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$M/M/1$ QUEUEING MODEL WITH DIFFERENTIATED VACATION AND INTERRUPTION

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Abstract. Consider a single server queueing model wherein the server can take two types of vacation, namely, type I vacation and type II vacation. The server will take type I vacation / longer duration vacation, once the server completes the service to all customers in the system and the server will take type II vacation / shorter duration vacation, once the server finds the system empty after returning from type I vacation. Also type II vacation can be interrupted if the number of customers in the system reaches some predefined thresholds. The explicit analytical expressions of transient state probabilities are derived using the generating function technique. To support the theoretical results, we added numerical illustrations.

MSC: 35Q99,60G65

KEYWORDS: Multiple Vacations, Vacation Interruption, Differentiated Vacation, Transient Analysis, Laplace Transform, Generating Function.

1. Introduction

1.1 Literature Survey

In a real-time scenario, it is quite natural that the server will not be available for a certain period of time due to sudden breakdown or due to the maintenance work or when the server is involved in some other secondary task and so on. The period during which server is unavailable is called vacation period of the server. The vacation queueing model was introduced by Levy and Yechialli [8]. To know more about the vacation queueing model, the readers may refer to Doshi [4,5], Takagi [14] and Tian and Zhang [15]. Further, Servi and Finn [11] were the first to introduce the idea of working vacation in an $M/M/1$ queueing model.

Li and Tian [9] introduce the concept of vacation interruption for an $M/M/1$ queue. Li, Tian and Ma [10] analyzed the $GI/M/1$ queue with working vacations and vacation interruption. Zhang and Hou [17] dealt with the concept of working vacations and vacation interruption in a $M/G/1$ queue. Ayyappan, Sekar and Ganapathi [2] made a similar study in an $M/M/1$ retrial queue. Sreenivasan, Chakravarthy and Krishnamoorthy [12] introduced the threshold concept in queueing models wherein the server on vacation may be interrupted when the queue size reaches some specified value, say $N(1 \leq N < \infty)$.

In recent times, Ibe and Isijola [6] introduced a different type of vacation called differentiated vacations in which a server can take two types of vacations: longer duration vacation and shorter duration vacation. Longer duration vacation means at the end of zero customers; the server can take a vacation of specific duration whereas shorter duration refers to the short break post the longer duration vacation when the server has no customer in the queue. For example, in our mobile phones, the sleep mode can be thought of as a differentiated vacation. Also, the reader may refer to papers by Alouf, Altman, and Azad [1], Ibe and Isijola[7], Vijayashree and Janani [16] and Suranga Sampath and Jicheng Liu [13] for an in-depth study on queues subject to differentiated vacation.

One of the potential applications of vacation queueing model is in Wireless Sensor Networks (WSN). WSN consists of several smaller devices called server nodes with multiple energy levels and various computational limits. A massive task in network design is minimizing energy consumption. To minimize energy consumption, the vacation queueing model plays a vital role in modeling and analyzing the sensor nodes (Boutoumi and Nawel Gharbi [3]). By considering the need for efficient energy-saving schemes in WSN, Isijola and Ibe [7] introduced vacation interruption with some threshold in differentiated vacation and derived the steady- state probabilities.

Certain other real-life situations are modeled according to this type of vacation policy. For example, a doctor in the hospital might take short break of specified duration amidst his busy schedule. However, there are situations wherein he may be necessitated to extend his duration of non-availability to a longer time. Yet, during an emergency need of the patient, the doctor's vacation can be interrupted depending upon the need. Similarly, the vacation of an active soldier may be interrupted due to urgent defense needs and many more such practical scenario does exist.

1.2 Importance of the considered model

Consider the sleep mode in IEEE 802.16E. The sleep modes in IEEE 802.16E power saving mechanism are of two types. The type 1 mode is based on binary increasing sleep window size while the type 2 mode has constant sleep window size. The Subscriber Station (SS) sleep mode mechanism is setup into repetitive sleep cycles to save energy. The sleep mode technology run as follows. The SS communicates with Base Station (BS) during busy period to send and receive data packets. The SS stays in an idle state for some random duration if there is no data traffic between SS and BS. Upon completion of idle state and still if there are no data traffic between SS and BS, then the SS will send a message to BS requesting permission to undergo sleep mode. After receiving mobile sleep response from BS station, it goes to the sleeping mode. Also, the sleep mode can be interrupted with some threshold when the data traffic is large. We have modelled the above scenario as differentiated vacation queueing model with interruption and we derived the transient probabilities of the same.

Most of the papers related to vacation queueing models provide only the steady -state analysis. However, steady- state analysis cannot be applied to systems that never approach equilibrium whereas transient analysis helps to analyse the system at any arbitrary time. Therefore, in this paper, the transient analysis of an $M/M/1$queue with differentiated vacation and partial interruption policy is addressed.

Furnished below are the details of the remaining sections of this paper.

- Section 2describes the model
- Section 3 provides the time-dependent probabilities of the model
- Section 4 presents the numerical illustrations
- Section 5 gives the concluding remarks

2. Model Description

Consider a single server queueing model. The pattern of arrivals, service and vacations are as follows.

- Arrivals follow Poisson distribution with parameter λ
- Service time follow exponential distribution with parameter μ
- Vacation time follow exponential distribution with two different parameters $\gamma_1$and$\gamma_2$for type I and type II vacationrespectively

The server takes type I vacation once the server completes the service to all customers in the system and the server takes type II vacation once the server finds the system empty after returning from type I vacation. Also, the type II vacation of the server can be interrupted if the number of customers in the system reaches some predefined threshold, say k_2.

Let $\mathbb{X}(t)$ represents the cumulative number of customers in the system and $\mathbb{J}(t)$denotes the server state. The state, $\mathbb{J}(t) = 0$ refers to the functional state of the server, $\mathbb{J}(t) = 1$ refers to type 1 vacation state of the server and $\mathbb{J}(t) = 2$ refers to type 2 vacation state of the server. Then, $\left(\mathbb{X}(t), \mathbb{J}(t)\right)$ defines a continuous time Markov process with state space, $S = \{(0,1) \cup (0,2) \cup (n, j); n = 1,2 \dots ; j = 0,1,2\}$. The pictorial representation of the model is given in Figure 1.

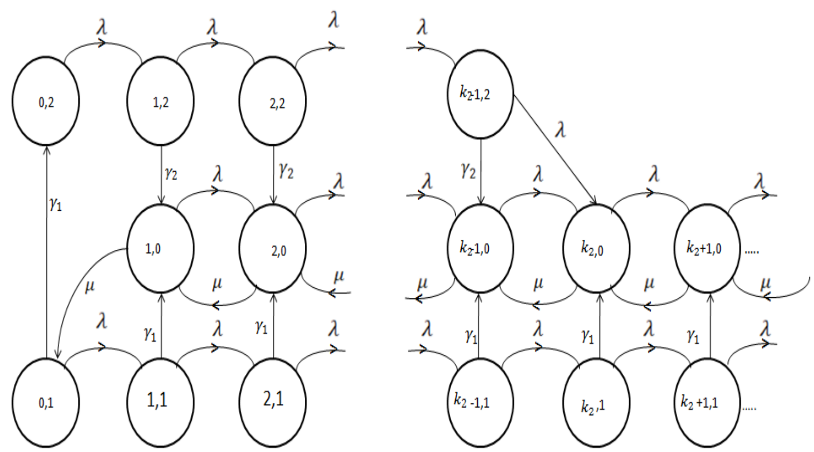

Figure 1. Representation of the Model

The meaning of $P_{n,j}(t)$ is the time-dependent probability for the system to be in state j with n customers at time t. Assume that initially, $P_{0,1}(0) = 1$. Then,

$$P'_{1,0}(t) = -(\mu + \lambda)P_{1,0}(t) + \mu P_{2,0}(t) + \gamma_2 P_{1,2}(t) + \gamma_1 P_{1,1}(t), \tag{2.1}$$

$$P'_{n,0}(t) = -(\mu + \lambda)P_{n,0}(t) + \mu P_{n+1,0}(t) + \lambda P_{n-1,0}(t) + \gamma_2 P_{n,2}(t) + \gamma_1 P_{n,1}(t); \quad n = 2,3 \dots (k_2 - 1), \tag{2.2}$$

$$P'_{k_2,0}(t) = -(\mu + \lambda)P_{k_2,0}(t) + \lambda P_{k_2-1,0}(t) + \mu P_{k_2+1,0}(t) + \gamma_1 P_{k_2,1}(t) + \lambda P_{k_2-1,2}(t), \tag{2.3}$$

$$P'_{n,0}(t) = -(\mu+\lambda)P_{n,0}(t) + \mu P_{n+1,0}(t) + \lambda P_{n-1,0}(t) + \gamma_1 P_{n,1}(t)\ ;$$
$$n = (k_2+1), (k_2+2) \ldots., \quad (2.4)$$

$$P'_{0,1}(t) = -(\lambda+\gamma_1)P_{0,1}(t) + \mu P_{1,0}(t), \quad (2.5)$$

$$P'_{n,1}(t) = -(\lambda+\gamma_1)P_{n,1}(t) + \lambda P_{n-1,1}(t); n = 1,2,3, \ldots, \quad (2.6)$$

$$P'_{0,2}(t) = -\lambda P_{0,2}(t) + \gamma_1 P_{0,1}(t), \quad (2.7)$$

and

$$P'_{n,2}(t) = -(\lambda+\gamma_2)P_{n,2}(t) + \lambda P_{n-1,2}(t); n = 1,2,3, \ldots (k_2-1). \quad (2.8)$$

3. Time Dependent Probabilities

This section provides explicit analytical expressions for the time dependent probabilities of the model described above. Taking Laplace transform of equations (2.5) to (2.8) leads to

$$s\hat{P}_{0,1}(s) = 1 - (\lambda+\gamma_1)\hat{P}_{0,1}(s) + \mu\hat{P}_{1,0}(s),$$
$$s\hat{P}_{n,1}(s) = -(\lambda+\gamma_1)\hat{P}_{n,1}(s) + \lambda\hat{P}_{n-1,1}(s)\ ; n = 1,2,3, \ldots$$
$$s\hat{P}_{0,2}(s) = -\lambda\hat{P}_{0,2}(s) + \gamma_1\hat{P}_{0,1}(s),$$

and

$$s\hat{P}_{n,2}(s) = -(\lambda+\gamma_2)\hat{P}_{n,2}(s) + \lambda\hat{P}_{n-1,2}(s), n = 1,2,3, \ldots (k_2-1).$$

The above equations can be rewritten as,

$$\hat{P}_{0,1}(s) = \frac{1}{s+\lambda+\gamma_1} + \frac{\mu}{s+\lambda+\gamma_1}\hat{P}_{1,0}(s), \quad (3.1)$$

$$\hat{P}_{n,1}(s) = \frac{\lambda}{s+\lambda+\gamma_1}\hat{P}_{n-1,1}(s) = \frac{\lambda^n}{(s+\lambda+\gamma_1)^n}\hat{P}_{0,1}(s)\ ; n = 1,2,3, \ldots$$

$$\hat{P}_{0,2}(s) = \frac{\gamma_1}{(s+\lambda)}\hat{P}_{0,1}(s),$$

and

$$\hat{P}_{n,2}(s) = \frac{\lambda}{s+\lambda+\gamma_2}\hat{P}_{n-1,2}(s) = \frac{\lambda^n}{(s+\lambda+\gamma_2)^n}\hat{P}_{0,2}(s)$$
$$= \frac{\lambda^n\gamma_1}{(s+\lambda)(s+\lambda+\gamma_2)^n}\hat{P}_{0,1}(s); n = 1,2 \ldots (k_2-1).$$

Substituting for $\hat{P}_{0,1}(s)$ from equation (3.1) in the above equations yields,

$$\hat{P}_{n,1}(s) = \frac{\lambda^n}{(s+\lambda+\gamma_1)^{n+1}} + \frac{\lambda^n\mu}{(s+\lambda+\gamma_1)^{n+1}}\hat{P}_{1,0}(s)\ ; n = 1,2,3, \ldots \quad (3.2)$$

$$\hat{P}_{0,2}(s) = \frac{\gamma_1}{(s+\lambda)(s+\lambda+\gamma_1)} + \frac{\gamma_1\mu}{(s+\lambda)(s+\lambda+\gamma_1)}\hat{P}_{1,0}(s), \quad (3.3)$$

and

$$\hat{P}_{n,2}(s) = \frac{\lambda^n \gamma_1}{(s+\lambda)(s+\lambda+\gamma_1)(s+\lambda+\gamma_2)^n} + \frac{\lambda^n \gamma_1 \mu}{(s+\lambda)(s+\lambda+\gamma_1)(s+\lambda+\gamma_2)^n}\hat{P}_{1,0}(s);$$
$$n = 1,2 \dots (k_2 - 1). \qquad (3.4)$$

Taking inverse Laplace transform of the equations (3.1) to (3.4) gives,

$$P_{0,1}(t) = e^{-(\lambda+\gamma_1)t} + \mu e^{-(\lambda+\gamma_1)t} * P_{1,0}(t), \qquad (3.5)$$

$$P_{n,1}(t) = \frac{\lambda^n t^n}{n!}\left(e^{-(\lambda+\gamma_1)t} + \mu e^{-(\lambda+\gamma_1)t} * P_{1,0}(t)\right); n = 1,2 \dots \qquad (3.6)$$

$$P_{0,2}(t) = \gamma_1 e^{-\lambda t} * \left(e^{-(\lambda+\gamma_1)t} + \mu e^{-(\lambda+\gamma_1)t} * P_{1,0}(t)\right), \qquad (3.7)$$

and

$$P_{n,2}(t) = \lambda^n \gamma_1 \left(e^{-\lambda t} * e^{-(\lambda+\gamma_1)t} * \frac{e^{-(\lambda+\gamma_2)t} t^{n-1}}{(n-1)!}\right) + \mu \lambda^n \gamma_1 \left(e^{-\lambda t} * e^{-(\lambda+\gamma_1)t} * \frac{e^{-(\lambda+\gamma_2)t} t^{n-1}}{(n-1)!} * P_{1,0}(t)\right);$$
$$n = 1,2 \dots (k_2 - 1). \qquad (3.8)$$

Hence, all the vacation state probabilities (both type I and type II) are expressed in terms of $P_{1,0}(t)$.

Evaluation of $P_{n,0}(t)$ for $n = 1, 2 \dots (k_2 - 1)$

Define

$$Q(z,t) = \sum_{n=1}^{k_2-1} P_{n,0}(t) z^n,$$

then

$$\frac{\partial Q(z,t)}{\partial t} = \sum_{n=1}^{k_2-1} P'_{n,0}(t) z^n.$$

Multiplying equation (2.1) by z and equation(2.2) by z^n and summing it over all possible values of n, we get after some algebra,

$$\frac{\partial Q(z,t)}{\partial t} - \left(-(\lambda+\mu) + \frac{\mu}{z} + \lambda z\right) Q(z,t) = -\mu P_{1,0}(t) + \mu P_{k_2,0}(t) z^{k_2-1} - \lambda P_{k_2-1,0}(t) z^{k_2} + \gamma_1 \sum_{n=1}^{k_2-1} P_{n,1}(t) z^n + \gamma_2 \sum_{n=1}^{k_2-1} P_{n,2}(t) z^n.$$

Upon integrating the above linear differential equation with respect to 't', we obtain

$$Q(z,t) = \int_0^t \left(-\mu P_{1,0}(y) + \mu P_{k_2,0}(y) z^{k_2-1} - \lambda P_{k_2-1,0}(y) z^{k_2} + \gamma_1 \sum_{n=1}^{k_2-1} P_{n,1}(y) z^n + \gamma_2 \sum_{n=1}^{k_2-1} P_{n,2}(y) z^n \right) e^{-(\lambda+\mu)(t-y)} e^{\left(\frac{\mu}{z}+\lambda z\right)(t-y)} dy. \quad (3.9)$$

If $\alpha = 2\sqrt{\lambda\mu}$and $\beta = \sqrt{\frac{\lambda}{\mu}}$, then $ex\,p\left(\frac{\mu}{z} + \lambda z\right)t = \sum_{n=-\infty}^{\infty} (\beta z)^n I_n(\alpha t)$, where $I_n(t)$ is the modified Bessel function of order n. Using the above fact in equation (3.9) and hence comparing the coefficients of z^n for $n = 1,2,3\ldots. k_2 - 1$ results in,

$$\begin{aligned} P_{n,0}(t) = & \int_0^t -\mu P_{1,0}(y) \beta^n I_n(\alpha(t-y)) e^{-(\lambda+\mu)(t-y)} dy \\ & + \int_0^t \mu P_{k_2,0}(y) \beta^{n-k_2+1} I_{n-k_2+1}(\alpha(t-y)) e^{-(\lambda+\mu)(t-y)} dy \\ & + \int_0^t -\lambda P_{k_2-1,0}(y) \beta^{n-k_2} I_{-(-n+k_2)}(\alpha(t-y)) e^{-(\lambda+\mu)(t-y)} dy \\ & + \gamma_1 \int_0^t \sum_{i=1}^{k_2-1} P_{i,1}(y) \beta^{n-i} I_{n-i}(\alpha(t-y)) e^{-(\lambda+\mu)(t-y)} dy \\ & + \gamma_2 \int_0^t \sum_{i=1}^{k_2-1} P_{i,2}(y) \beta^{n-i} I_{n-i}(\alpha(t-y)) e^{-(\lambda+\mu)(t-y)} dy ; \end{aligned}$$

$$1 \le n \le k_2 - 1 \quad (3.10)$$

Taking Laplace Transform on both the sides of the equation (3.10) leads to,

$$\begin{aligned} \hat{P}_{n,0}(s) = & -\mu \hat{P}_{1,0}(s) \left(\frac{\omega - \sqrt{\omega^2 - \alpha^2}}{2\mu} \right)^n \frac{1}{\sqrt{\omega^2 - \alpha^2}} \\ & + \mu \hat{P}_{k_2,0}(s) \left(\frac{\omega - \sqrt{\omega^2 - \alpha^2}}{2\mu} \right)^{n-k_2+1} \frac{1}{\sqrt{\omega^2 - \alpha^2}} \\ & - \lambda \hat{P}_{k_2-1,0}(s) \left(\frac{\omega - \sqrt{\omega^2 - \alpha^2}}{2\mu} \right)^{k_2-n} \frac{1}{\sqrt{\omega^2 - \alpha^2}} \\ & + \gamma_1 \sum_{i=1}^{k_2-1} \hat{P}_{i,1}(s) \left(\frac{\omega - \sqrt{\omega^2 - \alpha^2}}{2\mu} \right)^{n-i} \frac{1}{\sqrt{\omega^2 - \alpha^2}} \\ & + \gamma_2 \sum_{i=1}^{k_2-1} \hat{P}_{i,2}(s) \left(\frac{\omega - \sqrt{\omega^2 - \alpha^2}}{2\mu} \right)^{n-i} \frac{1}{\sqrt{\omega^2 - \alpha^2}}. \end{aligned}$$

Substituting for $\hat{P}_{i,1}(s)$ and $\hat{P}_{i,2}(s)$ from equation (3.2) and equation (3.4) in the above equation leads to,

$$
\begin{aligned}
\hat{P}_{n,0}(s) = & -\mu\hat{P}_{1,0}(s)\left(\frac{\omega-\sqrt{\omega^2-\alpha^2}}{2\mu}\right)^{n}\frac{1}{\sqrt{\omega^2-\alpha^2}} \\
& + \mu\hat{P}_{k_2,0}(s)\left(\frac{\omega-\sqrt{\omega^2-\alpha^2}}{2\mu}\right)^{n-k_2+1}\frac{1}{\sqrt{\omega^2-\alpha^2}} \\
& - \lambda\hat{P}_{k_2-1,0}(s)\left(\frac{\omega-\sqrt{\omega^2-\alpha^2}}{2\mu}\right)^{k_2-n}\frac{1}{\sqrt{\omega^2-\alpha^2}} \\
& + \gamma_1\sum_{i=1}^{k_2-1}\left[\frac{\lambda^i}{(s+\lambda+\gamma_1)^{i+1}}\left(1 + \mu\hat{P}_{1,0}(s)\right)\left(\frac{\omega-\sqrt{\omega^2-\alpha^2}}{2\mu}\right)^{n-i}\frac{1}{\sqrt{\omega^2-\alpha^2}}\right] \\
& + \gamma_2\sum_{i=1}^{k_2-1}\left[\frac{\lambda^i\gamma_1}{(s+\lambda)(s+\lambda+\gamma_1)(s+\lambda+\gamma_2)^i}\left(1+\mu\hat{P}_{1,0}(s)\right)\left(\frac{\omega-\sqrt{\omega^2-\alpha^2}}{2\mu}\right)^{n-i}\frac{1}{\sqrt{\omega^2-\alpha^2}}\right];
\end{aligned}
$$

$$n = 1,2,3.\dots k_2-1 \qquad (3.11)$$

With $n = k_2 - 1$ in equation (3.11), we get

$$
\begin{aligned}
& \hat{P}_{k_2-1,0}(s)\left(1+\lambda\hat{g}_{k_2-2}(s)\right) \\
& = -\mu\hat{P}_{1,0}(s)\hat{g}_0(s) + \mu\hat{P}_{k_2,0}(s)\frac{1}{\sqrt{\omega^2-\alpha^2}} \\
& + \left(1 + \mu\hat{P}_{1,0}(s)\right)\left[\sum_{i=1}^{k_2-1}\left(\frac{\gamma_1\lambda^i}{(s+\lambda+\gamma_1)^{i+1}} + \frac{\gamma_2\lambda^i\gamma_1}{(s+\lambda)(s+\lambda+\gamma_1)(s+\lambda+\gamma_2)^i}\right)\hat{g}_i(s)\right],
\end{aligned}
\qquad (3.12)
$$

where

$$\hat{g}_i(s) = \left(\frac{\omega-\sqrt{\omega^2-\alpha^2}}{2\mu}\right)^{k_2-1-i}\frac{1}{\sqrt{\omega^2-\alpha^2}};\, i = 0,1,2 \dots (k_2-1).$$

Taking inverse Laplace transform for the equation (3.12), we get

$$P_{k_2-1,0}(t) = \sum_{j=0}^{\infty} \left(-\lambda g_{k_2-2}(t)\right)^{*j} * \left[-\mu P_{1,0}(t) * g_0(t) + \mu P_{k_2,0}(t) * I_0(\alpha t) e^{-(\lambda+\mu)t} + \left(\delta(t) + \mu P_{1,0}(t)\right) * \left(\sum_{i=1}^{k_2-1} \left[\gamma_1 \lambda^i \frac{e^{-(\lambda+\gamma_1)t} t^i}{i!} * g_i(t)\right] + \sum_{i=1}^{k_2-1} \gamma_1 \gamma_2 \lambda^i \left(e^{-\lambda t} * e^{-(\lambda+\gamma_1)t} * \frac{e^{-(\lambda+\gamma_2)t} t^{i-1}}{(i-1)!} * g_i(t)\right)\right)\right], \tag{3.13}$$

where

$$g_i(t) = \frac{1}{(2\mu)^{k_2-1-i}} \alpha^{k_2-1-i} e^{-(\lambda+\mu)t} I_{k_2-1-i}(\alpha t), i = 0,1,2 \dots k_2 - 1.$$

In a similar way, inverse Laplace transform of equation (3.11) results in,

$$P_{n,0}(t) = \left[-\mu P_{1,0}(t) * \left(\frac{\alpha}{2\mu}\right)^n e^{-(\lambda+\mu)t} I_n(\alpha t)\right] + \left[\mu P_{k_2,0}(t) * \left(\frac{\alpha}{2\mu}\right)^{n-k_2+1} e^{-(\lambda+\mu)t} I_{n-k_2+1}(\alpha t)\right] - \left[\lambda P_{k_2-1,0}(t) * \left(\frac{\alpha}{2\mu}\right)^{k_2-n} e^{-(\lambda+\mu)t} I_{k_2-n}(\alpha t)\right] + \left[\left(\delta(t) + \mu P_{1,0}(t)\right) * \sum_{i=1}^{k_2-1} \left(\gamma_1 \lambda^i \frac{e^{-(\lambda+\gamma_1)t} t^i}{i!} + \gamma_1 \gamma_2 \lambda^i e^{-\lambda t} * e^{-(\lambda+\gamma_1)t} * \frac{e^{-(\lambda+\gamma_2)t} t^{i-1}}{(i-1)!}\right) * \left(\frac{\alpha}{2\mu}\right)^{n-i} e^{-(\lambda+\mu)t} I_{n-i}(\alpha t)\right]; n = 1,2,3.\dots k_2 - 1$$

Hence, $P_{n,0}(t), 1 \leq n \leq k_2 - 1$ is expressed in terms of $P_{k_2,0}(t), P_{k_2-1,0}(t)$ and $P_{1,0}(t)$ in the above equation where $P_{k_2-1,0}(t)$ is given by equation (3.13). Below, we present the method of finding $P_{n,0}(t)$ in terms of $P_{1,0}(t)$ for $n = k_2, k_2 + 1, \dots$.

Evaluation of $P_{n,0}(t)$ for $n = k_2, k_2 + 1, \dots$

Define

$$B(z,t) = \sum_{n=k_2}^{\infty} P_{n,0}(t) z^n ,$$

then

$$\frac{\partial B(z,t)}{\partial t} = \sum_{n=k_2}^{\infty} P_{n,0}^{'}(t)z^n.$$

Multiplying equation (2.3) by z and equation(2.4) by z^n and summing it over all possible values of n, we get after some algebra,

$$\frac{\partial B(z,t)}{\partial t} - \left(-(\lambda+\mu) + \frac{\mu}{z} + \lambda z\right) B(z,t) = \lambda P_{k_2-1,0}(t)z^{k_2} + \lambda P_{k_2-1,2}(t)z^{k_2} - \mu P_{k_2,0}(t)z^{k_2-1} + \gamma_1 \sum_{n=k_2}^{\infty} P_{n,1}(t)z^n.$$

After integrating the above equation with respect to 't' leads to,

$$B(z,t) = \int_0^t \left(\lambda P_{k_2-1,0}(t)z^{k_2} + \lambda P_{k_2-1,2}(t)z^{k_2} - \mu P_{k_2,0}(t)z^{k_2-1} + \gamma_1 \sum_{n=k_2}^{\infty} P_{n,1}(t)z^n \right) e^{-(\lambda+\mu)(t-y)} e^{\left(\frac{\mu}{z}+\lambda z\right)(t-y)} dy. \tag{3.14}$$

Along the similar lines as before, comparing the coefficients of z^n for $n = k_2, k_2+1 \dots$ leads to,

$$P_{n,0}(t) = \int_0^t \lambda P_{k_2-1,0}(y)\beta^{n-k_2} I_{n-k_2}\big(\alpha(t-y)\big)e^{-(\lambda+\mu)(t-y)}dy + \int_0^t \lambda P_{k_2-1,2}(y)\beta^{n-k_2} I_{n-k_2}\big(\alpha(t-y)\big)e^{-(\lambda+\mu)(t-y)}dy - \int_0^t \mu P_{k_2,0}(y)\beta^{n-k_2+1} I_{n-k_2+1}\big(\alpha(t-y)\big)e^{-(\lambda+\mu)(t-y)}dy + \gamma_1 \int_0^t \sum_{i=k_2}^{\infty} P_{i,1}(y)\beta^{n-i} I_{n-i}\big(\alpha(t-y)\big)e^{-(\lambda+\mu)(t-y)}\, dy. \tag{3.15}$$

Taking Laplace transform for the equation (3.15), we get

$$\hat{P}_{n,0}(s) = \lambda \hat{P}_{k_2-1,0}(s)\left(\frac{\omega-\sqrt{\omega^2-\alpha^2}}{2\mu}\right)^{n-k_2} \frac{1}{\sqrt{\omega^2-\alpha^2}} + \lambda \hat{P}_{k_2-1,2}(s)\left(\frac{\omega-\sqrt{\omega^2-\alpha^2}}{2\mu}\right)^{n-k_2} \frac{1}{\sqrt{\omega^2-\alpha^2}} - \mu \hat{P}_{k_2,0}(s)\left(\frac{\omega-\sqrt{\omega^2-\alpha^2}}{2\mu}\right)^{n-k_2+1} \frac{1}{\sqrt{\omega^2-\alpha^2}} + \gamma_1 \sum_{i=k_2}^{\infty} \hat{P}_{i,1}(s)\left(\frac{\omega-\sqrt{\omega^2-\alpha^2}}{2\mu}\right)^{n-i} \frac{1}{\sqrt{\omega^2-\alpha^2}}.$$

Substituting for $\hat{P}_{i,1}(s)$ and $\hat{P}_{k_2-1,2}(s)$ from equation (3.2) and equation (3.4) in the above equation leads to,

$$\hat{P}_{n,0}(s)$$
$$= \lambda \hat{P}_{k_2-1,0}(s)\left(\frac{\omega-\sqrt{\omega^2-\alpha^2}}{2\mu}\right)^{n-k_2}\frac{1}{\sqrt{\omega^2-\alpha^2}}$$
$$+\frac{\lambda^{k_2}\gamma_1\left(1+\mu\hat{P}_{1,0}(s)\right)}{(s+\lambda)(s+\lambda+\gamma_1)(s+\lambda+\gamma_2)^{k_2-1}}\left(\frac{\omega-\sqrt{\omega^2-\alpha^2}}{2\mu}\right)^{n-k_2}\frac{1}{\sqrt{\omega^2-\alpha^2}}$$
$$-\mu\hat{P}_{k_2,0}(s)\left(\frac{\omega-\sqrt{\omega^2-\alpha^2}}{2\mu}\right)^{n-k_2+1}\frac{1}{\sqrt{\omega^2-\alpha^2}}$$
$$+\gamma_1\sum_{i=k_2}^{\infty}\frac{\lambda^i}{(s+\lambda+\gamma_1)^{i+1}}\left(1+\mu\hat{P}_{1,0}(s)\right)\left(\frac{\omega-\sqrt{\omega^2-\alpha^2}}{2\mu}\right)^{n-i}\frac{1}{\sqrt{\omega^2-\alpha^2}};$$
$$n=k_2,k_2+1,\ldots. \qquad (3.16)$$

With $n = k_2$ in equation (3.16), we get after simplification

$$\hat{P}_{k_2,0}(s)\left(1+\mu\hat{g}_{k_2-2}(s)\right)$$
$$= \lambda\hat{P}_{k_2-1,0}(s)\frac{1}{\sqrt{\omega^2-\alpha^2}}$$
$$+\left(1+\mu\hat{P}_{1,0}(s)\right)\left[\frac{1}{\sqrt{\omega^2-\alpha^2}}\frac{\lambda^{k_2}\gamma_1}{(s+\lambda)(s+\lambda+\gamma_1)(s+\lambda+\gamma_2)^{k_2-1}}\right.$$
$$\left.+\gamma_1\sum_{i=k_2}^{\infty}\lambda^i\hat{l}_i(s)\right], \qquad (3.17)$$

where $\hat{l}_i(s) = \frac{1}{(s+\lambda+\gamma_1)^{i+1}}\left(\frac{\omega-\sqrt{\omega^2-\alpha^2}}{2\mu}\right)^{k_2-i}\frac{1}{\left(\sqrt{\omega^2-\alpha^2}\right)}, i = k_2, k_2+1, \ldots.$

Taking inverse Laplace transform for the equation (3.17), we get

$$P_{k_2,0}(t)$$
$$=\sum_{j=0}^{\infty}\left(-\mu g_{k_2-2}(t)\right)^{*j}$$
$$*\left\{-\lambda P_{k_2-1,0}(t)*e^{-(\lambda+\mu)t}I_0(\alpha t)+\left(\delta(t)+\mu P_{1,0}(t)\right)\right.$$
$$*\left[\lambda^{k_2}\gamma_1\left(e^{-(\lambda+\mu)t}I_0(\alpha t)*e^{-\lambda t}*e^{-(\lambda+\gamma_1)t}*\frac{e^{-(\lambda+\gamma_2)t}t^{k_2-2}}{(k_2-2)!}\right)\right.$$
$$\left.\left.+\gamma_1\sum_{i=k_2}^{\infty}\lambda^i l_i(t)\right]\right\}, \qquad (3.18)$$

where

$$l_i(t) = \frac{\alpha^{k_2-i}}{(2\mu)^{k_2-i}}e^{-(\lambda+\mu)t}I_{k_2-i}(\alpha t)*\frac{e^{-(\lambda+\gamma_1)t}t^i}{i!}, i = k_2, k_2+1\ldots \; .$$

Substituting the equation (3.17) in equation (3.12) leads to,

$$\hat{P}_{k_2-1,0}(s)\left(1+\lambda\hat{g}_{k_2-2}(s)\right)$$
$$= -\mu\hat{P}_{1,0}(s)\hat{g}_0(s)$$
$$+\frac{\mu}{\sqrt{\omega^2-\alpha^2}}\sum_{m=0}^{\infty}\left(-\mu\hat{g}_{k_2-2}(s)\right)^m\left[\lambda\hat{P}_{k_2-1,0}(s)\frac{1}{\sqrt{\omega^2-\alpha^2}}\right.$$
$$+\left(1+\mu\hat{P}_{1,0}(s)\right)\left[\frac{1}{\sqrt{\omega^2-\alpha^2}}\frac{\lambda^{k_2}\gamma_1}{(s+\lambda)(s+\lambda+\gamma_1)(s+\lambda+\gamma_2)^{k_2-1}}\right.$$
$$\left.\left.+\gamma_1\sum_{i=k_2}^{\infty}\lambda^i\hat{l}_i(s)\right]\right]$$
$$+\left(1\right.$$
$$\left.+\mu\hat{P}_{1,0}(s)\right)\left[\sum_{i=1}^{k_2-1}\left(\frac{\gamma_1\lambda^i}{(s+\lambda+\gamma_1)^{i+1}}\right.\right.$$
$$\left.\left.+\frac{\gamma_2\lambda^i\gamma_1}{(s+\lambda)(s+\lambda+\gamma_1)(s+\lambda+\gamma_2)^i}\right)\hat{g}_i(s)\right], \tag{3.19}$$

Taking inverse Laplace transform for the equation (3.19)

$$P_{k_2-1,0}(t)$$
$$=\sum_{j=0}^{\infty}(-F(t))^{*j}$$
$$*\left[-\mu P_{1,0}(t)*g_0(t)+\mu e^{-(\lambda+\mu)t}I_0(\alpha t)\right.$$
$$*\sum_{m=0}^{\infty}\left(-\mu g_{k_2-2}(t)\right)^{*m}*\left(\delta(t)+\mu P_{1,0}(t)\right)$$
$$*\left(\lambda^{k_2}\gamma_1\left(e^{-(\lambda+\mu)t}I_0(\alpha t)*e^{-\lambda t}*e^{-(\lambda+\gamma_1)t}*\frac{e^{-(\lambda+\gamma_2)t}t^{k_2-2}}{(k_2-2)!}\right)+\gamma_1\sum_{i=k_2}^{\infty}\left(\lambda^i l_i(t)\right)\right)$$
$$+\left(\delta(t)+\mu P_{1,0}(t)\right)*\sum_{i=1}^{k_2-1}\left(\frac{\gamma_1\lambda^i e^{-(\lambda+\gamma_1)t}t^i}{i!}+\frac{\gamma_2\lambda^i e^{-(\lambda+\gamma_2)t}t^{i-1}}{(i-1)!}*e^{-\lambda t}*e^{-(\lambda+\gamma_1)t}\right)$$
$$\left.*g_i(t)\right], \tag{3.20}$$

where

$$F(t)=\lambda g_{k_2-2}(t)-\left(\mu e^{-(\lambda+\mu)t}I_0(\alpha t)*\lambda e^{-(\lambda+\mu)t}I_0(\alpha t)*\sum_{m=0}^{\infty}(-\mu g_{k_2-2}(t))^{*m}\right).$$

In a similar way, substituting equation (3.12) in equation (3.17) leads to,

$$\hat{P}_{k_2,0}(s)\left(1+\mu\hat{g}_{k_2-2}(s)\right) = -\mu\hat{P}_{1,0}(s)\hat{g}_0(s)\frac{\lambda}{\sqrt{\omega^2-\alpha^2}}\sum_{m=0}^{\infty}\left(-\lambda\hat{g}_{k_2-2}(s)\right)^m + \frac{\lambda}{\sqrt{\omega^2-\alpha^2}}\sum_{m=0}^{\infty}\left(-\lambda\hat{g}_{k_2-2}(s)\right)^m\left[\mu\hat{P}_{k_2,0}(s)\frac{1}{\sqrt{\omega^2-\alpha^2}} + \left(1+\mu\hat{P}_{1,0}(s)\right)\left[\sum_{i=1}^{k_2-1}\left(\frac{\gamma_1\lambda^i}{(s+\lambda+\gamma_1)^{i+1}} + \frac{\gamma_2\lambda^i\gamma_1}{(s+\lambda)(s+\lambda+\gamma_1)(s+\lambda+\gamma_2)^i}\right)\hat{g}_i(s)\right]\right] + \left(1+\mu\hat{P}_{1,0}(s)\right)\left[\frac{1}{\sqrt{\omega^2-\alpha^2}}\frac{\lambda^{k_2}\gamma_1}{(s+\lambda)(s+\lambda+\gamma_1)(s+\lambda+\gamma_2)^{k_2-1}} + \gamma_1\sum_{i=k_2}^{\infty}\lambda^i\hat{l}_i(s)\right].$$

Taking inverse Laplace transform of the above equation results in,

$$P_{k_2,0}(t) = \sum_{j=0}^{\infty}\left(-G(t)\right)^{*j} * \left[-\mu P_{1,0}(t) * g_0(t) * \lambda e^{-(\lambda+\mu)t}I_0(\alpha t) * \sum_{m=0}^{\infty}\left(-\lambda g_{k_2-2}(t)\right)^{*m} + \lambda e^{-(\lambda+\mu)t}I_0(\alpha t) * \left(\delta(t)+\mu P_{1,0}(t)\right) * \sum_{m=0}^{\infty}\left(-\mu g_{k_2-2}(t)\right)^{*m} * \sum_{i=1}^{k_2-1} g_i(t) * \left(\frac{\gamma_1\lambda^i e^{-(\lambda+\gamma_1)t}t^i}{i!} + \frac{\gamma_2\lambda^i e^{-(\lambda+\gamma_2)t}t^{i-1}}{(i-1)!} * e^{-\lambda t} * e^{-(\lambda+\gamma_1)t}\right) + \left(\delta(t)+\mu P_{1,0}(t)\right) * \left(\lambda^{k_2}\gamma_1\left(e^{-(\lambda+\mu)t}I_0(\alpha t) * e^{-\lambda t} * e^{-(\lambda+\gamma_1)t} * \frac{e^{-(\lambda+\gamma_2)t}t^{k_2-2}}{(k_2-2)!}\right) + \gamma_1\sum_{i=k_2}^{\infty}\left(\lambda^i l_i(t)\right)\right)\right], \tag{3.21}$$

where

$$G(t) = \mu g_{k_2-2}(t) - \left(\mu e^{-(\lambda+\mu)t} I_0(\alpha t) * \lambda e^{-(\lambda+\mu)t} I_0(\alpha t) * \sum_{m=0}^{\infty} (-\lambda g_{k_2-2}(t))^{*m}\right).$$

Also, the inverse Laplace transform of equation (3.16) yields

$$\begin{aligned} P_{n,0}(t) = &\left[\lambda P_{k_2-1,0}(t) * \left(\frac{\alpha}{2\mu}\right)^{n-k_2} e^{-(\lambda+\mu)t}\, I_{n-k_2}(\alpha t)\right] \\ &+ \left[\lambda^n e^{-\lambda t} * e^{-(\lambda+\gamma_1)t} * e^{-(\lambda+\gamma_2)t} \frac{t^{n-1}}{(n-1)!} * \left(\delta(t) + P_{1,0}(t)\mu\right)\right. \\ &\left. * \left(\frac{\alpha}{2\mu}\right)^{n-k_2} e^{-(\lambda+\mu)t}\, I_{n-k_2}(\alpha t)\right] \\ &- \left[\mu P_{k_2,0}(t) * \left(\frac{\alpha}{2\mu}\right)^{n-k_2+1} e^{-(\lambda+\mu)t}\, I_{n-k_2+1}(\alpha t)\right] \\ &+ \sum_{i=k_2}^{\infty} \gamma_1 \lambda^i e^{-(\lambda+\gamma_1)t} \frac{t^i}{i!} * + \gamma_1\gamma_2 \lambda^i e^{-\lambda t} * \left(\delta(t) + P_{1,0}(t)\mu\right) \\ &* \left(\frac{\alpha}{2\mu}\right)^{n-i} e^{-(\lambda+\mu)t}\, I_{n-i}(\alpha t). \end{aligned}$$

Hence, $P_{n,0}(t), k_2 \le n \le \infty$ is expressed in terms of$P_{k_2-1,0}(t)$ and $P_{k_2,0}(t)$ which are expressed in terms of $P_{1,0}(t)$ in equation (3.20) and equation (3.21) respectively. Therefore, all the probabilities are expressed purely in terms of $P_{1,0}(t)$.Using the normalization condition given by

$$\sum_{n=0}^{\infty} P_{n,1}(t) + \sum_{n=0}^{k_2-1} P_{n,2}(t) + \sum_{n=1}^{k_2-1} P_{n,0}(t) + \sum_{n=k_2}^{\infty} P_{n,0}(t) = 1\,,$$

the term $P_{1,0}(t)$ can be explicitly determined.

Remark:

i) Let $\pi_{n,j}$ denote the steady state probability. Mathematically, let

$$\pi_{n,j} = \lim_{t\to\infty} P_{n,j}(t)$$

By the final value theorem of Laplace transform which states

$$\lim_{t\to\infty} P_{n,j}(t) = \lim_{s\to 0} s\hat{P}_{n,j}(s)$$

It is observed that $\pi_{n,j} = \lim_{s\to 0} s\hat{P}_{n,j}(s)$.Therefore from equation (3.2)

$$\lim_{s\to 0} s\hat{P}_{n,1}(s) = \lim_{s\to 0} s\left(\frac{\lambda^n}{(s+\lambda+\gamma_1)^{n+1}} + \frac{\lambda^n \mu}{(s+\lambda+\gamma_1)^{n+1}} \hat{P}_{1,0}(s)\right)$$

$$\pi_{n,1} = \frac{\lambda^n \mu}{(\lambda+\gamma_1)^{n+1}} \pi_{1,0}.$$

The above equation is same as equation obtained by Ibe and Isijola [14]

Similarly, from equation (3.4)

$$\lim_{s\to 0} s\hat{P}_{n,1}(s) = \lim_{s\to 0} s\left(\frac{\lambda^n\gamma_1}{(s+\lambda)(s+\lambda+\gamma_1)(s+\lambda+\gamma_2)^n} + \frac{\lambda^n\gamma_1\mu}{(s+\lambda)(s+\lambda+\gamma_1)(s+\lambda+\gamma_2)^n}\hat{P}_{1,0}(s)\right)$$

Hence,

$$\pi_{n,2} = \frac{\lambda^{n-1}\gamma_1\mu}{(\lambda+\gamma_1)(\lambda+\gamma_2)^n}\pi_{1,0}$$

which coincides with the steady state probability obtained by Ibe and Isijola [14].

ii) When there is no vacation interruption, $P_{n,0}(t)$ is given by

$$\begin{aligned} P_{n,0}(t) = {} & \gamma_1 \int_0^t \sum_{k=1}^{\infty} P_{k,1}(y)\beta^{n-k}(I_{n-k}(\alpha(t-y)) \\ & - I_{n+k}(\alpha(t-y)))e^{-(\lambda+\mu)(t-y)}\,dy \\ & + \gamma_2 \int_0^t \sum_{k=1}^{\infty} P_{k,2}(y)\beta^{n-k}(I_{n-k}(\alpha(t-y)) \\ & - I_{n+k}(\alpha(t-y)))e^{-(\lambda+\mu)(t-y)}\,dy. \end{aligned}$$

which coincides with the result obtained by Vijayashree and Janani[16].

4. Numerical Analysis

This section presents the way in which $P_{n,j}(t)$, mean and variance of the system behaves as time progresses for different parameter values. Even though the system under consideration is of infinite capacity, for the purpose of numerical analysis n is limited to 20.

Figure 2 presents the behavior of $P_{0,1}(t)$ against time for $\lambda = 0.4, \mu = 0.6$ and $\gamma_2 = 1$. The server is in Type I vacation initially with zero customers. Therefore, all the probabilities for $P_{0,1}(t)$ start at one. As we allow customers to join the system even during Type I vacation, we find that the possibility for the system to be empty decreases with an increase in time. Further, at a given instant of time, $P_{0,1}(t)$ decreases with increases in γ_1. This is because as γ_1 increases, the server is more likely to avail another vacation of shorter duration (Type II).

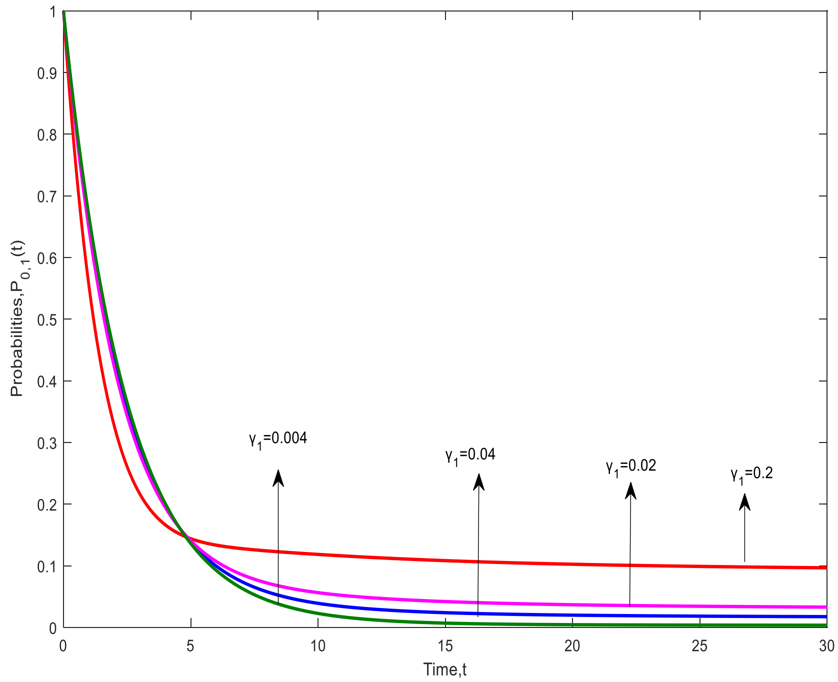

Figure 2. $P_{0,1}(t)$ **versus** t

Figure 3 and Figure 4 depicts the graph of $P_{n,1}(t)$ against time t for the same parameter values along with $\gamma_1 = 0.9$ and varying values of n. All the values for $P_{n,1}(t)$ start at zero as we assumed that the server is in type I vacation initially with zero customers.We observe that at a particular instant of time, the probability to find the system with fewer customers is relatively high. However, for a specific value of n, the probability value increases during the initial period, it reaches a peak and gradually decreases until it converges. Also, the peak probability reduces with an increase in n due to the specific choice of γ_1.

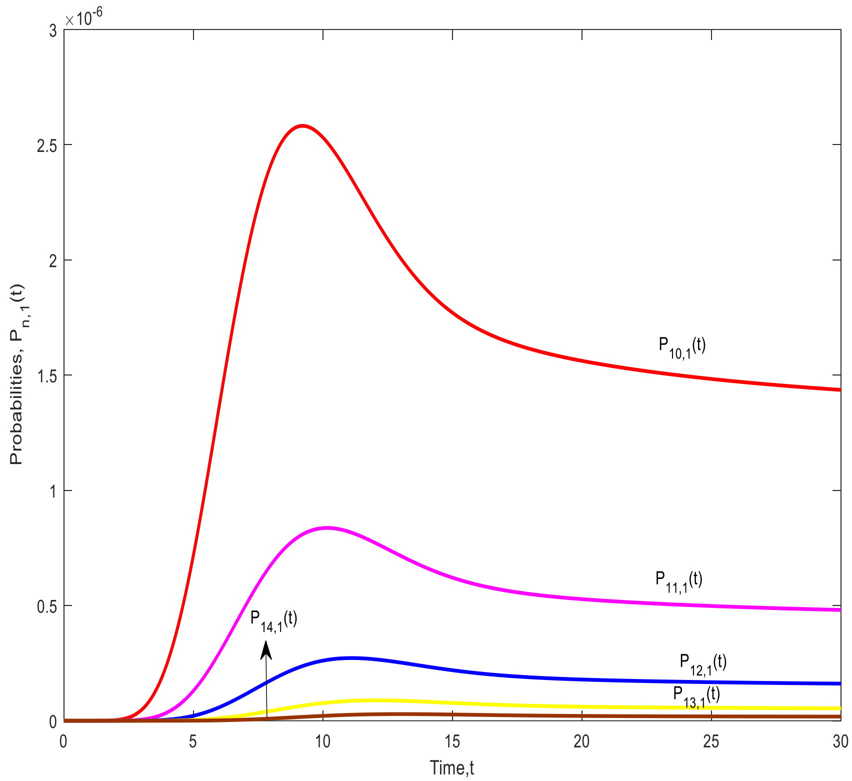

Figure 3. $\boldsymbol{P_{n,1}(t)}$ **versus** $\boldsymbol{t}$

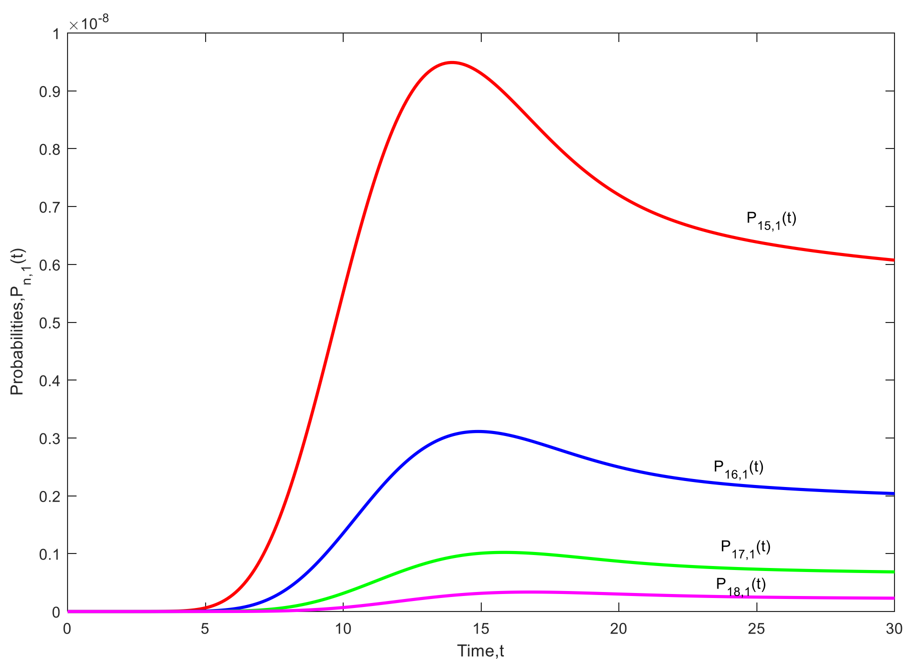

Figure 4. $\boldsymbol{P_{n,1}(t)}$ **versus** $\boldsymbol{t}$

Figure 5 and Figure 6 depicts the behavior of $P_{n,2}(t)$versus $t$for varying values of n. The values of $P_{n,2}(t)$ are equal to zero at the initial state, increase steadily till it reaches a specific value and gradually decreases. For our choice of the parameter values, we observe that for the low value of n, it is more likely for the system to be in state$(n,2)$ initially and the probability reduces drastically as time progresses. The behavior is justified by the fact that arrivals continue to join the system at the rate λ even during type II vacation. Furthermore with $\gamma_2 = 1$, the system is more likely to make a transition from $(n,0)$ to $(n,2)$ for the corresponding n as time progresses.

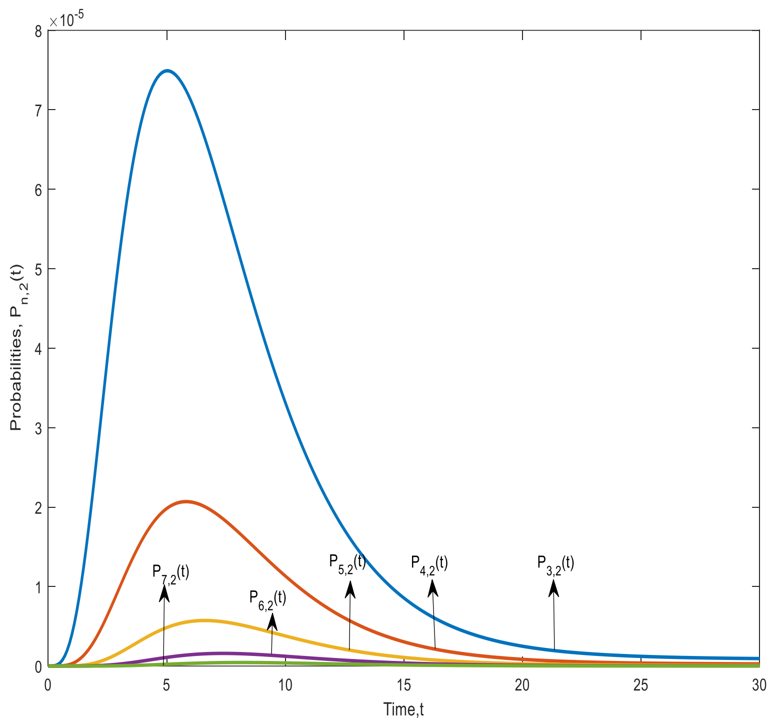

Figure 5. $P_{n,2}(t)$ versus t

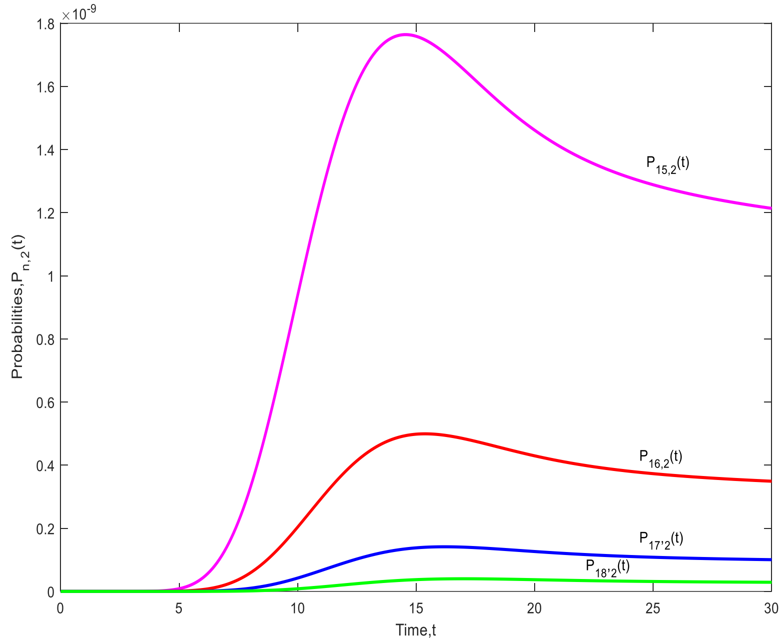

Figure 6. $\boldsymbol{P_{n,2}(t)}$ **versus** $\boldsymbol{t}$

Figure 7 and Figure 8 depicts the behavior of $P_{n,0}(t)$ against time for varying values of *n*. It is observed that for a particular value of *n* the transient state probability increases as time progresses and converges to the corresponding steady-state probabilities. Note that for a specific *n*, $P_{n,0}(t)$ increases steadily as the transition into the state $(n, 0)$ can take place either from $(n-1,0)$ at the rate λ or from $(n, 1)$ at the rate $\gamma_1$or from $(n, 2)$ at the rate γ_2.However, for a particular value of *t* the value of the probability decreases with increase in the number of customer in the system. This is because it is more probable to find the system with fewer customers at any time as service takes place continuously at an exponential rate with parameter,$\mu = 0.6$.

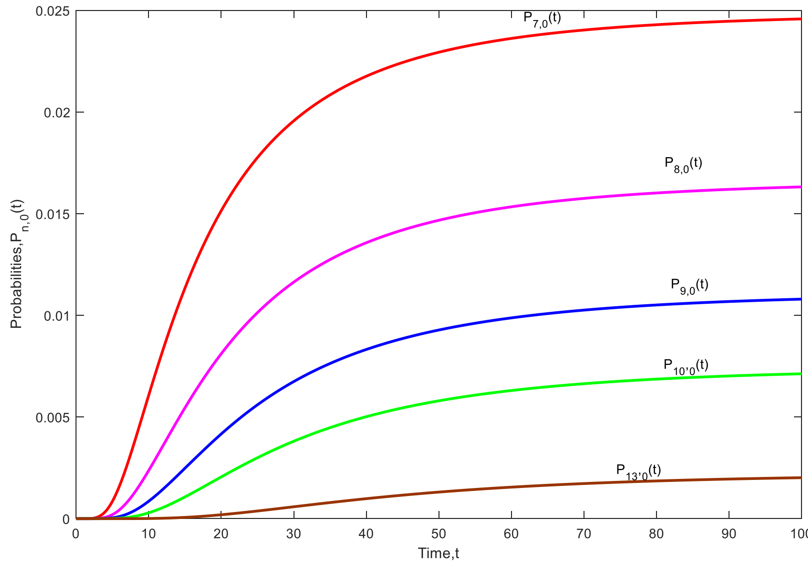

Figure 7. $P_{n,0}(t)$ versus t

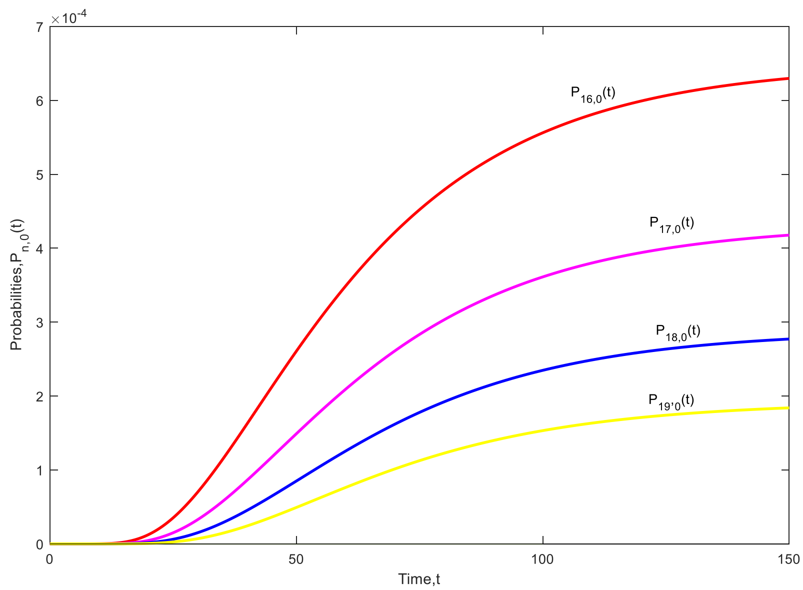

Figure 8. $P_{n,0}(t)$ versus t

Table 1 presents the values of the steady state probabilities for the different choices of n as depicted in the various figures as converging probabilities.

n	$P_{n,1}$	n	$P_{n,2}$	n	$P_{n,0}$
10	5.6974e-07	3	0.4094*10^{-2}	7	0.2426*10^{-1}
11	1.7550e-07	4	0.1173*10^{-2}	8	0.1610*10^{-1}
12	5.4063e-08	5	3.3630e-04	9	0.1065*10^{-1}
13	1.6655e-08	6	9.6411e-05	10	0.7023*10^{-2}
14	5.1311e-09	7	2.7645e-05	13	0.1984*10^{-2}
15	1.5809e-09	15	1.1336e-09	16	6.2081e-04
16	4.8711e-09	16	3.2393e-10	17	4.1174e-04
17	1.5010e-10	17	9.2566e-11	18	2.7316e-04
18	4.6254e-11	18	2.6452e-11	19	1.8136e-04

Table 1: Steady State Probabilities for varying values of n

Figure 9 describes how the mean value changes according to varying values of γ_1. The mean value increases when the value of $\gamma_1$decreases. Figure 10 describes how the variance value changes according to varying values of γ_1.The variance value also increases when the value of $\gamma_1$decreases.

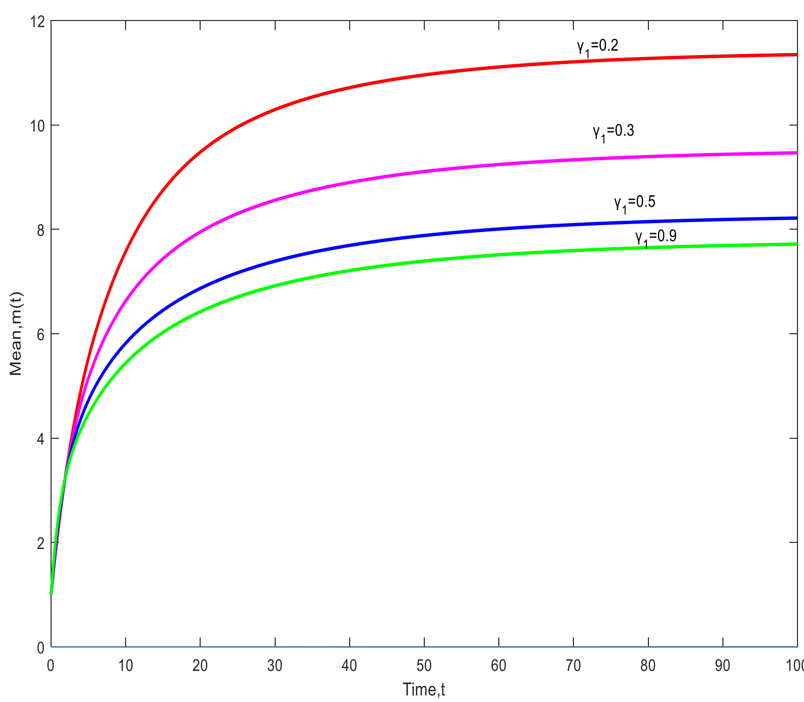

Figure 9. Mean versus t

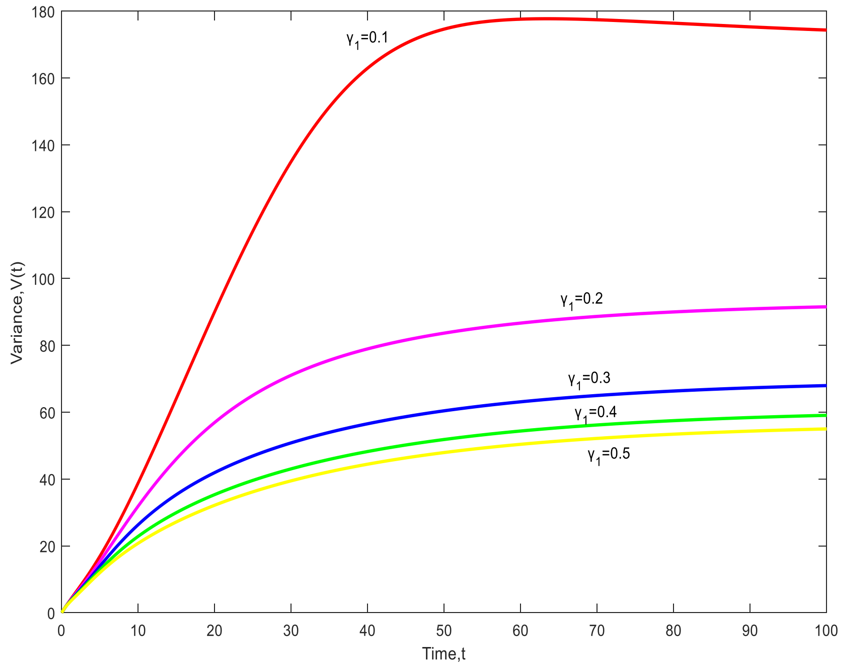

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Figure 10. Variance versus t

References

1. Alouf, S., Altman, E., and Azad, A. P, "Analysis of an *z* queue with repeated inhomogeneous vacations – application to IEEE 802.16e power saving" (Research Report RR-6488). INRIA,2008.
2. Ayyappan, G, Sekar G. and Ganapathi, A.M," *M/M/1* retrial queueing system with vacation interruptions under Erlang-K service", International Journal of Computer Applications, vol.2, pp. 52-57, 2010.
3. Boutoumi B and Gharbi N, "Two Thresholds Working Vacation Policy for Improving Energy Consumption and Latency in WSNs", In: Takahashi Y., Phung-Duc T., Wittevrongel S., Yue W. (eds) Queueing Theory and Network Applications. QTNA 2018. Lecture Notes in Computer Science, vol 10932. Springer, Cham
4. Doshi, B.T, "Queueing systems with vacation – A survey", Queueing System, vol.1, pp. 29-66, 1986.
5. Doshi, B. T, "Single server with vacations", Stochastic Analysis of Computer and Communication Systems, H. Takagi, Ed. Amsterdam, The Netherlands: Elsevier, 1990.
6. Ibe, O.C. and Isijola, O. A, "$M/M/1$ multiple vacation queueing systems with differentiated vacations," Modelling and Simulation in Engineering, vol.2014, Article ID 158247, 2014a.
7. Isijola O. A.and Ibe O.C,"$M/M/1$ multiple vacation queueing systems with differentiated vacations and vacation interruptions," IEEE Access, vol.2, pp. 1384-1395, 2014b.
8. Levy, Y. and Yechiali, U, "Utilization of idle time in an *M/G/1* queueing system", Management Science, vol.22, pp.202-211,1975.
9. Li, J. and Tian, N, "The $M/M/1$ queue with working vacations and vacation interruption," Journal of Systems Science and Systems Engineering, vol.16, pp. 121-127, 2007.
10. Li, J.H, Tian, N.S. and Ma, Z.Y, "Performance analysis of $GI/M/1$ queue with working vacations and vacation interruption," Applied Mathematical Modelling, vol.32, pp. 2715-2730, 2008.

11. Servi, L. D., and Finn,S. G.,"$M/M/1$ queues with working vacations($M/M/1/WV$)",Performance Evaluation, vol. 50, pp. 41–52, 2002.
12. Sreenivasan, C. Chakravarthy, S.R. and Krishnamoorthy, A," $MAP/PH/1$ queue with working vacations, vacations interruptions and N policy," Applied Mathematical Modelling, vol.37, pp. 3879-3893, 2013.
13. Suranga Sampath, M. I. G. and Jicheng Liu," Impact of customers' impatience on an $M/M/1$ queueing system subject to differentiated vacations with a waiting server, Quality Technology and Quantitative Management, vol. 17, pp.125-148, 2020.
14. Takagi, H. Queueing Analysis: A Foundation of Performance Analysis, Volume I: Vacation and Priority Systems. Part 1 Amsterdam. The Netherlands: Elsevier, 1991.
15. Tian, N. and Zhang, G. Vacation Queueing Models: Theory and Applications, New York, Springer Verlag, 2006.
16. Vijayashree, K.V. and Janani, B, 'Transient Analysis of an $M/M/1$ Queueing Systems subject to Differentiated Vacations', Quality Technology and Quantitative Management, vol.15, pp. 730-748,2018.
17. Zhang, M. and Hou, Z, "Performance analysis of $M/G/1$ queue with working vacations and vacation interruption," Journal of Computational and Applied Mathematics, vol.234, pp. 2977-2985, 2010.

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