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A NEW SUBCLASS OF MEROMORPHIC FUNCTIONS ASSOCIATED WITH RAPID OPERATOR

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Abstract. In this paper, we introduce and study new class $M_n(\omega,\vartheta,\mu,\theta)$ of meromorphic univalent functions defined in $U^*=\{z: z\in C \text{ and } 0<|z|<1\}=\mathrm{U}\setminus\{0\}$.We obtain coefficients inequaities, distortion theorems, extreme points, closure theorems, radius of convexity estimates and modified Hadamard products.

1. Introduction

Let Σ^* denote the class of meromorphic function of the form

$$f(z)=\frac{1}{z}+\sum_{n=1}^{\infty}a_nz^n,\ (a_n\geq 0) \tag{1.1}$$

which are analytic in the punctured unit disc $U^*=\{z: z\in C \text{ and } 0<|z|<1\}=\mathrm{U}\setminus\{0\}$. Let $g(z)\in\Sigma^*$ be given by

$$g(z)=\frac{1}{z}+\sum_{n=1}^{\infty}b_nz^n \tag{1.2}$$

then the Hadamard product (or convolution) of $f(z)$ and $g(z)$ is given by

$$(f*g)(z)=\frac{1}{z}+\sum_{n=1}^{\infty}a_nb_nz^n=(g*f)(z) \tag{1.3}$$

A function $f\in\Sigma^*$ is meromorphic starlike of order $\omega(0\leq\omega<1)$, if

$$-Re\left\{\frac{zf'(z)}{f(z)}\right\}>\omega,\ (z\in U). \tag{1.4}$$

The class of such functions is denoted by $\Sigma^*(\omega)$. A function $f\in\Sigma^*$ is meromorphic convex of order $\omega(0\leq\omega<1)$, if

$$-Re\left\{1+\frac{zf''(z)}{f'(z)}\right\}>\omega,\ (z\in U). \tag{1.5}$$

The class of such functions is denoted by $\Sigma_k^*(\omega)$. The classes $\Sigma^*(\omega)$ and $\Sigma_k^*(\omega)$ were introduced and studied by Pommerenke [5], Miller [3], Mogra et al. [4], Cho [2] , Venkateswarlu et al. [8].
In [1], Atshan and Kulkarni introduced Rapid-operator for analytic functions and

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Rosy and Sunil Varma [6] modified their operator to meromorphic functions as follows.

Lemma 1.1. *For $f \in \Sigma^*$ given by (1.1), $0 \leq \mu \leq 1$ and $0 \leq \theta \leq 1$, if the operator $S_\mu^\theta : \Sigma^* \to \Sigma^*$ is defined by*

$$S_\mu^\theta f(z) = \frac{1}{(1-\mu)^\theta \Gamma(\theta+1)} \int_0^\infty t^{1+\theta} e^{\frac{-t}{1-\mu}} f(zt)dt \tag{1.6}$$

then

$$S_\mu^\theta f(z) = \frac{1}{z} + \sum_{n=1}^{\infty} L(n,\mu,\theta) a_n z^n \tag{1.7}$$

where $L(n,\mu,\theta) = (1-\mu)^{n+1} \frac{\Gamma(n+\theta+2)}{\Gamma(\theta+1)}$ and Γ is the familar Gamma function.

Definition 1.2. For $0 \leq \omega < 1$, $0 < \vartheta \leq 1$, $0 \leq \mu \leq 1$, $0 \leq \theta \leq 1$ and $n \in N$, we denote by $M_n(\omega, \vartheta, \mu, \theta)$, the subclass of Σ^* consisting of functions of the form (1.1) and satisfying the analytic criterion

$$\left| \frac{z^2 (S_\mu^\theta f(z))' + 1}{z^2 (S_\mu^\theta f(z))' + (2\omega - 1)} \right| < \vartheta \ (z \in U^*) \tag{1.8}$$

Unless otherwise mentioned, we assume throughout this paper that $0 \leq \omega < 1$, $0 < \vartheta \leq 1$, $0 \leq \mu \leq 1$, $0 \leq \theta \leq 1$, $n \in N$ and $z \in U^*$

2. Coefficient Estimates

Theorem 2.1. *The function $f(z) \in M_n(\omega, \vartheta, \mu, \theta)$ if and only if*

$$\sum_{n=1}^{\infty} [n(1+\vartheta)] L(n,\mu,\theta) a_n \leq 2\vartheta(1-\omega). \tag{2.1}$$

Proof. Suppose (2.1) holds, so

$$|z^2 (S_\mu^\theta f(z))' + 1| - \vartheta |(z^2 (S_\mu^\theta f(z))' + (2\omega - 1)|$$

$$= |\sum_{n=1}^{\infty} nL(n,\mu,\theta) a_n z^{n+1}| - \vartheta \left| 2(\omega-1) + \sum_{n=1}^{\infty} nL(n,\mu,\theta) a_n z^{n+1} \right|$$
$$\leq \sum_{n=1}^{\infty} nL(n,\mu,\theta) a_n r^{n+1}| - \vartheta \left\{ 2(\omega-1) + \sum_{n=1}^{\infty} nL(n,\mu,\theta) a_n r^{n+1} \right\}$$
$$= \sum_{n=1}^{\infty} n(1+\vartheta) L(n,\mu,\theta) a_n r^{n+1} - 2\vartheta(1-\omega)$$

Since the above inequality holds for all $r, 0 < r < 1$,

letting $r \to 1^-$, we have

$$\sum_{n=1}^{\infty} n(1+\vartheta) L(n,\mu,\theta) a_n - 2\vartheta(1-\omega) \leq 0$$

by (2.1), hence $f(z) \in M_n(\omega, \vartheta, \mu, \theta)$.

Conversely, suppose that $f(z)$ is in the class $M_n(\omega, \vartheta, \mu, \theta)$, then

$$\left|\frac{z^2(S_\mu^\theta f(z))' + 1}{z^2(S_\mu^\theta f(z))' + (2\omega - 1)}\right| = \left|\frac{\sum\limits_{n=1}^{\infty} nL(n,\mu,\theta)a_n z^{n+1}}{2(1-\omega) - \sum\limits_{n=1}^{\infty} nL(n,\mu,\theta)a_n z^{n+1}}\right| \leq \vartheta.$$

Using the fact that $Re(z) \leq |z|$ for all z, we have

$$\left|\frac{z^2(S_\mu^\theta f(z))' + 1}{z^2(S_\mu^\theta f(z))' + (2\omega - 1)}\right| \leq \left\{\frac{\sum\limits_{n=1}^{\infty} nL(n,\mu,\theta)a_n z^{n+1}}{2(1-\omega) - \sum\limits_{n=1}^{\infty} nL(n,\mu,\theta)a_n z^{n+1}}\right\} \leq \vartheta. \quad (2.2)$$

If we choose z to be real so that $z^2(S_\mu^\theta f(z))'$ is real. Upon cleaning the denominator in (2.2) and letting $z \to 1^-$ through positive values, we obtain

$$\sum_{n=1}^{\infty} n[1+\vartheta]L(n,\mu,\theta)a_n \leq 2\vartheta(1-\omega).$$

This completes the proof of the theorem. □

Corollary 2.2. *Let the function $f(z)$ denoted by (1.1) be in $M_n(\omega, \vartheta, \mu, \theta)$, then*

$$a_n \leq \frac{2\vartheta(1-\omega)}{n[1+\vartheta]L(n,\mu,\theta)} \ (n \geq 1),$$

with equality for the function

$$f(z) = \frac{1}{z} + \frac{2\vartheta(1-\omega)}{n[1+\vartheta]L(n,\mu,\theta)} z^n \quad (2.3)$$

3. Distortion Theorems

Theorem 3.1. *Let the function $f(z) \in M_n(\omega, \vartheta, \mu, \theta)$, then for $0 < |z| = r < 1$, we have*

$$\frac{1}{r} - \frac{2\vartheta(1-\omega)}{(1+\vartheta)(1-\mu)^2(\theta+1)(\theta+2)} r \leq |f(z)| \leq \frac{1}{r} - \frac{2\vartheta(1-\omega)}{(1+\vartheta)(1-\mu)^2(\theta+1)(\theta+2)} r \quad (3.1)$$

with equality for the function

$$f(z) = \frac{1}{z} + \frac{2\vartheta(1-\omega)}{(1+\vartheta)(1-\mu)^2(\theta+1)(\theta+2)} z^n \quad (3.2)$$

Proof. Suppose that f is in $M_n(\omega, \vartheta, \mu, \theta)$. In view of Theorem 2.3, we have

$$(1+\vartheta)(1-\mu)^2(\theta+1)(\theta+2)\sum_{n=1}^{\infty} a_n \leq \sum_{n=1}^{\infty} n[1+\vartheta]L(n,\mu,\theta)a_n \leq 2\vartheta(1-\omega)$$

. Then

$$\sum_{n=1}^{\infty} a_n \leq \frac{2\vartheta(1-\omega)}{(1+\vartheta)(1-\mu)^2(\theta+1)(\theta+2)}. \quad (3.3)$$

Consequently, we obtain

$$\begin{aligned}|f(z)| &= \left|\frac{1}{z} + \sum_{n=1}^{\infty} a_n z^n\right| \leq \frac{1}{|z|} + \sum_{n=1}^{\infty} a_n |z|^n \\ &\leq \frac{1}{r} + r\sum_{n=1}^{\infty} a_n \\ &\leq \frac{1}{r} - \frac{2\vartheta(1-\omega)}{(1+\vartheta)(1-\mu)^2(\theta+1)(\theta+2)} r \qquad (3.4)\end{aligned}$$

Also,

$$\begin{aligned}|f(z)| &= \left|\frac{1}{z} + \sum_{n=1}^{\infty} a_n z^n\right| \geq \frac{1}{|z|} - \sum_{n=1}^{\infty} a_n |z|^n \\ &\geq \frac{1}{r} - r\sum_{n=1}^{\infty} a_n \\ &\geq \frac{1}{r} - \frac{2\vartheta(1-\omega)}{(1+\vartheta)(1-\mu)^2(\theta+1)(\theta+2)} r. \qquad (3.5)\end{aligned}$$

Hence, (3.1) follows. □

Theorem 3.2. *Let the function $f \in M_n(\omega, \vartheta, \mu, \theta)$, then for $0 < |z| = r < 1$, we have*

$$\begin{aligned}\frac{1}{r^2} - \frac{2\vartheta(1-\omega)}{(1+\vartheta)(1-\mu)^2(\theta+1)(\theta+2))} &\leq |f'(z)| \\ &\leq \frac{1}{r^2} + \frac{2\vartheta(1-\omega)}{(1+\vartheta)(1-\mu)^2(\theta+1)(\theta+2))} \qquad (3.6)\end{aligned}$$

with equality for the function $f(z)$ given by (3.2).

Proof. From theorem (2.1) and (3.3), we have,

$$\sum_{n=1}^{\infty} n a_n \leq \frac{2\vartheta(1-\omega)}{(1+\vartheta)(1-\mu)^2(\theta+1)(\theta+2)}. \qquad (3.7)$$

The remaining part of the proof is similar to the proof of Theorem 3.1, so we omit the details. □

4. Closure Theorems

Let the functions $f_j(z)$ be defined for j = 1,2,...,m by

$$f_j(z) = \frac{1}{z} + \sum_{n=1}^{\infty} a_{n,j} z^n, \ (a_{n,j} \geq 0) \qquad (4.1)$$

Theorem 4.1. *Let $f_j(z) \in M_n(\omega, \vartheta, \mu, \theta), (j = 1, 2, ...m)$. Then the function*

$$h(z) = \frac{1}{z} + \sum_{n=1}^{\infty} \left(\frac{1}{m}\sum_{j=1}^{\infty} a_{n,j}\right) z^n \qquad (4.2)$$

is in $M_n(\omega, \vartheta, \mu, \theta)$.

Proof. Since$f_j(z) \in M_n(\omega, \vartheta, \mu, \theta), (j = 1, 2, ...m)$, it follows from Theorem (2.1), that

$$\sum_{n=1}^{\infty} n[1 + \vartheta]L(n, \mu, \theta)a_{n,j} \le 2\vartheta(1 - \omega),$$

for every j = 1,2,....,m. Hence

$$\sum_{n=1}^{\infty} n[1 + \vartheta]L(n, \mu, \theta) \left(\frac{1}{m} \sum_{j=1}^{\infty} a_{n,j} \right)$$
$$= \frac{1}{m} \sum_{j=1}^{\infty} \left[\sum_{n=1}^{\infty} n[1 + \vartheta]L(n, \mu, \theta)a_{n,j} \right] \le 2\vartheta(1 - \omega).$$

From Theorem 2.1, it follows that $h(z) \in M_n(\omega, \vartheta, \mu, \theta)$
This completes the proof. □

Theorem 4.2. *The class* $M_n(\omega, \vartheta, \mu, \theta)$ *is closed under convex linear combinations.*

Proof. Let $f_j(z), (j = 1, 2)$ defined by (4.1) be in the class $M_n(\omega, \vartheta, \mu, \theta)$, then it is sufficient to show that

$$h(z) = \xi f_1(z) + (1 - \xi)f_2(z), \; (0 \le \xi \le 1) \tag{4.3}$$

is in the class $M_n(\omega, \vartheta, \mu, \theta)$. Since

$$h(z) = \frac{1}{z} + \sum_{n=1}^{\infty} [\xi a_{n,1} + (1 - \xi)a_{n,2}]z^n, \tag{4.4}$$

then, we have from Theorem 2.1, that

$$\sum_{n=1}^{\infty} n[1 + \vartheta]L(n, \mu, \theta)[\xi a_{n,1} + (1 - \xi)a_{n,2}]$$
$$\le 2\xi\vartheta(1 - \omega) + 2\vartheta(1 - \xi)(1 - \omega) = 2\vartheta(1 - \omega)$$

So, $h(z) \in M_n(\omega, \vartheta, \mu, \theta)$. □

Theorem 4.3. *Let* $0 \le \rho < 1$*, then*

$$M_n(\omega, \vartheta, \mu, \theta) \subseteq M_n(\rho, \vartheta, \mu, \theta)$$

where

$$\rho = 1 - \frac{(1 + \vartheta)(1 - \omega)}{(1 + \vartheta)}. \tag{4.5}$$

Proof. Let$f(z) \in M_n(\omega, \vartheta, \mu, \theta)$, then

$$\sum_{n=1}^{\infty} \frac{n[1 + \vartheta]L(n, \mu, \theta)}{2\vartheta(1 - \omega)} a_n \le 1. \tag{4.6}$$

We need to find the value of ρ such that

$$\sum_{n=1}^{\infty} \frac{n(1+\vartheta)}{2\vartheta(1-\rho)} L(n,\mu,\theta) a_n \leq 1. \tag{4.7}$$

In view of equations (4.6) and (4.7), we have

$$\frac{n[1+\vartheta]}{2\vartheta(1-\rho)} L(n,\mu,\theta) \leq \frac{n[1+\vartheta]L(n,\mu,\theta)}{2\vartheta(1-\omega)}.$$

That is

$$\rho \leq 1 - \frac{\gamma(1+\vartheta)(1-\omega)}{(1+2\vartheta\gamma-\vartheta)}$$

. which completes the proof of theorem. □

Theorem 4.4. *Let $f_0(z) = \frac{1}{z}$ and*

$$f_n(z) = \frac{1}{z} + \frac{2\vartheta(1-\omega)}{n[1+\vartheta]L(n,\mu,\theta)} z^n, \ n \geq 1. \tag{4.8}$$

Then $f(z)$ is in the class $M_n(\omega,\vartheta,\mu,\theta)$ if and only if, it can be expressed in the form

$$f(z) = \sum_{n=0}^{\infty} \mu_n f_n(z) \tag{4.9}$$

where $\mu_n \geq 0$ and $\sum\limits_{n=0}^{\infty} \mu_n = 1$.

Proof. Assume that

$$f(z) = \sum_{n=0}^{\infty} \mu_n f_n(z) = \frac{1}{z} + \sum_{n=1}^{\infty} \frac{2\vartheta(1-\omega)}{n[1+\vartheta]L(n,\mu,\theta)} \mu_n z^n. \tag{4.10}$$

Then it follows that

$$\sum_{n=1}^{\infty} \frac{2\vartheta(1-\omega)}{n[1+\vartheta]L(n,\mu,\theta)} \mu_n . \frac{n[1+\vartheta]L(n,\mu,\theta)}{2\vartheta(1-\omega)}$$
$$= \sum_{n=1}^{\infty} \mu_n = 1 - \mu_0 \leq 1,$$

which implies that $f(z) \in M_n(\omega,\vartheta,\mu,\theta)$.

Conversely, assume that the function $f(z)$ defined by (1.1) be in the class $M_n(\omega,\vartheta,\mu,\theta)$.

Then

$$a_n \leq \frac{2\vartheta(1-\omega)}{n[1+\vartheta]L(n,\mu,\theta)}.$$

Setting

$$\mu_n = \frac{n[1+\vartheta]L(n,\mu,\theta)}{2\vartheta(1-\omega)}, \ n \geq 1$$

and

$$\mu_0 = 1 - \sum_{n=1}^{\infty} \mu_n,$$

we can see that $f(z)$ can be expressed in the form (4.9).

This completes the proof of the theorem.

5. Integral Operators

Theorem 5.1. *Let the function $f(z) \in M_n(\omega, \vartheta, \mu, \theta)$. Then the integral operator*

$$F_c(z) = c \int_0^1 u^c f(u, z) dz, \ (0 < u \leq 1; c > 0) \tag{5.1}$$

is in the class $M_0(\xi)$, where

$$\xi = 1 - \frac{2\vartheta c(1-\omega)}{1+\vartheta)(c+2)}. \tag{5.2}$$

The result is sharp for the function $f(z)$ given by (3.2)

Proof. Let $f(z) \in M_0(\xi)$, then

$$F_c(z) = c \int_0^1 u^c f(u, z) dz = \frac{1}{z} + \sum_{n=1}^{\infty} \frac{c}{n+c+1} a_n z^n \tag{5.3}$$

It is sufficient to show that

$$\sum_{n=1}^{\infty} \frac{nc}{(n+c+1)(1-\xi)} a_n \leq 1 \tag{5.4}$$

Since $f(z) \in M_n(\omega, \vartheta, \mu, \theta)$, then

$$\sum_{n=1}^{\infty} \frac{n(1+\vartheta)L(n, \mu, \theta)}{2\vartheta(1-\omega)} a_n \leq 1 \tag{5.5}$$

From (5.4) and (5.5), we have

$$\frac{nc}{(n+c+1)(1-\xi)} \leq \frac{n(1+\vartheta)L(n, \mu, \theta)}{2\vartheta(1-\omega)},$$

Then

$$\xi \leq 1 - \frac{2\vartheta c(1-\omega)}{n(1+\vartheta)(n+c+1)}.$$

Since

$$H(n) = 1 - \frac{2\vartheta c(1-\omega)}{n(1+\vartheta)(n+c+1)}$$

is an increasing function of n $(n \geq 1)$, we obtain

$$\xi \leq H(1) = 1 - \frac{2\vartheta c(1-\omega)}{n(1+\vartheta)(c+2)}$$

and hence the proof of theorem 5.1 is completed.

6. Radius of Convexity

Theorem 6.1. *Let the function $f(z) \in M_n(\omega, \vartheta, \mu, \theta)$. Then $f(z)$ is meromorphically convex of order δ $(0 \leq \delta < 1)$ in $0 < |z| < r$, where*

$$r \leq \left\{ \frac{(1+\vartheta)(1-\delta)L(n,\mu,\theta)}{2\vartheta(n+2-\delta)(1-\omega)} \right\}^{1/n+1} \tag{6.1}$$

The result is sharp.

Proof. We must show that

$$\left| 2 + \frac{zf''(z)}{f'(z)} \right| \leq 1 - \delta \text{ for } 0 < |z| < r, \tag{6.2}$$

where r is given by (6.1). Indeed, we find from (6.2) that

$$\left| 2 + \frac{zf''(z)}{f'(z)} \right| \leq \sum_{n=1}^{\infty} \frac{n(n+1)a_n|z|^{n+1}}{1 - \sum\limits_{n=1}^{\infty} na_n|z|^{n+1}}$$

This will be bounded by $1-\delta$, if

$$\sum_{n=1}^{\infty} \frac{n(n+2-\delta)}{1-\delta} a_n r^{n+1} \leq 1. \tag{6.3}$$

But by using Theorem 2.1, (6.3) will be true, if

$$\frac{n(n+2-\delta)}{1-\delta} r^{n+1} \leq \frac{n[1+\vartheta]L(n,\mu,\theta)}{2\vartheta(1-\omega)}$$

. Then

$$r \leq \left\{ \frac{(1+\vartheta)(1-\delta)L(n,\mu,\theta)}{2\vartheta(n+2-\delta)(1-\omega)} \right\}^{1/n+1}$$

This completes the proof of theorem. □

7. Modified Hadamard Product

For $f_j(z)$ $(j = 1, 2)$ defined by (4.1), the modified Hadamard product of $f_1(z)$ and $f_2(z)$ defined by

$$(f_1 * f_2)(z) = \frac{1}{z} + \sum_{n=1}^{\infty} a_{n,1}a_{n,2}z^n = (f_2 * f_1)(z) \tag{7.1}$$

Theorem 7.1. *Let $f_j(z) \in M_n(\omega, \vartheta, \mu, \theta)$ $(j = 1, 2)$.*
Then $(f_1 * f_2)(z) \in M_n(\phi, \vartheta, \mu, \theta)$, where

$$\phi = 1 - \frac{2\vartheta(1-\omega)^2}{(1+\vartheta)(1-\mu)^2(\theta+1)(\theta+2)} \tag{7.2}$$

The result is sharp for the functions $f_j(z), (j = 1, 2)$ given by

$$f_j(z) = \frac{1}{z} + \frac{2\vartheta(1-\omega)}{(1+\vartheta)(1-\mu)^2(\theta+1)(\theta+2)} \tag{7.3}$$

Proof. Using the technique for Schild and Silverman [7], we need to find the largest ϕ such that

$$\sum_{n=1}^{\infty} \frac{n[1+\vartheta]L(n,\mu,\theta)}{2\vartheta(1-\phi)} a_{n,1}a_{n,2} \leq 1 \tag{7.4}$$

Since $f_j(z) \in M_n(\omega,\vartheta,\mu,\theta)$, $(j=1,2)$, we readily see that

$$\sum_{n=1}^{\infty} \frac{n[1+\vartheta]L(n,\mu,\theta)}{2\vartheta(1-\omega)} a_{n,1} \leq 1 \tag{7.5}$$

and

$$\sum_{n=1}^{\infty} \frac{n[1+\vartheta]L(n,\mu,\theta)}{2\vartheta(1-\omega)} a_{n,2} \leq 1. \tag{7.6}$$

By the Cauchy Schwarz inequality, we have

$$\sum_{n=1}^{\infty} \frac{n[1+2\vartheta\vartheta]}{2\vartheta(1-\omega)} \sqrt{a_{n,1}a_{n,2}} \leq 1. \tag{7.7}$$

Thus it is sufficient to show that

$$\frac{n[1+\vartheta]L(n,\mu,\theta)}{2\vartheta(1-\phi)} a_{n,1}a_{n,2} \leq \frac{n[1+\vartheta]L(n,\mu,\theta)}{2\vartheta(1-\omega)} \sqrt{a_{n,1}a_{n,2}} \tag{7.8}$$

or equivalently that

$$\sqrt{a_{n,1}a_{n,2}} \leq \frac{1-\phi)}{(1-\omega)} \tag{7.9}$$

Connecting with (7.7), it is sufficient to prove that

$$\frac{2\vartheta(1-\omega)}{n[1+\vartheta]L(n,\mu,\theta)} \leq \frac{(1-\phi)}{(1-\omega)}. \tag{7.10}$$

It follows from (7.10) that

$$\phi \leq 1 - \frac{2\vartheta(1-\omega)^2}{n[1+\vartheta]L(n,\mu,\theta)}. \tag{7.11}$$

Now defining the function $G(n)$ by

$$G(n) = 1 - \frac{2\vartheta(1-\omega)^2}{n[1+\vartheta]L(n,\mu,\theta)}. \tag{7.12}$$

We see that G(n) is an increasing function of $n(n \geq 1)$.
Therefore, we conclude that

$$\phi \leq G(1) = 1 - \frac{2\vartheta(1-\omega)^2}{[1+\vartheta](1-\mu)^2(\theta+1)(\theta+2)}. \tag{7.13}$$

which evidently completes the proof of the theorem.

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