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A STUDY ON (ρ, ζ)-CIRCULANT POLYNOMIAL MATRICES

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Abstract. (ρ, ζ)- Circulant polynomial matrices are defined. Its additive properties are investigated and characterizations are also given.

1. Introduction

Let $(a_1(\alpha), a_2(\alpha), ...a_n(\alpha))$ be an ordered n-tuple of polynomials with coefficients in the field of complex numbers and let them generate the circulant polynomial matrix [1][3] [4] of order n:

$$A(\alpha) = \begin{pmatrix} a_1(\alpha) & a_2(\alpha) & ... & a_n(\alpha) \\ a_n(\alpha) & a_1(\alpha) & ... & a_2(\alpha) \\ ... & ... & ... & ... \\ a_2(\alpha) & a_3(\alpha) & ... & a_1(\alpha) \end{pmatrix} \tag{1.1}$$

We shall often denote this circulant polynomial matrix as

$$A(\alpha) = Circ(a_1(\alpha), a_2(\alpha), ..., a_n(\alpha)) \tag{1.2}$$

In this paper, we define the (ρ, ζ)-circulant polynomial matrix and also, we examine some fundamental properties.

We found a characterization of (ρ, ζ)-circulant polynomial matrix. Let $I_n(\alpha)$ be the unit n x n polynomial matrix.

Let $A(\alpha) \in C_{n\times n}(\alpha)$, then $A^T(\alpha), A^*(\alpha)$ and $|A(\alpha)|$ be its transpose, adjoint and the determinant respectively.

2. (ρ, ζ)-Circulant Polynomial Matrices

Here we define (ρ, ζ)-circulant polynomial matrix. Also, we generalize some properties of (ρ, ζ)-circulant matrices found in [2], [5], [6], [7].

Definition 2.1. If a polynomial matrix is of the form,

$$A(\alpha) =$$

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$$\begin{pmatrix} a_0(\alpha) & a_1(\alpha) & a_2(\alpha) & ... & a_{n-2}(\alpha) & a_{n-1}(\alpha) \\ \rho a_{n-1}(\alpha) & a_0(\alpha)-\zeta a_{n-1}(\alpha) & a_1(\alpha) & ... & a_{n-3}(\alpha) & a_{n-2}(\alpha) \\ \rho a_{n-2}(\alpha) & \rho a_{n-1}(\alpha)-\zeta a_{n-2}(\alpha) & a_0(\alpha)-\zeta a_{n-1}(\alpha) & ... & a_{n-4}(\alpha) & a_{n-3}(\alpha) \\ \rho a_{n-3}(\alpha) & \rho a_{n-2}(\alpha)-\zeta a_{n-3}(\alpha) & a_{n-1}(\alpha)-\zeta a_{n-2}(\alpha) & ... & a_{n-5}(\alpha) & a_{n-4}(\alpha) \\ ... & ... & ... & ... & ... & ... \\ \rho a_2(\alpha) & \rho a_3(\alpha)-\zeta a_2(\alpha) & \rho a_4(\alpha)-\zeta a_3(\alpha) & ... & a_0(\alpha)-\zeta a_{n-1}(\alpha) & a_1(\alpha) \\ \rho a_1(\alpha) & \rho a_2(\alpha)-\zeta a_1(\alpha) & \rho a_3(\alpha)-\zeta a_2(\alpha) & ... & a_{n-1}(\alpha)-\zeta a_{n-2}(\alpha) & a_0(\alpha)-\zeta a_{n-1}(\alpha) \end{pmatrix}$$

it is known as a (ρ,ζ)-circulant polynomial matrix. which is denoted by $A(\alpha) = C_{(\rho,\zeta)}(a_0(\alpha), a_1(\alpha), ..., a_{n-1}(\alpha))$.

Remark 2.2. (i) If $\zeta = 0$, then $A(\alpha)$ is a ρ-circulant polynomial matrix.

(ii)The polynomial matrix $b(\alpha) = C_{(\rho,\zeta)}(0, 1, 0, ...0)$ is referred to as fundamental (ρ,ζ) circulant matrix.

Example 2.3. A 4X4 (3,2)-circulant polynomial matrix is given below.

$$A(\alpha) = \begin{pmatrix} \alpha+\alpha^2 & 1-\alpha & -3+\alpha-2\alpha^2 & 2+2\alpha+3\alpha^2 \\ 6+6\alpha+9\alpha^2 & -4-3\alpha-5\alpha^2 & 1-\alpha & -3+\alpha-2\alpha^2 \\ -9+3\alpha-6\alpha^2 & 12+4\alpha+13\alpha^2 & -4-3\alpha-5\alpha^2 & 1-\alpha \\ 3-3\alpha & 11-5\alpha-6\alpha^2 & 12+4\alpha-13\alpha^2 & -4-3\alpha+5\alpha^2 \end{pmatrix}$$

$= A_0 + A_1\alpha + A_2\alpha^2$ where $A_0 = C_{(3,2)}(0, 1, -3, 2)$, $A_1 = C_{(3,2)}(1, -1, 1, 2)$ and $A_2 = C_{(3,2)}(1, 0, -2, 3)$.
that is

$$A_0 = \begin{pmatrix} 0 & 1 & -3 & 2 \\ 6 & -4 & 1 & -3 \\ -9 & 12 & -4 & 1 \\ 3 & -11 & 12 & -4 \end{pmatrix} A_1 = \begin{pmatrix} 1 & -1 & 1 & 2 \\ 6 & -3 & -1 & 1 \\ 3 & 4 & -3 & -1 \\ -3 & 5 & 4 & -3 \end{pmatrix} A_2 = \begin{pmatrix} 1 & 0 & -2 & 3 \\ 9 & -5 & 0 & -2 \\ -6 & 13 & -5 & 0 \\ 0 & -6 & 13 & -5 \end{pmatrix}$$

Proposition 2.4. *If $A(\alpha), B(\alpha)$ are (ρ,ζ)-circulant polynomial matrices, then $A(\alpha) + B(\alpha)$, $A(\alpha) - B(\alpha)$, $\alpha A(\alpha)$ where α is a scalar, are also (ρ,ζ)-circulant polynomial matrices.*

Proposition 2.5. *A polynomial matrix $A(\alpha)$ is a (ρ,ζ)-circulant polynomial matrix if and only if $A(\alpha) = f_{A(\alpha)}(b(\alpha)) = \left(\sum_{i=0}^{n-1} a_i(\alpha)b^i(\alpha)\right)$.*

Theorem 2.6. *A matrix with polynomial coefficients $A(\alpha) \in C^{(n\times n)}(\alpha)$ is a (ρ,ζ)-circulant polynomial matrix if and only if $A(\alpha)b(\alpha) = b(\alpha)A(\alpha)$.*

Proof. Assume $A(\alpha)$ is a (ρ,ζ)-circulant polynomial matrix.
We must demonstrate our worth $A(\alpha)b(\alpha) = b(\alpha)A(\alpha)$
Let $A(\alpha)b(\alpha) = C_{(\rho,\zeta)}(a_0(\alpha), a_1(\alpha), \ldots a_{n-1}(\alpha))$ be a (ρ,ζ)-circulant polynomial matrix. Then $A(\alpha) = \left(\sum_{i=0}^{n-1} a_i(\alpha)b^i(\alpha)\right)$.
$\Rightarrow (\alpha)b(\alpha) = b(\alpha)A(\alpha)$.

Conversely, assume that $A(\alpha)b(\alpha) = b(\alpha)A(\alpha)$. Let us prove $A(\alpha)$ is a (ρ,ζ)-circulant polynomial matrix.

If $A(\alpha)b(\alpha) = b(\alpha)A(\alpha)$, then

$$b^T(\alpha)A^T(\alpha) = A^T(\alpha)b^T(\alpha)$$
$$(b^T)^i(\alpha)A^T(\alpha) = A^T(\alpha)(b^T)^i(\alpha), \text{ i = 1,2,...}$$

If $e_i(\alpha)$ is the i^{th} column of $I_n(\alpha)$, then

$$b^T(\alpha)e_i(\alpha) = e_{i+1}(\alpha) \text{ for } i = 1, 2, ...n-1.$$

Thus, we have $(b^T)^i(\alpha)e_i(\alpha) = e_{i+1}(\alpha)$ for $i = 1, 2, ...n-1$.

$$\begin{aligned}
\text{Now } A^T(\alpha) &= A^T(\alpha)I_n(\alpha)\\
&= A^T(\alpha)[e_1(\alpha), e_2\alpha, \dots, e_n(\alpha)]\\
&= A^T(\alpha)[e_1(\alpha), b^T(\alpha)e_1\alpha, \dots, (b^T)^{n-1}(\alpha)e_1(\alpha)]\\
&= [A^T(\alpha)e_1(\alpha), A^T(\alpha)b^T(\alpha)e_1\alpha, \dots, A^T(\alpha)(b^T)^{n-1}(\alpha)e_1(\alpha)]\\
&= [A^T(\alpha)e_1(\alpha), b^T(\alpha)A^T(\alpha)e_1\alpha, \dots, (b^T)^{n-1}(\alpha)A^T(\alpha)e_1(\alpha)]\\
&= [\beta(\alpha), (b^T)(\alpha)\beta(\lambda), \dots, (b^T)^{n-1}(\alpha)\beta(\lambda)]
\end{aligned}$$

where $\lambda^T(\alpha)$ is the first row of $A(\alpha)$.

Let $\lambda^T(\alpha) = (a_0(\alpha), a_1(\alpha, \dots, a_{n-1}(\alpha)))$

$$\text{Thus } \lambda(\alpha) = \left(\sum_{i=0}^{n-1} a_i(\alpha)e_{i+1}(\alpha)\right)$$

$$\begin{aligned}
A^T(\alpha) &= \left(\sum_{i=0}^{n-1} a_i(\alpha)e_{i+1}(\alpha), \sum_{i=0}^{n-1} a_i(\alpha)b^T(\alpha)e_{i+1}(\alpha), \cdots \sum_{i=0}^{n-1} a_i(\alpha)(b^T)^{n-1}(\alpha)e_{i+1}(\alpha)\right)\\
&= \sum_{i=0}^{n-1} a_i \left(e_{i+1}(\alpha), b^T(\alpha)e_{i+1}(\alpha), \dots, (b^T)^{n-1}(\alpha)e_{i+1}(\alpha)\right)\\
&= \sum_{i=0}^{n-1} a_i \left((b^T)^i(\alpha)e_1(\alpha), (b^T)^{i+1}(\alpha)e_1(\alpha), \dots, (b^T)^{n+i-1}(\alpha)e_1(\alpha)\right)\\
&= \sum_{i=0}^{n-1} a_i(b^T)^i(\alpha)(e_1(\alpha), e_2(\alpha), \dots, e_n(\alpha))\\
&= \sum_{i=0}^{n-1} a_i(\alpha)(b^T)^i(\alpha)
\end{aligned}$$

$$\Rightarrow A(\alpha) = \sum_{i=0}^{n-1} a_i(\alpha)b(\alpha)$$

Hence $A(\alpha)$ is a (ρ, ζ)-circulant polynomial matrix. □

Corollary 2.7. $|A(\alpha)| \neq 0$ *is a* (ρ, ζ)*-circulant polynomial matrix if and only if* $A^{-1}(\alpha)$ *is a* (ρ, ζ)*-circulant polynomial matrix.*

Proof. Given that $|A(\alpha)| \neq 0$ is a (ρ, ζ)-circulant polynomial matrix.
$\iff A(\alpha)b(\alpha) = b(\alpha)A(\alpha)$
$\iff A^{-1}(\alpha)b(\alpha) = b(\alpha)A^{-1}(\alpha)$
$\iff A^{-1}(\alpha)$ is a (ρ, ζ)-circulant polynomial matrix. □

Theorem 2.8. *If* $A(\alpha), B(\alpha)$ *are* (ρ, ζ)*-circulant polynomial matrices, then* $A(\alpha)B(\alpha)$ *and* $B(\alpha)A(\alpha)$ *are* (ρ, ζ)*-circulant polynomial matrices and* $A(\alpha)B(\alpha) = B(\alpha)A(\alpha)$.

Proof. Given that $A(\alpha), B(\alpha)$ are (ρ, ζ)-circulant polynomial matrices.
From theorem (2.6), we have $A(\alpha)b(\alpha) = b(\alpha)A(\alpha)$ and $B(\alpha)b(\alpha) = b(\alpha)B(\alpha)$.

$$\begin{aligned}
\text{Now } [A(\alpha)B(\alpha)]b(\alpha) &= A(\alpha)[B(\alpha)b(\alpha)]\\
&= A(\alpha)[b(\alpha)B(\alpha)]\\
&= [A(\alpha)b(\alpha)]B(\alpha)\\
&= b(\alpha)[A(\alpha)B(\alpha)]
\end{aligned}$$

Thus $A(\alpha)B(\alpha)$ is a (ρ, ζ)-circulant polynomial matrix.

Also, $[B(\alpha)A(\alpha)]b(\alpha) = B(\alpha)[A(\alpha)b(\alpha)]$
$= B(\alpha)[b(\alpha)A(\alpha)]$
$= [B(\alpha)b(\alpha)]A(\alpha)]$
$= [b(\alpha)B(\alpha)]A(\alpha)]$
$= b(\alpha)[B(\alpha)A(\alpha)]$
Hence $B(\alpha)A(\alpha)$ is a (ρ, ζ)-circulant polynomial matrix.
We can deduce from proposition that (2.5), we assume that $A(\alpha) = f(b(\alpha))$ and $B(\alpha) = g(b(\alpha))$.
$\Rightarrow A(\alpha)B(\alpha) = B(\alpha)A(\alpha)$ □

3. Conclusion

some of the characterization of (ρ, ζ)-circulant polynomial matrices are discussed here. In the same way, the other properties can be extended.

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