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SECOND HANKEL DETERMINANT BOUNDS FOR BI-UNIVALENT FUNCTIONS WITH RESPECT TO SYMMETRIC CONJUGATE POINTS DEFINED BY HORADAM POLYNOMIALS

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ABSTRACT. In this investigation, we propose to make use of the Horadam polynomials, we consider a class of bi-univalent functions with respect to symmetric conjugate points. For functions belonging to this class, we find the second Hankel determinant inequality for defined class of bi-univalent functions. Some interesting remarks of the results presented here are also investigated.

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1. Introduction and definitions

Let $\mathcal{A}$ denote the class of functions of the form

$$f(z) = z + a_2 z^2 + \cdots \tag{1.1}$$

which are analytic in

$$\mathbb{U} = \{z : z \in \mathbb{C} \text{ and } |z| < 1\}.$$

We denote $\mathcal{S}$ by the class of univalent functions in $\mathbb{U}$. Further, we know that every univalent function has an inverse f^{-1}, defined by

$$f^{-1}(f(z)) = z, \qquad z \in \mathbb{U}$$

and

$$f(f^{-1}(w)) = w, |w| < r_0(f);\ r_0(f)\frac{1}{4},$$

where

$$f^{-1}(w) = w - a_2 w^2 + (2a_2^2 - a_3)w^3 - (5a_2^3 - 5a_2 a_3 + a_4)w^4 + \cdots .$$

A function $f \in \mathcal{A}$ is said to be bi-univalent in $\mathbb{U}$ if both a function f and it's inverse f^{-1} are univalent in $\mathbb{U}$. Let σ denote the class of bi-univalent functions in $\mathbb{U}$ given by (1.1). After Brannan and Taha [12],Srivastava et al[43] only revived the study of bi-univalent functions and its related work. After this paper we could see huge number of papers. [1-3,5,7-9,11,15,17,24,25,28,30,34-36,38,40,41,46-48] (also, the references therein) in this line by defining various subclasses of bi-univalent

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functions to discuss the initial TaylorMaclaurin coefficient estimates $|a_2|$, $|a_3|$ and $|a_4|$.

For analytic functions f and g in $\mathbb{U}$, f is said to be subordinate to g if there exists an analytic function w such that

$$w(0) = 0, \qquad |w(z)| < 1 \quad \text{and} \quad f(z) = g(w(z)), \qquad z \in \mathbb{U}.$$

It is denoted by

$$f \prec g \qquad (z \in \mathbb{U}) \qquad \text{that is} \qquad f(z) \prec g(z), \qquad z \in \mathbb{U}.$$

In particular, when g is univalent in $\mathbb{U}$,

$$f \prec g \qquad (z \in \mathbb{U}) \;\Leftrightarrow\; f(0) = g(0) \quad \text{and} \quad f(\mathbb{U}) \subset g(\mathbb{U}).$$

For $a,\ b,\ p,\ q \in \mathbb{R}$, the Horadam polynomials $\hbar_n(x,\ a,\ b;\ p,\ q) := \hbar_n(x)$ are given by the recurrence relation (see [23, 24]):

$$\hbar_n(x) \quad = \quad px\hbar_{n-1}(x) + q\hbar_{n-2}(x), \qquad n \in \mathbb{N} \tag{1.2}$$

here

$$\hbar_1(x) = a; \qquad \hbar_2(x) = bx. \tag{1.3}$$

The generating function of the Horadam polynomials $\hbar_n(x)$ (see [24]) is given by

$$\Pi(x,\ z) := \sum_{n=1}^{\infty} \hbar_n(x) z^{n-1} = \frac{a + (b - ap)xz}{1 - pxz - qz^2}\ . \tag{1.4}$$

Here, and in what follows, the argument $x \in \mathbb{R}$ is independent of the argument $z \in \mathbb{C}$; that is, $x \neq \Re(z)$.

It is observed that for special values of the parameters involved in this polynomial leads to different some other polynomials like the Lucas polynomials, the Fibonacci polynomials, the Pell polynomials, the Pell-Lucas polynomials and the Chebyshev polynomials for more details (see, [1, 2, 24, 41]).

A function $f \in \mathcal{S}$ is said to be starlike with respect to symmetric points ($\mathcal{S}^*_{sc}$) if

$$\Re\left(\frac{zf'(z)}{f(z) - \overline{f(-\bar{z})}}\right) > 0, \qquad z \in \mathbb{U}.$$

A function $f \in \mathcal{S}$ is convex with respect to symmetric conjugate points ($\mathcal{C}_{sc}$) if

$$\Re\left(\frac{(zf'(z))'}{\left(f(z) - \overline{f(-\bar{z})}\right)'}\right) > 0, \qquad z \in \mathbb{U}.$$

The classes $\mathcal{S}^*_{sc}$ and $\mathcal{C}_{sc}$ were studied by El-Ashwah and Thomas [21]. For more details of the above said classes and its subclasses one could refer [6, 38, 40, 45, 46].

Definition 1.1. A function $f \in \sigma$ is said to be in the class $\mathcal{P}^{\sigma}_{Sc}(\alpha,\ x)$, if

$$\frac{2zf'(z)}{f(z) - \overline{f(-\bar{z})}} + \frac{2\,(zf'(z))'}{\left(f(z) - \overline{f(-\bar{z})}\right)'} - \frac{2\alpha z^2 f''(z) + 2zf'(z)}{\alpha z\left(f(z) - \overline{f(-\bar{z})}\right)' + (1-\alpha)\left(f(z) - \overline{f(-\bar{z})}\right)}$$
$$\prec \Pi(x,\ z) + 1 - a, \qquad z \in \mathbb{U}$$

and

$$\frac{2wg'(w)}{g(w)-\overline{g(-\bar{w})}}+\frac{2\left(wg'(w)\right)'}{\left(g(w)-\overline{g(-\bar{w})}\right)'}-\frac{2\alpha w^2 g''(w)+2wg'(w)}{\alpha w\left(g(w)-\overline{g(-\bar{w})}\right)'+(1-\alpha)\left(g(w)-\overline{g(-\bar{w})}\right)}$$
$$\prec \Pi(x,\ w)+1-a, \qquad w\in\mathbb{U}$$

hold.

Various, results for the special values of the parameters involved given as follws:

(1) In particular, when $\alpha=1$, we have $\mathcal{P}^{\sigma}_{sc}(1,\ x):=\mathcal{S}^{\sigma}_{sc}(x)$, if

$$\frac{2zf'(z)}{f(z)-\overline{f(-\bar{z})}}\prec \Pi(x,\ z)+1-a, \qquad z\in\mathbb{U}$$

and

$$\frac{2wg'(w)}{g(w)-\overline{g(-\bar{w})}}\prec \Pi(x,\ w)+1-a, \qquad w\in\mathbb{U}$$

hold.

(2) When $\alpha=0$, $\mathcal{P}^{\sigma}_{sc}(0,\ x):=\mathcal{C}^{\sigma}_{sc}(x)$ if

$$\frac{2\left(zf'(z)\right)'}{\left(f(z)-\overline{f(-\bar{z})}\right)'}\prec \Pi(x,\ z)+1-a, \qquad z\in\mathbb{U}$$

and

$$\frac{2\left(wg'(w)\right)'}{\left(g(w)-\overline{g(-\bar{w})}\right)'}\prec \Pi(x,\ w)+1-a, \qquad w\in\mathbb{U}$$

hold.

(3) If $a=p=x=1$, $b=2$ and $q=0$, then we have

$$\frac{2zf'(z)}{f(z)-\overline{f(-\bar{z})}}\prec \frac{1+z}{1-z} \qquad z\in\mathbb{U}.$$

defined as

$$H_q(n)=\begin{vmatrix} a_n & a_{n+1} & \cdots & a_{n+q-1}\\ a_{n+1} & a_{n+2} & \cdots & a_{n+q-2}\\ \vdots & \vdots & \vdots & \vdots\\ a_{n+q-1} & a_{n+q-2} & \cdots & a_{n+2q-2}\end{vmatrix} \qquad (a_1=1).$$

The Hankel determinants $H_2(1)=a_3-a_2^2$ and $H_2(2)=a_2a_4-a_2^3$ are well-known as Fekete-Szegö and second Hankel determinant functionals respectively. Further Fekete and Szegö [22] introduced the generalized functional $a_3-\delta a_2^2$, where δ is some real number. In 1969, Keogh and Merkes [27] studied the Fekete-Szegö problem for the classes $\mathcal{S}^*$ and $\mathcal{K}$. In 2001, Srivastava et al. [43] solved completely the Fekete-Szegö problem for the family $\mathcal{C}_1:=\{f\in\mathcal{A}\ :\ \Re\left(e^{i\eta}f'(z)\right)>0,\ -\frac{\pi}{2}<\eta<\frac{\pi}{2},\ z\in\mathbb{D}\}$ and obtained improvement of $|a_3-a_2^2|$ for the smaller set $\mathcal{C}_1$. Recently, Kowalczyk et al. [28] discussed the developments involving the Fekete-Szegö functional $|a_3-\delta a_2^2|$, where $0\le\delta\le 1$ as well as the corresponding Hankel determinant for the Taylor-Maclaurin coefficients $\{a_n\}_{n\in\mathbb{N}\setminus\{1\}}$ of normalized univalent functions of the form (1.1). Similarly, several authors have investigated upper bounds for the Hankel determinant of functions belonging to various subclasses of univalent functions [3, 4, 11, 12, 15, 30, 32, 34] and the references therein. On the other hand, Zaprawa [49, 50] extended the study on Fekete-Szegö problem to some specific classes of bi-univalent functions. Very recently, the upper bounds of $H_2(2)$ for the classes $S^*_\sigma(\beta)$ and $K_\sigma(\beta)$ were discussed by Deniz et al. [19]. Later, the upper bounds of $H_2(2)$ for various subclasses of σ were obtained by Altınkaya and Yalçın [9], Çağlar et al. [16], Kanas et al. [25], Karthiyayini and Sivasankari [26], Motamednezhad et al. [31], Orhan et al. [35, 36, 37] and Srivastava et al. [42].

2. Main Result

In the following theorem, we estimate the second Hankel determinant inequality for functions functions in $\mathcal{P}^{\sigma}_{Sc}(\alpha,\ x)$. To prove our result we need the following lemmas.

Lemma 2.1. [20] *Let u be analytic function in the unit disk $\mathbb{U}$, with $u(0) = 0$, and $|u(z)| < 1$ for all $z \in \mathbb{D}$, with the power series expansion $u(z) = u_1 z + u_2 z^2 + \cdots$. Then, $|u_n| \leq 1$ for all $n \in \mathbb{N}$. Furthermore, $|u_n| = 1$ for some $n \in \mathbb{N}$ if and only if $u(z) = e^{i\theta} z^n$, $\theta \in \mathbb{R}$.*

Lemma 2.2. [25] *If $\psi(z) = \psi z + \psi_2 z^2 + \cdots$, $z \in \mathbb{U}$, is a Schwarz function, then*

$$\psi_2 = x(1 - \psi_1^2)\ ,$$
$$\psi_3 = (1 - \psi_1^2)(1 - |x|^2)s - \psi_1(1 - \psi_1^2)x^2,$$

for some x, s, with $|x| \leq 1$ and $|s| \leq 1$.

Theorem 2.3. *For $0 < \alpha \leq 1$ and let $f(z) = z + a_2 z^2 + \cdots$ be in the class $\mathcal{P}^{\sigma}_{Sc}(\alpha,\ x)$. Then*

$$|a_2 a_4 - a_3^2| \leq \begin{cases} R, & \text{if } Q \leq 0, \quad P \leq -Q \\ P + Q + R, & \text{if } \left(Q \geq 0, P \geq -\frac{Q}{2}\right) \text{ or } (Q \leq 0, P \geq -Q) \\ \frac{4PR - Q^2}{4P}, & \text{if } Q > 0, P \leq \frac{-Q}{2}, \end{cases}$$

where,

$$P = \frac{|8bx(bp^2x^3 + (apq + bq)x)(2-\alpha)^3 - b^4x^4(\alpha^2 - 12\alpha + 2)|}{64(4-3\alpha)(2-\alpha)^4} + \frac{b^2x^2(2\alpha^2 - 8\alpha + 7)}{8(3-2\alpha)^2(4-3\alpha)(2-\alpha)} - \frac{bx(aq + bpx^2)}{4(4-3\alpha)(2-\alpha)} - \frac{b^3x^3(3\alpha^2 + 60\alpha - 70)}{32(4-3\alpha)(3-2\alpha)(2-\alpha)^2}$$

$$Q = \frac{bx(aq + bpx^2)}{4(4-3\alpha)(2-\alpha)} + \frac{b^3x^3(3\alpha^2 + 60\alpha - 70)}{32(4-3\alpha)(3-2\alpha)(2-\alpha)^2} - \frac{b^2x^2(8\alpha^2 - 28\alpha + 23)}{8(4-3\alpha)(2-\alpha)(3-2\alpha)^2}$$

$$R = \frac{b^2x^2}{4(3-2\alpha)^2}.$$

Proof. For $0 < \alpha \leq 1$ let f of the form (1.1) be in the class $\mathcal{P}^{\sigma}_{Sc}(\alpha,\ x)$, then there exists Schwarz functions $u(z)$ and $v(w)$ given by

$$u(z) = u_1 z + u_2 z^2 + u_3 z^3 + \cdots \qquad \forall z \in \mathbb{U}$$

and

$$v(w) = v_1 w + v_2 w^2 + v_3 w^3 + \cdots \qquad \forall w \in \mathbb{U}$$

such that

$$\frac{2zf'(z)}{f(z) - \overline{f(-\bar{z})}} + \frac{2\left(zf'(z)\right)'}{\left(f(z) - \overline{f(-\bar{z})}\right)'} - \frac{2\alpha z^2 f''(z) + 2zf'(z)}{\alpha z \left(f(z) - \overline{f(-\bar{z})}\right)' + (1-\alpha)\left(f(z) - \overline{f(-\bar{z})}\right)}$$
$$= \Pi(x, u(z)) + 1 - a$$
$$\frac{2wg'(w)}{g(w) - \overline{g(-\bar{w})}} + \frac{2\left(wg'(w)\right)'}{\left(g(w) - \overline{g(-\bar{w})}\right)'} - \frac{2\alpha w^2 g''(w) + 2wg'(w)}{\alpha w \left(g(w) - \overline{g(-\bar{w})}\right)' + (1-\alpha)\left(g(w) - \overline{g(-\bar{w})}\right)}$$
$$\prec \Pi(x,\ v(w)) + 1 - a.$$

Or, equivalently,

$$\frac{2zf'(z)}{f(z) - \overline{f(-\bar{z})}} + \frac{2\left(zf'(z)\right)'}{\left(f(z) - \overline{f(-\bar{z})}\right)'} - \frac{2\alpha z^2 f''(z) + 2zf'(z)}{\alpha z \left(f(z) - \overline{f(-\bar{z})}\right)' + (1-\alpha)\left(f(z) - \overline{f(-\bar{z})}\right)}$$
$$= 1 + h_2(x)u_1 z + [h_2(x)u_2 + h_3(x)u_1^2]z^2 + \cdots \tag{2.1}$$
$$\frac{2wg'(w)}{g(w) - \overline{g(-\bar{w})}} + \frac{2\left(wg'(w)\right)'}{\left(g(w) - \overline{g(-\bar{w})}\right)'} - \frac{2\alpha w^2 g''(w) + 2wg'(w)}{\alpha w \left(g(w) - \overline{g(-\bar{w})}\right)' + (1-\alpha)\left(g(w) - \overline{g(-\bar{w})}\right)}$$
$$= 1 + h_2(x)v_1 w + [h_2(x)v_2 + h_3(x)v_1^2]w^2 + \cdots. \tag{2.2}$$

It is fairly we known that,

$$|u(z)| = |u_1 z + u_2 z^2 + u_3 z^3 + \cdots| < 1$$

and

$$|v(w)| = |v_1 w + v_2 w^2 + v_3 w^3 + \cdots| < 1$$

then

$$|u_k| \leq 1 \qquad \text{and} \qquad |v_k| \leq 1, \qquad k \in \mathbb{N}.$$

Also, from the Definition 1.1, we have

$$\frac{2zf'(z)}{f(z)-\overline{f(-\bar{z})}}+\frac{2\left(zf'(z)\right)'}{\left(f(z)-\overline{f(-\bar{z})}\right)'}-\frac{2\alpha z^2 f''(z)+2zf'(z)}{\alpha z\left(f(z)-\overline{f(-\bar{z})}\right)'+(1-\alpha)\left(f(z)-\overline{f(-\bar{z})}\right)}$$
$$=1+2(2-\alpha)a_2z+2(3-2\alpha)a_3z^2+[4(4-3\alpha)a_4+2a_2a_3(2\alpha^2+3\alpha-6)]z^3+\cdots \tag{2.3}$$

$$\frac{2wg'(w)}{g(w)-\overline{g(-\bar{w})}}+\frac{2\left(wg'(w)\right)'}{\left(g(w)-\overline{g(-\bar{w})}\right)'}-\frac{2\alpha w^2 g''(w)+2wg'(w)}{\alpha w\left(g(w)-\overline{g(-\bar{w})}\right)'+(1-\alpha)\left(g(w)-\overline{g(-\bar{w})}\right)}$$
$$=1-2(2-\alpha)a_2w+2(3-2\alpha)(2a_2^2-a_3)w^2$$
$$-[4(4-3\alpha)(5a_2^3-5a_2a_3+a_4)+2(\alpha^2+3\alpha+8)(2a_2^3-a_2a_3)]w^3+\cdots. \tag{2.4}$$

Comparing the coefficients of like terms in equations (2.1),(2.3) and (2.2), (2.4), we have

$$2(2-\alpha)a_2=h_2(x)u_1 \tag{2.5}$$
$$2(3-2\alpha)a_3=h_2(x)u_2+h_3(x)u_1^2 \tag{2.6}$$
$$4(4-3\alpha)a_4+2a_2a_3(2\alpha^3+3\alpha-6)=h_2(x)u_3+2h_3(x)u_1u_2+h_4(x)u_1^3 \tag{2.7}$$
$$-2(2-\alpha)a_2=h_2(x)v_1 \tag{2.8}$$
$$2(3-2\alpha)(2a_2^2-a_3)=h_2(x)v_2+h_3(x)v_1^2 \tag{2.9}$$

and

$$-\left[4(4-3\alpha)(5a_2^3-5a_2a_3+a_4)+2(\alpha^2+3\alpha+8)(2a_2^3-a_2a_3)\right]$$
$$=h_2(x)v_3+2h_3(x)v_1v_2+h_4(x)v_1^3. \tag{2.10}$$

From equations (2.5) and (2.8), we get

$$u_1=-v_1. \tag{2.11}$$

It follows from (2.5) that

$$a_2=\frac{h_2(x)u_1}{2(2-\alpha)}. \tag{2.12}$$

Subtracting (2.9) from (2.6), we have

$$a_3=\frac{h_2(x)(u_2-v_2)}{4(3-2\alpha)}+a_2^2=\frac{h_2(x)(u_2-v_2)}{4(3-2\alpha)}+\frac{h_2^2(x)u_1^2}{4(2-\alpha)^2} \tag{2.13}$$

Subtracting (2.10) from (2.7), we have

$$a_4=\frac{2(\alpha^2+3\alpha+8)}{8(4-3\alpha)}(2a_2^3-a_2a_3)-\frac{2(2\alpha^2+3\alpha-6)}{8(4-3\alpha)}a_2a_3-\frac{4(4-3\alpha)}{8(4-3\alpha)}(5a_2^3-5a_2a_3) \tag{2.14}$$

$$+\frac{1}{8(4-3\alpha)}[h_2(x)(u_3-v_3)+2h_3(x)u_1(u_2+v_2)+2h_4(x)u_1^3]. \tag{2.15}$$

Now, from the equations (2.12), (2.13) and (2.15), we have

$$a_2a_4 - a_3^2 = \left(\frac{8h_4(x)(2-\alpha)^3 - h_2^3(x)(\alpha^2-12\alpha+2)}{64(4-3\alpha)(2-\alpha)^4}\right)h_2(x)u_1^4 - \frac{(3\alpha^2+60\alpha-70)h_2^3(x)u_1^2(u_2-v_2)}{64(4-3\alpha)(3-2\alpha)(2-\alpha)^2} + \frac{h_2(x)^2u_1(u_3-v_3)}{16(4-3\alpha)(2-\alpha)} + \frac{h_2(x)h_3(x)u_1^2(u_2+v_2)}{8(4-3\alpha)(2-\alpha)} - \frac{h_2^2(x)(u_2-v_2)^2}{16(3-2\alpha)^2}. \tag{2.16}$$

By the Lemma 2.2, we find that

$$\begin{aligned} u_2 - v_2 &= (x-y)(1-u_1^2) \quad \textit{since} \quad u_1 = -v_1 \\ u_2 + v_2 &= (x+y)(1-u_1^2) \\ u_3 &= (1-u_1^2)(1-|x|^2)s - u_1(1-u_1^2)x^2 \\ v_3 &= (1-v_1^2)(1-|y|^2)t - v_1(1-v_1^2)y^2 \\ u_3 - v_3 &= (1-u_1^2)[(1-|x|^2)s - (1-|y|^2)t] - u_1(1-u_1^2)(x^2+y^2) \end{aligned}$$

for some x, y, s, t with $|x| \leq 1, |y| \leq 1, |s| \leq 1, |t| \leq 1$. Taking modulus on both sides, then equation (2.16) becomes

$$\begin{aligned} |a_2a_4 - a_3^2| = \Big| &\left(\frac{8h_4(x)(2-\alpha)^3 - h_2^3(x)(\alpha^2-12\alpha+2)}{64(4-3\alpha)(2-\alpha)^4}\right)h_2(x)u_1^4 \\ &- \frac{(3\alpha^2+60\alpha-70)h_2^3(x)u_1^2(1-u_1^2)(x-y)}{64(4-3\alpha)(3-2\alpha)(2-\alpha)^2} \\ &+ \frac{h_2^2(x)u_1}{16(4-3\alpha)(2-\alpha)}\big((1-u_1^2)[(1-|x|^2)s-(1-|y|^2)t] - u_1(1-u_1^2)(x^2+y^2)\big) \\ &+ \frac{h_2(x)h_3(x)u_1^2(x+y)(1-u_1^2)}{8(4-3\alpha)(2-\alpha)} - \frac{h_2^2(x)(x-y)^2(1-u_1^2)^2}{16(3-2\alpha)^2}\Big| \\ \leq &\frac{|8h_4(x)(2-\alpha)^3 - h_2^3(x)(\alpha^2-12\alpha+2)|}{64(4-3\alpha)(2-\alpha)^4}h_2(x)u_1^4 + \frac{h_2^2(x)u_1(1-u_1^2)}{8(4-3\alpha)(2-\alpha)} \\ &+ \left(\frac{h_2(x)h_3(x)}{8(4-3\alpha)(2-\alpha)} + \frac{(3\alpha^2+60\alpha-70)h_2^3(x)}{64(4-3\alpha)(3-2\alpha)(2-\alpha)^2}\right)u_1^2(1-u_1^2)(|x|+|y|) \\ &+ \frac{h_2^2(x)(u_1^2-u_1)(1-u_1^2)}{16(4-3\alpha)(2-\alpha)}(|x|^2+|y|^2) + \frac{h_2^2(x)(1-u_1^2)^2}{16(3-2\alpha)^2}(|x|+|y|)^2. \end{aligned}$$

Since, $|u_1| \leq 1$, we may assume that $u_1 = u \in [0,1]$, so we have

$$\begin{aligned} |a_2a_4 - a_3^2| \leq &\frac{|8h_4(x)(2-\alpha)^3 - h_2^3(x)(\alpha^2-12\alpha+2)|}{64(4-3\alpha)(2-\alpha)^4}h_2(x)u^4 + \frac{h_2^2(x)u(1-u^2)}{8(4-3\alpha)(2-\alpha)} \\ &+ \left(\frac{h_2(x)h_3(x)}{8(4-3\alpha)(2-\alpha)} + \frac{(3\alpha^2+60\alpha-70)h_2^3(x)}{64(4-3\alpha)(3-2\alpha)(2-\alpha)^2}\right)u^2(1-u^2)(|x|+|y|) \\ &+ \frac{h_2^2(x)(u^2-u)(1-u^2)}{16(4-3\alpha)(2-\alpha)}(|x|^2+|y|^2) + \frac{h_2^2(x)(1-u^2)^2}{16(3-2\alpha)^2}(|x|+|y|)^2. \end{aligned}$$

Now, for $\gamma_1 = |x| \leq 1$ and $\gamma_2 = |y| \leq 1$, we obtain

$$|a_2a_4 - a_3^2| \leq F(\gamma_1,\ \gamma_2) = S_1 + S_2(\gamma_1+\gamma_2) + S_3(\gamma_1^2+\gamma_2^2) + S_4(\gamma_1+\gamma_2)^2$$

where,

$$S_1 = \frac{|8h_4(x)(2-\alpha)^3 - h_2^3(x)(\alpha^2-12\alpha+2)|}{64(4-3\alpha)(2-\alpha)^4}h_2(x)u^4 + \frac{h_2^2(x)u(1-u^2)}{8(4-3\alpha)(2-\alpha)} \geq 0$$

$$S_2 = \left(\frac{h_2(x)h_3(x)}{8(4-3\alpha)(2-\alpha)} + \frac{(3\alpha^2+60\alpha-70)h_2^3(x)}{64(4-3\alpha)(3-2\alpha)(2-\alpha)^2}\right)u^2(1-u^2) \geq 0$$

$$S_3 = \frac{h_2(x)^2(u^2-u)(1-u^2)}{16(4-3\alpha)(2-\alpha)} \leq 0$$

$$S_4 = \frac{h_2(x)^2(1-u^2)^2}{16(3-2\alpha)^2} \geq 0.$$

Now we need to maximize the function $F(\gamma_1,\gamma_2)$ on the closed square $[0,1]\times[0,1]$ for $u\in[0,1]$.With regards to $F(\gamma_1,\gamma_2) = F(\gamma_2,\gamma_1)$, it is sufficient that, we investigate maximum of

$$G(\gamma_2) = F(\gamma_2,\gamma_2) = S_1 + 2\gamma_2 S_2 + 2\gamma_2^2(S_3+2S_4) \quad on \quad \gamma_2\in[0,1] \tag{2.17}$$

according to $u\in(0,\ 1)$, $u=0$ and $u=1$. Firstly, if we let $u=1$, then we obtain

$$max\Big\{G(\gamma_2);\gamma_2\in[0,1]\Big\} = \frac{|8h_4(x)(2-\alpha)^3 - h_2^3(x)(\alpha^2-12\alpha+2)|}{64(4-3\alpha)(2-\alpha)^4}h_2(x).$$

Secondly, letting $u=0$, we get

$$G(\gamma_2) = \frac{4h_2^2(x)\gamma_2^2}{16(3-2\alpha)^2} = \frac{h_2^2(x)\gamma_2^2}{4(3-2\alpha)^2}.$$

Hence we can see that

$$max\Big\{G(\gamma_2);\gamma_2\in[0,1]\Big\} = G(1) = \frac{h_2^2(x)}{4(3-2\alpha)^2}.$$

Finally, we let $u\in(0,1)$. Considering the equation (2.17) for $0\leq\gamma_2\leq1$, we get

(1) If $S_3+2S_4\geq0$, it is clear that

$$G'(\gamma_2) = 4(S_3+2S_4)\gamma_2 + 2S_2 > 0$$

for $0<\gamma_2<1$ and any fixed $u\in(0,1)$ ie., $G(\gamma_2)$ is an increasing function. Hence $max\Big\{G(\gamma_2);\gamma_2\in[0,1]\Big\} = G(1) = S_1+2S_2+2S_3+4S_4$

(2) If $S_3+2S_4<0$, then we consider for critical point

$$\gamma_{2_0} = -\frac{S_2}{2(S_3+2S_4)} = \frac{S_2}{2K}$$

for any fixed $u\in(0,1)$, where $K = -(S_3+2S_4) > 0$ the following two cases

Case 1: For $\gamma_{2_0} = \frac{S_2}{2K} > 1$, it follows that $K < \frac{S_2}{2} \leq S_2$ and so $S_2+S_3+2S_4\geq0$, therefore $G(0) = S_1 \leq S_1+2S_2+2S_3+4S_4 = G(1)$.

Case 2: For $\gamma_{2_0} = \frac{S_2}{2K} \leq 1$, since $S_2\geq0$, we get $\frac{S_2^2}{2K}\leq S_2$. Therefore, $G(0) = S_1 \leq S_1 + \frac{S_2^2}{2K} = G(\gamma_{2_0}) < S_1+S_2$. Considering the above cases for point of u, it follows that the function $G(\gamma_2)$ get, its maximum when $S_3+2S_4\geq 0$, it means

$$max\Big\{G(\gamma_2);\gamma_2\in[0,1]\Big\}=G(1)=S_1+2S_2+2S_3+4S_4$$

Therefore, $maxF(\gamma_1,\gamma_2)=F(1,1)$ on the boundary of the square. Let $K:[0,1]\to\mathbb{R}$

$$K(u)=maxF(\gamma_1,\gamma_2)=F(1,1)=S_1+2S_2+2S_3+4S_4.$$

By replacing the values of S_2,S_2,S_3,S_4 in the above function K, we have

$$\begin{aligned}S_1+2S_2+2S_3+4S_4=&\Bigg[\frac{|8h_2(x)h_4(x)(2-\alpha)^3-(\alpha^2-12\alpha+2)h_2^4(x)|}{64(4-3\alpha)(2-\alpha)^4}\\&+\frac{h_2^2(x)}{4(3-2\alpha)^2}-\frac{h_2^2(x)}{8(4-3\alpha)(2-\alpha)}\\&-\frac{h_2(x)h_3(x)}{4(4-3\alpha)(2-\alpha)}-\frac{h_2^3(x)(3\alpha^2+60\alpha-70)}{32(4-3\alpha)(3-2\alpha)(2-\alpha)^2}\Bigg]u^4\\&+\Bigg[\frac{h_2(x)h_3(x)}{4(4-3\alpha)(2-\alpha)}+\frac{h_2^3(x)(3\alpha^2+60\alpha-70)}{32(4-3\alpha)(3-2\alpha)(2-\alpha)^2}\\&+\frac{h_2^2(x)}{8(4-3\alpha)(2-\alpha)}-\frac{h_2^2(x)}{2(3-2\alpha)^2}\Bigg]u^2+\frac{h_2^2(x)}{4(3-2\alpha)^2}\end{aligned}$$

Letting $u^2=c$, we get

$$S_1+2S_2+2S_3+S_4=Pc^2+Qc+R, \tag{2.18}$$

where,

$$\begin{aligned}P=&\frac{|8h_2(x)h_4(x)(2-\alpha)^3-(\alpha^2-12\alpha+2)h_2^4(x)|}{64(4-3\alpha)(2-\alpha)^4}+\frac{h_2^2(x)}{4(3-2\alpha)^2}\\&-\frac{h_2^2(x)}{8(4-3\alpha)(2-\alpha)}-\frac{h_2(x)h_3(x)}{4(4-3\alpha)(2-\alpha)}-\frac{h_2^3(x)(3\alpha^2+60\alpha-70)}{32(4-3\alpha)(3-2\alpha)(2-\alpha)^2}\\Q=&\frac{h_2(x)h_3(x)}{4(4-3\alpha)(2-\alpha)}+\frac{h_2^3(x)(3\alpha^2+60\alpha-70)}{32(4-3\alpha)(3-2\alpha)(2-\alpha)^2}\\&+\frac{h_2^2(x)}{8(4-3\alpha)(2-\alpha)}-\frac{h_2^2(x)}{2(3-2\alpha)^2}\\R=&\frac{h_2^2(x)}{4(3-2\alpha)^2}\end{aligned}$$

Then with the help of optimal value of quadratic expression, further, using (1.2) and (1.3), we get the required result. This completes the proof of the theorem. □

Corollary 2.4. *Let $f(z)=z+a_2z^2+\cdots$ be in the class $\mathcal{S}^{\sigma}_{sc}(x)$. Then*

$$|a_2a_4-a_3^2|\leq\begin{cases}R_1, & if\,Q_1\leq 0,\quad P_1\leq -Q_1\\P_1+Q_1+R_1, & if\,\left(Q_1\geq 0,P_1\geq-\frac{Q_1}{2}\right)or\,(Q_1\leq 0,\ P_1\geq -Q_1)\\\frac{4P_1R_1-Q_1^2}{4P_1}, & if\,Q_1>0,P_1\leq\frac{-Q_1}{2},\end{cases}$$

where,

$$P_1 = \frac{|8bx(bp^2x^3 + (apq + bq)x) + 9b^4x^4|}{64(4-3\alpha)(2-\alpha)^4} + \frac{b^2x^2}{8} - \frac{bx(aq + bpx^2)}{4} + \frac{7b^3x^3}{32}$$

$$Q_1 = \frac{bx(aq + bpx^2)}{4} + \frac{7b^3x^3}{32} - \frac{3b^2x^2}{8}$$

$$R_1 = \frac{b^2x^2}{4}.$$

Corollary 2.5. *For $0 < \alpha \le 1$ and let $f(z) = z + a_2z^2 + \cdots$ be in the class $\mathcal{C}_{sc}^{\sigma}(x)$. Then*

$$|a_2a_4 - a_3^2| \le \begin{cases} R_2, & \text{if } Q_1 \le 0, \quad P_2 \le -Q_2 \\ P_2 + Q_2 + R_2, & \text{if } \left(Q_2 \ge 0, P_2 \ge -\frac{Q_2}{2}\right) \text{ or } (Q_2 \le 0,\ P_2 \ge -Q_2) \\ \frac{4P_2R_2 - Q_2^2}{4P_2}, & \text{if } Q_2 > 0, P_2 \le \frac{-Q_2}{2}, \end{cases}$$

where,

$$P_2 = \frac{|32bx(bp^2x^3 + (apq + bq)x) - b^4x^4|}{2048} + \frac{7b^2x^2}{576} - \frac{bx(aq + bpx^2)}{32} + \frac{35b^3x^3}{768}$$

$$Q_2 = \frac{bx(aq + bpx^2)}{32} - \frac{35b^3x^3}{768} - \frac{23b^2x^2}{576}$$

$$R_2 = \frac{b^2x^2}{36}.$$

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