k & a_6 \leq x \leq a_7 \\ m_7(x) & a_7 \leq x \leq a_8 \\ 0 & x \geq a_8 \end{cases}$$

where $0 < k < w$, $w = height(\tilde{A})$ $(w > k)$, $y = m_1(x)$ is the line joining the points $(a_1, 0)$ and (a_2, k), $y = m_3(x)$ is that of (a_3, k) and (a_4, w), $y = m_5(x)$ is that of (a_5, w) and (a_6, k), $y = m_7(x)$ is that of (a_7, k) and $(a_8, 0)$.

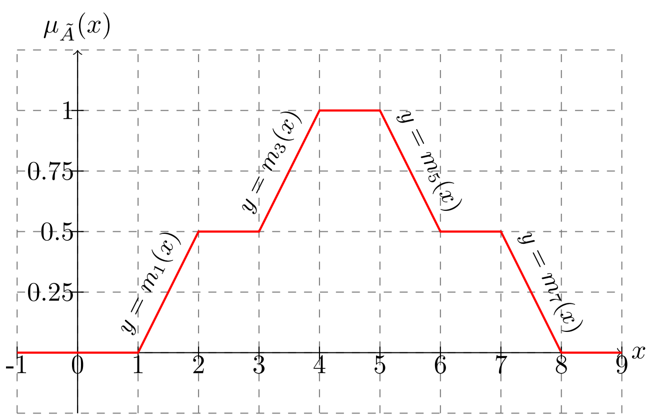


FIGURE 3. Graphical representation of octagonal fuzzy number $\tilde{A} = (1, 2, 3, 4, 5, 6, 7, 8; 0.5, 1)$

3. Addition and Scalar Multiplication

Let $\tilde{A} = (a_1, a_2, ..., a_n)$ and $\tilde{B} = (b_1, b_2, ..., b_n)$ be two linear fuzzy numbers, then there are two ways of defining the arithmetic operations on $\tilde{A}$ and $\tilde{B}$:

(1) $\alpha-$ cut approach, where the interval arithmetic is used on the α-cuts of $\tilde{A}$ and $\tilde{B}$
(2) co-ordinate wise approach, where the operations are performed on the co-ordinates of $\tilde{A}$ and $\tilde{B}$

Definition 3.1. Let $\tilde{A} = (a_1, a_2, ..., a_n)$ and $\tilde{B} = (b_1, b_2, ..., b_n)$ be two linear fuzzy numbers and λ be any real number, then the co-ordinate wise addition and scalar multiplication are defined as follows:
(i) $\tilde{A} + \tilde{B} = (a_1 + b_1, a_2 + b_2, ..., a_n + b_n)$

(ii) $\lambda\tilde{A} = \begin{cases} (\lambda a_1, \lambda a_2, ..., \lambda a_n) & \lambda \geq 0 \\ (\lambda a_n, \lambda a_{n-1}, ..., \lambda a_1) & \lambda < 0 \end{cases}$

Proposition 3.2. *The α-cut approach and co-ordinate wise approach of the addition and scalar multiplication of the octagonal fuzzy numbers yields the same octagonal fuzzy number.*

Proof. Addition: Let $\tilde{A} = (a_1, a_2, ..., a_8; k, w)$ and $\tilde{B} = (b_1, b_2, ..., b_8; k, w)$ be the two octagonal fuzzy numbers, then the $\alpha-$ cut of $\tilde{A} + \tilde{B}$ is as follows:

$$
\begin{aligned}
&[\tilde{A} + \tilde{B}]_\alpha \\
&= [\tilde{A}]_\alpha + [\tilde{B}]_\alpha \\
&= [A^L_\alpha, A^R_\alpha] + [B^L_\alpha, B^R_\alpha] \\
&= \begin{cases} [(A^L_\alpha)_1, (A^R_\alpha)_1] & \alpha \in [0, k] \\ [(A^L_\alpha)_2, (A^R_\alpha)_2] & \alpha \in (k, w] \end{cases} + \begin{cases} [(B^L_\alpha)_1, (B^R_\alpha)_1] & \alpha \in [0, k] \\ [(B^L_\alpha)_2, (B^R_\alpha)_2] & \alpha \in (k, w] \end{cases} \\
&= \begin{cases} [(A^L_\alpha)_1 + (B^L_\alpha)_1, (A^R_\alpha)_1 + (B^R_\alpha)_1] & \alpha \in [0, k] \\ [(A^L_\alpha)_2 + (B^L_\alpha)_2, (A^R_\alpha)_2 + (B^R_\alpha)_2] & \alpha \in (k, w] \end{cases} \\
&= \begin{cases} [a_1 + \frac{\alpha}{k}(a_2 - a_1) + b_1 + \frac{\alpha}{k}(b_2 - b_1), a_8 - \frac{\alpha}{k}(a_8 - a_7) + b_8 - \frac{\alpha}{k}(b_8 - b_7)] & \alpha \in [0, k] \\ [a_3 + \frac{\alpha-k}{w-k}(a_4 - a_3) + b_3 + \frac{\alpha-k}{w-k}(b_4 - b_3), a_5 + \frac{\alpha-w}{k-w}(a_6 - a_5) + b_5 + \frac{\alpha-w}{k-w}(b_6 - b_5)] & \alpha \in (k, w] \end{cases} \\
&= \begin{cases} [a_1 + b_1 + \frac{\alpha}{k}(a_2 + b_2 - a_1 - b_1), a_8 + b_8 - \frac{\alpha}{k}(a_8 + b_8 - a_7 - b_7)] & \alpha \in [0, k] \\ [a_3 + b_3 + \frac{\alpha-k}{w-k}(a_4 + b_4 - a_3 - b_3), a_5 + b_5 + \frac{\alpha-w}{k-w}(a_6 + b_6 - a_5 - b_5)] & \alpha \in (k, w] \end{cases} \\
&= [(a_1 + b_1, a_2 + b_2, ..., a_8 + b_8; k, w)]_\alpha
\end{aligned}
$$

Scalar Multiplication: Let $\tilde{A} = (a_1, a_2, ..., a_8; k, w)$ and $\lambda \in \mathbb{R}$, then according to the interval arithmetic, the scalar multiplication on the interval $[A]_\alpha$ is

$$
\begin{aligned}
\lambda[A]_\alpha &= \begin{cases} [\lambda A^L_\alpha, \lambda A^R_\alpha] & \lambda \geq 0 \\ [\lambda A^R_\alpha, \lambda A^L_\alpha] & \lambda < 0 \end{cases} \\
&= \begin{cases} \begin{cases} [\lambda(A^L_\alpha)_1, \lambda(A^R_\alpha)_1] & \alpha \in [0, k] \\ [\lambda(A^L_\alpha)_2, \lambda(A^R_\alpha)_2] & \alpha \in (k, w] \end{cases} & \lambda \geq 0 \\ \begin{cases} [\lambda(A^R_\alpha)_1, \lambda(A^L_\alpha)_1] & \alpha \in [0, k] \\ [\lambda(A^R_\alpha)_2, \lambda(A^L_\alpha)_2] & \alpha \in (k, w] \end{cases} & \lambda < 0 \end{cases} \\
&= \begin{cases} \begin{cases} [\lambda a_1 + \frac{\alpha}{k}(\lambda a_2 - \lambda a_1), \lambda a_8 - \frac{\alpha}{k}(\lambda a_8 - \lambda a_7)] & \alpha \in [0, k] \\ [\lambda a_3 + \frac{\alpha-k}{w-k}(\lambda a_4 - \lambda a_3), \lambda a_5 + \frac{\alpha-w}{k-w}(\lambda a_6 - \lambda a_5)] & \alpha \in (k, w] \end{cases} & \lambda \geq 0 \\ \begin{cases} [\lambda a_8 - \frac{\alpha}{k}(\lambda a_8 - \lambda a_7), \lambda a_1 + \frac{\alpha}{k}(\lambda a_2 - \lambda a_1)] & \alpha \in [0, k] \\ [\lambda a_5 + \frac{\alpha-w}{k-w}(\lambda a_6 - \lambda a_5), \lambda a_3 + \frac{\alpha-k}{w-k}(\lambda a_4 - \lambda a_3)] & \alpha \in (k, w] \end{cases} & \lambda < 0 \end{cases} \\
&= \begin{cases} [(\lambda a_1, \lambda a_2, ..., \lambda a_8; k, w)]_\alpha & \lambda \geq 0 \\ [(\lambda a_8, \lambda a_7, ..., \lambda a_1; k, w)]_\alpha & \lambda < 0 \end{cases} \\
&= [\lambda A]_\alpha
\end{aligned}
$$

Hence, the resultant of addition and scalar multiplication performed by α-cut approach or co-ordinate wise approach yield the same octagonal fuzzy number.

□

Remark 3.3. Proposition 3.2 holds good for any linear fuzzy number, whose membership function can be determined by the $n-$tuple $(a_1, a_2, ..., a_n)$, where $a_1 \leq a_2 \leq ... \leq a_n$

Remark 3.4. Proposition 3.2 implies that the addition and non-negative scalar multiplication on the collection of linear fuzzy numbers is similar to that in $\mathbb{R}^n$ and hence the collection of linear fuzzy numbers can be considered as a subset of $\mathbb{R}^n$.

4. Convex Cones

Some basic definitions of cones are as follows:

Definition 4.1 (cone)**.** The set $C \subseteq \mathbb{R}^n$ is a cone if $\lambda y \in C$ for all $y \in C$ and $0 \leq \lambda \in \mathbb{R}$.

Definition 4.2 (Convex Cone)**.** A cone C is said to be a convex cone if $y_1 + y_2 \in C$, $\forall y_1, y_2 \in C$.

Definition 4.3 (Pointed Cone)**.** A cone C is said to be pointed if $C \cap -C = \{0\}$.

Definition 4.4 (Order w.r.t. a cone)**.** For a pointed cone C, we write $y_1 \leq_C y_2$ if $y_2 - y_1 \in C$ and $y_1 <_C y_2$ if $y_2 - y_1 \in C - \{0\}$.

Definition 4.5. Let $C \subset \mathbb{R}^n$ be a closed convex cone. The polar cone and the dual cone of C are respectively, the sets,

$$\begin{aligned} C^{\perp} &= \{x \in \mathbb{R}^n \mid \langle x, y \rangle \leq 0, \forall y \in C\} \\ C^{*} &= \{x \in \mathbb{R}^n \mid \langle x, y \rangle \geq 0, \forall y \in C\} \end{aligned}$$

where $\langle . , . \rangle$ is the inner product defined on $\mathbb{R}^n$.

Definition 4.6 (Polyhedral Cone)**.** A polyhedral cone is an intersection of finitely many linear halfspaces, defined by an $m \times n$ matrix A such that $C = \{x \in \mathbb{R}^n \mid Ax \preceq 0\}$

Definition 4.7 (Finitely Generated Cones)**.**

$$cone\{v_1, v_2, ..., v_m\} = \{\lambda_1 v_1 + \lambda_2 v_2 + ... + \lambda_m v_m \mid \lambda_1, \lambda_2, ..., \lambda_m \geq 0\}$$

is called a finitely generated cone generated by $\{v_1, v_2, ..., v_m\}$

Definition 4.8 (Miniedral)**.** [11] The convex cone $C \subseteq \mathbb{R}^n$ is said to be a miniedral when $C = cone\{v_1, v_2, ..., v_n\}$ with $v_1, v_2, ..., v_n$ linearly independent elements in $\mathbb{R}^n$

Definition 4.9. Denote $\mathbb{R}^n_+ = \{x = (x_1, x_2, ..., x_n) \in \mathbb{R}^n | x_i \geq 0, i = 1, 2, ..., n\}$ the nonnegative orthant. Let $A \in \mathbb{R}^{n \times n}$ be a non-singular matrix. Then the cone

$$K = A\mathbb{R}^n_+ = \{Ax | x = (x_1, x_2, ..., x_n) \in \mathbb{R}^n_+\}$$

is called a simplicial cone.

4.1. Linear Fuzzy Numbers as cones. From remark 3.4 the collection of non-negative linear fuzzy numbers can be considered as a subset of $\mathbb{R}^n_+$ defined by

$$LFN = \{x \in \mathbb{R}^n \mid 0 \leq x_1 \leq x_2 \leq \ldots \leq x_n\}$$

which is closed under addition and non-negative scalar multiplication and hence forms a convex cone in $\mathbb{R}^n$.
Also,

$$\begin{aligned} LFN &= \{x \in \mathbb{R}^n \mid 0 \leq x_1 \leq x_2 \leq \ldots \leq x_n\} \\ &= \{x \in \mathbb{R}^n \mid Px \succeq 0\} \\ &= \{x \in \mathbb{R}^n \mid Qx \preceq 0\} \end{aligned}$$

where the columns of P are given by $P_{.,i} = e_i - e_{i+1}$ for $i = 1, 2, ..., n-1$ and $P_{.,n} = e_n$ and the matrix Q is obtained by transposing P and letting $Q_{n,n} = -1$. Here $e_1, e_2, ..., e_n$ are the standard unit vectors of $\mathbb{R}^n$. The above gives the half-space description for LFN, which shows that LFN forms a polyhedral cone.
For example, for the collection of octagonal fuzzy number with same k and w, we get

$$P = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 & 1 \end{pmatrix}$$

$$Q = \begin{pmatrix} 1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 \end{pmatrix}$$

For $u_1 = (1, 1, ..., 1), u_2 = (0, 1, ..., 1), ..., u_n = (0, 0, ..., 1)$, we see that $LFN = cone\{u_1, u_2, ..., u_n\}$ that is, collection of linear fuzzy numbers form a finitely generated cone. Also, since the vectors $u_1, u_2, ..., u_n$ are linearly independent in $\mathbb{R}^n$, LFN forms a miniedral.
Arrange these vectors in the columns of a matrix A and we see that $LFN = A\mathbb{R}^n_+$, a simplicial cone.

Proposition 4.10. *The convex cone LFN is closed.*

Proof. Let $\{\tilde{x}^k\}$ be a sequence such that $\tilde{x}^k \in LFN, k = 1, 2, ...$ Then each $\tilde{x}^k$ can be represented by

$$\tilde{x}^k = \sum_{i=1}^{n} \lambda_i^k u_i$$

as LFN is a finitely generated cone. The convergence of $\tilde{x}^k$ in LFN is guaranteed as the convergence of each $\{\lambda_i^k\}$ to some λ_i which is non-negative as each $\{\lambda_i^k\}$ is non-negative. □

Problem
Obtain the dual cone and polar cone of the cone LFN.
Solution
The dual cone and the polar cone of the cone LFN are given by

$$\begin{aligned} LFN^* &= \{y \in \mathbb{R}^n \mid \langle y, x\rangle \geq 0, \forall x \in LFN\} \\ LFN^{\perp} &= \{y \in \mathbb{R}^n \mid \langle y, x\rangle \leq 0, \forall x \in LFN\} \end{aligned}$$

Since LFN is finitely generated, any element of the cone LFN is a conic combination of the finite number of vectors $u_1, u_2, ..., u_n$ and hence,

$$\begin{aligned} LFN^* &= \{y \in \mathbb{R}^n \mid \langle y, u_i\rangle \geq 0, i = 1, 2, ..., n\} \\ LFN^{\perp} &= \{y \in \mathbb{R}^n \mid \langle y, u_i\rangle \leq 0, i = 1, 2, ..., n\} \end{aligned}$$

Let $y = (y_1, y_2, ..., y_n)$ then $\langle y, u_i\rangle = \sum_{j=i}^{n} y_j$ for $i = 1, 2, ..., n$. Thus,

$$LFN^* = \{y \in \mathbb{R}^n \mid \sum_{j=i}^{n} y_j \geq 0,\ i = 1, 2, ..., n\} \tag{4.1}$$

$$LFN^{\perp} = \{y \in \mathbb{R}^n \mid \sum_{j=i}^{n} y_j \leq 0,\ i = 1, 2, ..., n\} \tag{4.2}$$

Lemma 4.11. [16] *Let $W \subset \mathbb{R}^n$ be a cone and $A \in \mathbb{R}^{n\times n}$ be a non-singular matrix. Then $K = AW$ is a cone whose dual is $K^* = (A^{-1})^T W^*$.*

Lemma 4.12. [3] *Let $A \in \mathbb{R}^{n\times n}$ be a non-singular matrix. Then $(A\mathbb{R}^n_+)^{\perp} = -(A^T)^{-1}\mathbb{R}^n_+$.*

From the above argument, we see that LFN is a simplicial cone given by $LFN = A\mathbb{R}^n_+$, where

$$A = \begin{pmatrix} 1 & 0 & 0 & ... & 0 \\ 1 & 1 & 0 & ... & 0 \\ 1 & 1 & 1 & ... & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & 1 & 1 & ... & 1 \end{pmatrix} \tag{4.3}$$

Let $W = \mathbb{R}^n_+$, $W^* = \mathbb{R}^n_+$ as $\mathbb{R}^n_+$ is self-dual, then from Lemma 4.11, the dual cone of LFN is given by $LFN^* = (A^{-1})^T\mathbb{R}^n_+$ and from Lemma 4.12 the polar cone is given by $LFN^{\perp} = -(A^{-1})^T\mathbb{R}^n_+$ which are same as Equations 4.1 and 4.2.

5. Linear Complementarity Problem on LFN

For the mapping $F : \mathbb{R}^n \to \mathbb{R}^n$ and the cone K, the complementarity problem $CP(F, K)$ is to find $x \in \mathbb{R}^n$ such that

$$x \in K, F(x) \in K^*, \langle x, F(x) \rangle = 0$$

The solution set of $CP(F, K)$ is denoted by $sol(F, K)$.
Using Lemma 4.11 the following results are obtained in [16]:

Proposition 5.1. *If W is a cone, $A \in GL(m, \mathbb{R})$ and $K = AW$, then $sol(F, K) = A(sol(A^T FA, W))$.*

The complementarity problem $CP(F, K)$ with $F(x) = Mx + q$, where $M \in \mathbb{R}^{n\times n}$ and $q \in \mathbb{R}$ is denoted by $LCP(K, M, q)$ called linear complementarity problem.
The solution set of $LCP(K, M, q)$ is denoted by $sol(K, M, q)$. In this case, Proposition 5.1 becomes

Proposition 5.2. [16] *If W is a cone, $A \in GL(n, \mathbb{R})$ and $K = AW$, then $sol(K, M, q) = A(sol(W, A^T MA, A^T q))$.*

For simplicial cone, Proposition 5.2 reduces to

Proposition 5.3. *If $K = A\mathbb{R}^n_+$ is a simplicial cone, then*

$$sol(K, M, q) = A(sol(\mathbb{R}^n_+, A^T MA, A^T q))$$

Remark 5.4. Using Proposition 5.3, we see that the linear complementarity problem on LFN is equivalent to the linear complementarity problem on $\mathbb{R}^n_+$. To be more precise, the linear complementarity problem on the collection of non-negative linear fuzzy numbers is equivalent to the classical linear complementarity problem. Given the linear transformation $M : \mathbb{R}^n \to \mathbb{R}^n$ and $q \in \mathbb{R}^n$, the problem of finding $x \in \mathbb{R}^n$ such that

$$x \in LFN, y = Mx + q \in LFN^*, \langle x, y \rangle = 0$$

is simply equal to the classical linear complementarity problem of finding $x' \in \mathbb{R}^n$ such that

$$x' \geq 0, y' = A^T MAx' + A^T q \geq 0, \langle x', y' \rangle = 0$$

and thus $x = Ax'$, where A is as given in equation 4.3.

6. Projection onto the LFN

Definition 6.1. The operator $P : \mathbb{R}^n \to LFN$ is said to be a projection operator if

$$\|\bar{x} - P\bar{x}\| = \inf_{\tilde{y}\in LFN} \|\bar{x} - \tilde{y}\|$$

where $\|.\|$ is the usual Euclidean norm.

Theorem 6.2. *Let P be the projection operator on LFN, then P satisfies:*

(1) $P\tilde{y} = \tilde{y}$
(2) $\langle \bar{x} - \bar{y}, P\bar{x} - P\bar{y} \rangle \geq \|P\bar{x} - P\bar{y}\|^2, \ \forall\ \bar{x}, \bar{y} \in \mathbb{R}^n$

(3) $\langle \bar{x}-\bar{y}, P\bar{x}-P\bar{y}\rangle \geq 0,\ \forall\ \bar{x},\bar{y}\in\mathbb{R}^n$
(4) $\|\bar{x}-\bar{y}\|^2 \geq \|P\bar{x}-P\bar{y}\|^2 + \|(\bar{x}-\bar{y})-(P\bar{x}-P\bar{y})\|^2,\ \forall\ \bar{x},\bar{y}\in\mathbb{R}^n$
(5) $\|P\bar{x}-P\bar{y}\| \leq \|\bar{x}-\bar{y}\|,\ \forall\ \bar{x},\bar{y}\in\mathbb{R}^n$

Proof. (1) This follows because each $\tilde{y}\in LFN$ is its best approximation in LFN, so that $P\tilde{y}=\tilde{y}$.

(2) We have

$$\begin{aligned}\langle \bar{x}-\bar{y}, P\bar{x}-P\bar{y}\rangle &= \langle \bar{x}-P\bar{x}, P\bar{x}-P\bar{y}\rangle\\ &+\langle P\bar{x}-P\bar{y}, P\bar{x}-P\bar{y}\rangle\\ &+\langle P\bar{y}-\bar{y}, P\bar{x}-P\bar{y}\rangle\end{aligned}$$

The first and third term on the right hand side are nonnegative by eq. (6.1) and the second term is $\|P\bar{x}-P\bar{y}, P\bar{x}-P\bar{y}\|^2$. Hence (2).

(3) An immediate consequence of (2).

(4) Using (2), we obtain for each $\bar{x},\bar{y}\in\mathbb{R}^n$ that

$$\begin{aligned}\|\bar{x}-\bar{y}\|^2 &= \|[\bar{x}-P\bar{x}]+[P\bar{x}-P\bar{y}]+[P\bar{y}-\bar{y}]\|^2\\ &= \|P\bar{x}-P\bar{y}\|^2+\|[\bar{x}-P\bar{x}]-[\bar{y}-P\bar{y}]\|^2\\ &\quad +2\langle P\bar{x}-P\bar{y}, [\bar{x}-P\bar{x}]-[\bar{y}-P\bar{y}]\rangle\\ &= \|P\bar{x}-P\bar{y}\|^2+\|[\bar{x}-P\bar{x}]-[\bar{y}-P\bar{y}]\|^2\\ &\quad +2\langle P\bar{x}-P\bar{y}, \bar{x}-\bar{y}\rangle-2\|P\bar{x}-P\bar{y}\|^2\\ &\geq \|P\bar{x}-P\bar{y}\|^2+\|[\bar{x}-P\bar{x}]-[\bar{y}-P\bar{y}]\|^2\end{aligned}$$

This proves (4).

(5) Follows immediately from (4).

□

Theorem 6.3. *A point $\tilde{y}_{\bar{x}}\in LFN$ is the projection of $\bar{x}\in\mathbb{R}^n$ if and only if*

$$\langle \tilde{y}_{\bar{x}}\bar{x}, \tilde{y}-\tilde{y}_{\bar{x}}\rangle \leq 0,\ \forall \tilde{y}\in LFN \tag{6.1}$$

Proof. Let $\tilde{y}_{\bar{x}}$ be the solution of

$$\inf\{\frac{1}{2}\|\tilde{y}-\bar{x}\|^2 \mid \tilde{y}\in LFN\} \tag{6.2}$$

i.e. $\tilde{y}_{\bar{x}}$ is a point of LFN closest to $\bar{x}\in\mathbb{R}^n$ for the Euclidean distance.
Let $f_{\bar{x}}: LFN\to\mathbb{R}$ be the function defined by

$$f_{\bar{x}}(\tilde{y}) = \frac{1}{2}\|\tilde{y}-\bar{x}\|^2 \tag{6.3}$$

Let $\tilde{y} \in LFN$ be an arbitrary point. Since, LFN is a closed(proposition 4.10), convex cone, we have $\tilde{y}_{\bar{x}} + \alpha(\tilde{y} - \tilde{y}_{\bar{x}}) \in LFN$ for any $\alpha \in (0,1)$. Then, by eq. (6.3)

$$\begin{aligned} f_{\bar{x}}(\tilde{y}_{\bar{x}}) \quad &\leq \quad f_{\bar{x}}(\tilde{y}_{\bar{x}} + \alpha(\tilde{y} - \tilde{y}_{\bar{x}})) \\ &= \frac{1}{2}\|\tilde{y}_{\bar{x}} - \bar{x} + \alpha(\tilde{y} - \tilde{y}_{\bar{x}})\|^2 \\ &= \frac{1}{2}\langle \tilde{y}_{\bar{x}} - \bar{x} + \alpha(\tilde{y} - \tilde{y}_{\bar{x}}), \tilde{y}_{\bar{x}} - \bar{x} + \alpha(\tilde{y} - \tilde{y}_{\bar{x}})\rangle \\ &= \frac{1}{2}\left\{\|\tilde{y}_{\bar{x}} - \bar{x}\|^2 + \|\alpha(\tilde{y} - \tilde{y}_{\bar{x}})\|^2 + 2\langle \tilde{y}_{\bar{x}} - \bar{x}, \alpha(\tilde{y} - \tilde{y}_{\bar{x}})\rangle\right\} \\ &= \alpha\langle \tilde{y}_{\bar{x}} - \bar{x}, \tilde{y} - \tilde{y}_{\bar{x}}\rangle + \frac{1}{2}\|\tilde{y}_{\bar{x}} - \bar{x}\|^2 + \frac{1}{2}\alpha^2\|\tilde{y} - \tilde{y}_{\bar{x}}\|^2 \\ &= \alpha\langle \tilde{y}_{\bar{x}} - \bar{x}, \tilde{y} - \tilde{y}_{\bar{x}}\rangle + f_{\bar{x}}(\tilde{y}_{\bar{x}}) + \frac{1}{2}\alpha^2\|\tilde{y} - \tilde{y}_{\bar{x}}\|^2 \end{aligned}$$

$$\Rightarrow 0 \leq \alpha\langle \tilde{y}_{\bar{x}} - \bar{x}, \tilde{y} - \tilde{y}_{\bar{x}}\rangle + \frac{1}{2}\alpha^2\|\tilde{y} - \tilde{y}_{\bar{x}}\|$$

Dividing by α and let $\alpha \to 0$, we get

$$\langle \tilde{y}_{\bar{x}} - \bar{x}, \tilde{y} - \tilde{y}_{\bar{x}}\rangle \geq 0$$

Conversely, suppose that $\tilde{y}_{\bar{x}} \in LFN$ satisfies eq. (6.1).
If $\tilde{y}_{\bar{x}} = \bar{x}$, then $\tilde{y}_{\bar{x}}$ obviously solves eq. (6.2). Now consider $\tilde{y}_{\bar{x}} \neq \bar{x}$, then

$$\begin{aligned} 0 \quad &\leq \quad \langle \tilde{y}_{\bar{x}} - \bar{x}, \tilde{y} - \tilde{y}_{\bar{x}}\rangle \\ \Longrightarrow 0 \quad &\geq \quad \langle \bar{x} - \tilde{y}_{\bar{x}}, \tilde{y} - \tilde{y}_{\bar{x}}\rangle \\ &= \langle \bar{x} - \tilde{y}_{\bar{x}}, \tilde{y} - \bar{x} + \bar{x} - \tilde{y}_{\bar{x}}\rangle \\ &= \|\bar{x} - \tilde{y}_{\bar{x}}\|^2 + \langle \bar{x} - \tilde{y}_{\bar{x}}, \tilde{y} - \bar{x}\rangle \\ &= \|\bar{x} - \tilde{y}_{\bar{x}}\|^2 - \langle \bar{x} - \tilde{y}_{\bar{x}}, \bar{x} - \tilde{y}\rangle \\ &\geq \|\bar{x} - \tilde{y}_{\bar{x}}\|^2 - \|\bar{x} - \tilde{y}\|\|\bar{x} - \tilde{y}_{\bar{x}}\| \end{aligned}$$

using Cauchy-Schwarz inequality

Dividing by $\|\bar{x} - \tilde{y}_{\bar{x}}\|$, we see that

$$0 \geq \|\bar{x} - \tilde{y}_{\bar{x}}\| - \|\bar{x} - \tilde{y}\|$$

$$\Rightarrow \|\bar{x} - \tilde{y}_{\bar{x}}\| \leq \|\bar{x} - \tilde{y}\| \ \forall \tilde{y} \in LFN$$

$$\Rightarrow \|\tilde{y}_{\bar{x}} - \bar{x}\| \leq \|\bar{x} - \tilde{y}\| \ \forall \tilde{y} \in LFN$$

implies $\tilde{y}_{\bar{x}}$ is the solution to eq. (6.2).

□

Theorem 6.4. *A point $\tilde{y}_{\bar{x}} \in LFN$ is the projection of $\bar{x} \in \mathbb{R}^n$ if and only if*

$$\tilde{y}_{\bar{x}} \in LFN, \ \tilde{z} = \tilde{y}_{\bar{x}} - \bar{x} \in LFN^*, \ \langle \tilde{z}, \tilde{y}_{\bar{x}}\rangle = 0 \tag{6.4}$$

Proof. From theorem 6.3 we know that $\tilde{y}_{\bar{x}} = P\bar{x}$ satisfies

$$\langle \bar{x} - \tilde{y}_{\bar{x}}, \tilde{y} - \tilde{y}_{\bar{x}}\rangle \leq 0, \ \forall \tilde{y} \in LFN \tag{6.5}$$

Let $\tilde{y} = \alpha\tilde{y}_{\bar{x}}$, with arbitrary $\alpha \geq 0$, then eq. (6.5) implies $(\alpha - 1)\langle \bar{x} - \tilde{y}_{\bar{x}}, \tilde{y}_{\bar{x}}\rangle \leq 0, \ \forall \alpha \geq 0$. Since $\alpha - 1$ can have either sign, we have $\langle \bar{x} - \tilde{y}_{\bar{x}}, \tilde{y}_{\bar{x}}\rangle = 0 \implies$

$\langle \tilde{y}_{\bar{x}} - \bar{x}, \tilde{y}_{\bar{x}} \rangle = 0$, that is $\langle \tilde{z}, \tilde{y}_{\bar{x}} \rangle = 0$.
Also,

$$\begin{array}{rrcl} & \langle \bar{x} - \tilde{y}_{\bar{x}}, \tilde{y} - \tilde{y}_{\bar{x}} \rangle & \leq & 0 \\ \Rightarrow & \langle \bar{x} - \tilde{y}_{\bar{x}}, \tilde{y} \rangle - \langle \bar{x} - \tilde{y}_{\bar{x}}, \tilde{y}_{\bar{x}} \rangle & \leq & 0 \\ \Rightarrow & \langle \bar{x} - \tilde{y}_{\bar{x}}, \tilde{y} \rangle & \leq & 0 \ \forall y \in LFN \\ \Rightarrow & \langle \tilde{y}, \tilde{y}_{\bar{x}} - \bar{x} \rangle & \geq & 0 \ \forall y \in LFN \\ \Rightarrow & \tilde{y}_{\bar{x}} - \bar{x} & \in & LFN^* \end{array}$$

Conversely, let $\tilde{y}_{\bar{x}}$ satisfy eq. (6.4). Using eq. (6.3),

$$\begin{array}{rcl} f_{\bar{x}}(\tilde{y}) & = & \frac{1}{2}\|\tilde{y} - \bar{x}\|^2 \\ & = & \frac{1}{2}\|\bar{x} - \tilde{y}\|^2 \\ & = & \frac{1}{2}\|\bar{x} - \tilde{y}_{\bar{x}} + \tilde{y}_{\bar{x}} - \tilde{y}\|^2 \\ & \geq & \frac{1}{2}\|\bar{x} - \tilde{y}_{\bar{x}}\|^2 + \frac{1}{2}\|\tilde{y}_{\bar{x}} - \tilde{y}\|^2 + \langle \bar{x} - \tilde{y}_{\bar{x}}, \tilde{y}_{\bar{x}} - \tilde{y} \rangle \\ & \geq & \frac{1}{2}\|\bar{x} - \tilde{y}_{\bar{x}}\|^2 + \langle \bar{x} - \tilde{y}_{\bar{x}}, \tilde{y}_{\bar{x}} - \tilde{y} \rangle \\ & = & f_{\bar{x}}(\tilde{y}_{\bar{x}}) + \langle \bar{x} - \tilde{y}_{\bar{x}}, \tilde{y}_{\bar{x}} - \tilde{y} \rangle \\ & = & f_{\bar{x}}(\tilde{y}_{\bar{x}}) + \langle \bar{x} - \tilde{y}_{\bar{x}}, \tilde{y}_{\bar{x}} \rangle - \langle \bar{x} - \tilde{y}_{\bar{x}}, \tilde{y} \rangle \\ & = & f_{\bar{x}}(\tilde{y}_{\bar{x}}) - \langle \bar{x} - \tilde{y}_{\bar{x}}, \tilde{y} \rangle \text{ as } \langle \bar{x} - \tilde{y}_{\bar{x}}, \tilde{y}_{\bar{x}} \rangle = 0 \\ & = & f_{\bar{x}}(\tilde{y}_{\bar{x}}) + \langle \tilde{y}_{\bar{x}} - \bar{x}, \tilde{y} \rangle \\ & \geq & f_{\bar{x}}(\tilde{y}_{\bar{x}}) \text{ as } \tilde{y}_{\bar{x}} - \bar{x} \in LFN^* \\ \Rightarrow \tilde{y}_{\bar{x}} & = & P\bar{x} \end{array}$$

□

Remark 6.5. Theorem 6.4 shows that the problem of projection onto LFN is same as solving the linear complementarity problem on the convex cone LFN given by: Given any $\bar{x} \in \mathbb{R}^n$ find $\tilde{y} \in \mathbb{R}^n$ such that

$$\tilde{y} \in LFN, \tilde{z} = I\tilde{y} - \bar{x} \in LFN^*, \langle \tilde{z}, \tilde{y} \rangle = 0 \tag{6.6}$$

and by Remark 5.4 this reduces the classical linear complementarity problem of finding $\tilde{y}' \in \mathbb{R}^n$ such that

$$\tilde{y}' \geq 0, \tilde{z}' = A^T I A \tilde{y}' + A^T(-\bar{x}) \geq 0, \langle \tilde{z}', \tilde{y}' \rangle = 0 \tag{6.7}$$

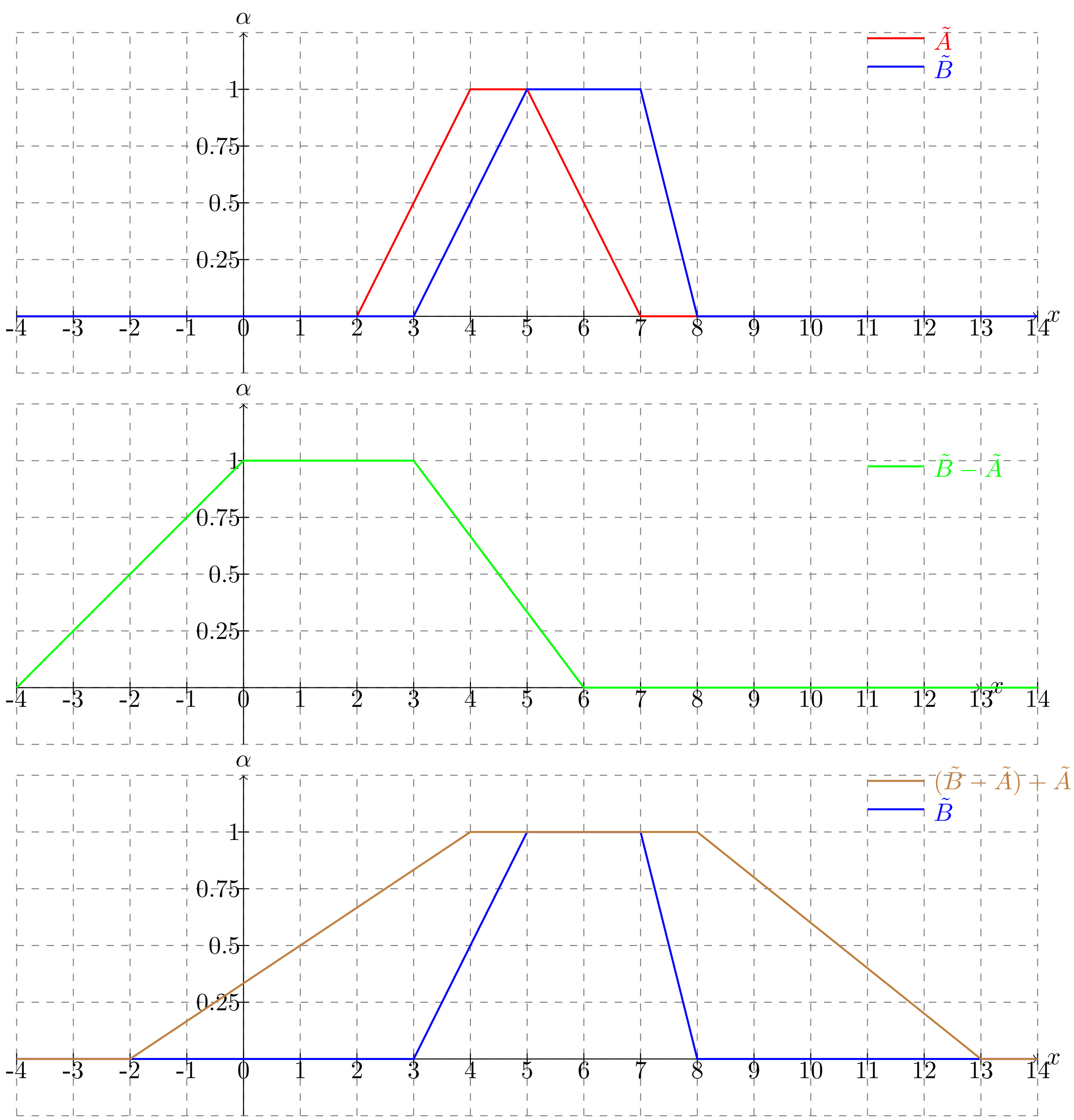
and $\tilde{y} = A\tilde{y}'$, where A is as given in equation 4.3.

7. Subtraction in Fuzzy Numbers

According to the definition 3.1 and the following proposition 3.2 and remark 3.3, the spread of the difference of two fuzzy numbers is more than the given fuzzy numbers. Also this type of subtraction yields $\tilde{B} \neq (\tilde{B} - \tilde{A}) + \tilde{A}$. For example, let us consider two trapezoidal fuzzy numbers $\tilde{A}$ and $\tilde{B}$, then

$$\begin{aligned} \tilde{A} &= (2,4,5,7) \\ \tilde{B} &= (3,5,7,8) \\ \tilde{B}-\tilde{A} &= (-4,0,3,6) \\ (\tilde{B}-\tilde{A})+\tilde{A} &= (-2,4,8,13) \end{aligned}$$

Graphically, the representation of these numbers are as follows:



In this paper, we propose a new subtraction on linear fuzzy numbers using projection and LCP.

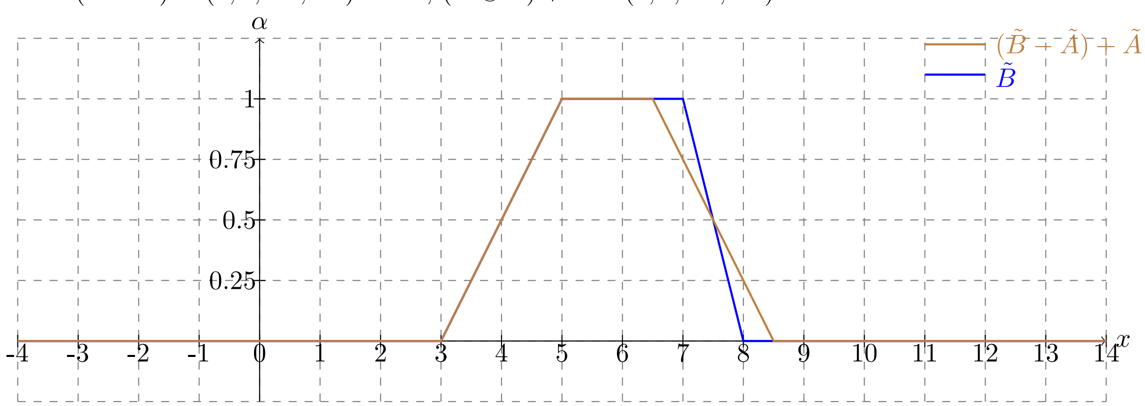
Definition 7.1. Let $\tilde{A}, \tilde{B} \in LFN$, where LFN is a cone in $\mathbb{R}^n$ and let $\tilde{A} = (a_1, a_2, ..., a_n)$ and $\tilde{B} = (b_1, b_2, ..., b_n)$, then $\tilde{C} = \tilde{B} \ominus \tilde{A}$ is defined as $\tilde{C} = P(\tilde{B} - \tilde{A})$.

Remark 7.2. The projection involved in the above definition is obtained by solving the classical LCP:
Given $\tilde{B} - \tilde{A} \in \mathbb{R}^n$ finding $\tilde{C} \in LFN$(equivalently finding $\tilde{c} \in \mathbb{R}^n$) such that

$$\tilde{c} \geq 0, \ \tilde{d} = A^T I A \tilde{c} + A^T(\tilde{A} - \tilde{B}) \geq 0, \langle \tilde{d}, \tilde{c} \rangle = 0$$

and $\tilde{C} = A\tilde{c}$, where the matrix A is as given in equation 4.3

For the above problem, $\tilde{B} - \tilde{A} = (1, 1, 2, 1)$ considered with usual subtraction in $\mathbb{R}^4$ which is not a trapezoidal fuzzy number. Hence $\tilde{C} = \tilde{B} \ominus \tilde{A}$ which is $\tilde{C} = P(\tilde{B} - \tilde{A}) = (1, 1, 1.5, 1.5)$. Also, $(\tilde{B} \ominus \tilde{A}) + \tilde{A} = (3, 5, 6.5, 8.5)$



8. Conclusion

In this paper, we generalized some of the well-known fuzzy numbers as a subset of $\mathbb{R}^n$ with a closed, convex cone structure. We proved that the operations of addition and scalar multiplication on linear fuzzy numbers is same as that performed in $\mathbb{R}^n$, which simplifies the computation for problems invloving only these operations. Various cone forms, such as polyhedral cone, minidrel, dual and polar cone, simplicial cone, of LFN were discussed. We have derived a classical linear complementarity problem for the linear complementarity problem on the convex cone LFN. We have discussed the projection of any element in $\mathbb{R}^n$ on the cone LFN of order n as a LCP problem on LFN.

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FELBIN C KENNEDY: Research Guide & Associate Professor, Stella Maris College(Autonomous), Chennai, Tamil Nadu, India
E-mail address: felbinckennedy@gmail.com

DHANALAKSHMI V: Research Scholar & Assistant Professor, Stella Maris College(Autonomous), Chennai, Tamil Nadu, India
E-mail address: dhanavdl24@gmail.com