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MODEL BUILDING OF NONLINEAR FEEDBACK CONTROL

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Abstract. Building a nonlinear feedback control is a complex task, the solution of which requires the development of various procedures to achieve certain control goals. A single chosen procedure cannot be successfully applied to all non-linear systems. To solve this problem, the researcher needs a whole set of analysis tools, design tools and methods for constructing nonlinear feedback. Consider the control for systems that satisfy the consistency condition, i.e. when in the system state equation the indefinite terms and the control terms are in the same place. The control law in the sliding mode ensures the motion of the trajectories of a system of some manifold at all future moments of time. The sliding manifold is defined using the reduced order model and corresponds to the stated control objective.

1. Introduction

Let us consider a method for constructing control laws based on the Lyapunov theory. In this method, for a nominal system, an additional component of the control law [1],[2] is used, which ensures the robustness of this law with respect to uncertainties that satisfy the consistency condition. The control laws based on the use of two methods – the sliding mode method [3] and the Lyapunov synthesis method [4] – are discontinuous functions, which is a disadvantage that manifests itself in the presence of delays and unsimulated high-frequency dynamics in the system. Therefore, in recent years, researchers have been developing "continuous" versions of these control laws. Therefore, researchers [5],[6] have developed "continuous" versions of these control laws. For example, a method for constructing control laws using the Lyapunov synthesis method [7].

The constraints associated with the consistency condition can be relaxed using the backstepping method. Backstepping is a recursive procedure that combines the problems of finding the Lyapunov function and the corresponding control law. According to this method, the task of developing a control law for the entire system is divided into a sequence of corresponding subtasks for subsystems of a smaller order. Since the researcher has more freedom when analyzing scalar and low-order systems, the backstepping method often makes it relatively easy to solve stabilization and tracking problems using robust control under conditions that are less restrictive than in the case of using other techniques.

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The passivity-based control method is based on using the passivity property of an open-loop system. Stabilization of a passive system at an equilibrium point is actually equivalent to introducing damping into the system, using which a non-passive system is transformed into a passive one by replacing the feedback.

2. Sliding mode control

Let's consider a second-order system

$$\begin{cases} \dot{x}_1 = x_2, \\ \dot{x}_2 = h(x) + g(x)u, \end{cases}$$

where h and g are unknown nonlinear functions and $g(x) \geq g_0 > 0$ for all x. The problem is to construct a state feedback control law that ensures the stabilization of the origin. Assume that a control law is known that holds the trajectories of the system on the manifold (or surface) $s = a_1x_1 + x_2 = 0$. Choosing $a_1 > 0$ ensures that $x(t)$ tends to zero as t tends to infinity. In this case, the rate of convergence can be controlled by choosing the value of a_1. The motion on the manifold $s = 0$ does not depend on h and g. Let us show how it is possible to ensure the reduction of the trajectory of the system to the manifold $s = 0$ and its retention on it. The variable s satisfies the equation

$$\dot{s} = a_1\dot{x}_1 + \dot{x}_2 = a_1x_2 + h(x) + g(x)u.$$

Assume that h and g satisfy the inequality

$$\left| \frac{a_1x_2 + h(x)}{g(x)} \right| \leq \varrho(x), \ \forall \, x \in \ R^2.$$

for some well-known function $\varrho(x)$. Using $V = \frac{1}{2}s^2$ as the Lyapunov function for the system $a_1x_2 + h(x) + g(x)u$, we obtain

$$\dot{V} = s\dot{s} = s[a_1x_2 + h(x)] + g(x)su \leq g(x)s\varrho(x) + g(x) + g(x)su.$$

Assuming $u = -\beta(x)sign(s)$, where $\beta\,(x) \geq \varrho\,(x) + \beta_0$, $\beta_0 > 0$ and

$$sign(s) = \begin{cases} 1, & s > 0 \\ 0, & s = 0 \\ -1, & s < 0, \end{cases}$$

we get

$$\dot{V} \leq g\,(x)\,|s|\,\varrho\,(x) + g\,(x)\,[\varrho\,(x) + \beta_0]\,s\ sign\,(s) = -g\,(x)\,\beta_0\,|s| \leq -g_0\beta_0\,|s|\,.$$

Note that the presented control law is applied only for $s \neq 0$, since in the ideal sliding mode the control u is not defined on the sliding surface $s = 0$. Alternatively, one can write $u = -\beta(x)\ sign(s)$ for all s. This remark remains valid in all cases corresponding to the ideal sliding mode control.

Thus, the function $W = \sqrt{V} = |s|$ satisfies the differential inequality $D^+W \leq -g_0\beta_0$, and it follows from the comparison lemma that

$$W\ (s(t))\ \leq W\ (s(0)) - g_0\beta_0\ t.$$

Summarizing the above, we can conclude that the motion of the system consists of two phases: first, the trajectory, starting outside the manifold $s = 0$, moves towards this manifold and reaches it in a finite time (the reaching phase), and then the sliding mode control (the sliding phase), during which movement is carried out movement is carried out on the manifold $s = 0$ and the dynamics is determined by the reduced order model $\dot{x}_1 = -a_1 x_1$. The phase portrait of the system is shown in the figure 1. The manifold $s = 0$ is called the sliding manifold,

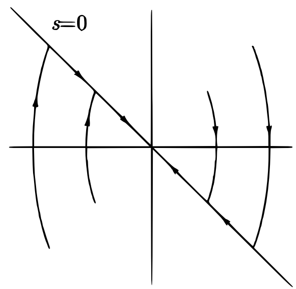

FIGURE 1. Phase portrait of the system under sliding mode control

and the corresponding control law $u = -\beta(x)sign(s)$ is called the sliding mode control. The striking feature of sliding mode control is that this control law is robust with respect to h and g.

The sliding mode control law is simplified if, in the domain under consideration, the functions h and g satisfy the inequality

$$\left|\frac{a_1 x_2 + h(x)}{g(x)}\right| \leq k_1$$

for some known non-negative constant k_1. In this case, the control law is a simple relay and takes the form

$$u = -k\ sign(s), \quad k > k_1.$$

However, in this case the attraction domain will be finite and its estimate can be obtained as follows: The condition $s\dot{s} \leq 0$ in the $\{|s| \leq c\}$ makes it positively invariant. From the equation

$$\dot{x_1} = x_2 = -a_1 x_1 + s$$

and using the function $V_1 = \frac{1}{2}x_1^2$ we get

$$\dot{V_1} = x_1 \dot{x_1} = a_1 x_1^2 + x_1 s \leq -a_1 x_1^2 + |x_1|\, c \leq 0, \quad \forall\, |x_1| \geq \frac{c}{a_1}.$$

Then

$$|x_1\,(0)| \leq \frac{c}{a_1} \Rightarrow |x_1\,(t)| \leq \frac{c}{a_1}, \quad \forall t \geq 0,$$

and the set

$$\Omega = \left\{ |x_1| \leq \frac{c}{a_1}, |s| \leq c \right\},$$

will be positively invariant if

$$\left|\frac{a_1 x_2 + h(x)}{g(x)}\right| \leq k_1 \quad \forall x \in \Omega.$$

Moreover, any trajectory starting at Ω tends to the origin at $t \to \infty$. By choosing a sufficiently large constant c, any compact set in the plane can be included in Ω. Therefore, if the constant k can be chosen arbitrarily large, the above control law can provide semi-global stabilization.

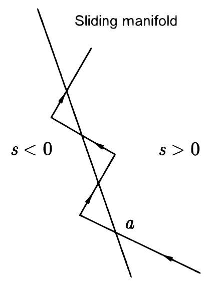

FIGURE 2. Chattering due to delay in control switching

Because of imperfections in switching devices and delays, sliding mode control suffers from chattering. The sketch of figure 2 shows how delays can cause chattering. With an ideally implemented control in the sliding mode, this trajectory should, starting from this point, move along the sliding manifold. In practice, however, there is a delay between the point at which the control mode is actually switched. When switching the control mode, the trajectory changes direction again and crosses it again. As a result of repeating this cycle, the trajectory becomes zigzag (see figure 2). This oscillatory mode leads to a decrease in control accuracy, heat losses in electrical networks and increased wear of the moving parts of mechanisms. It can also introduce unsimulated high-frequency dynamics into the system, which can degrade system performance or even lead to instability.

3. Computer model

For a better understanding of the chattering effect, consider the results of computer simulation of the pendulum model

$$\begin{cases} \dot{x}_1 = x_2, \\ \dot{x}_2 = -\dfrac{g_0}{l}\sin(x_1+\delta_1) - \dfrac{k_0}{m}x_2 + \dfrac{1}{ml^2}u, \\ u = -k\ sign\,(s) = -k\ sign\,(a_1x_1 + x_2)\,, \end{cases}$$

where $x_1 = \theta - \delta_1$ and $x_2 = \dot{\theta}$. The problem is to stabilize the position of the pendulum in the position $\delta_1 = \frac{\pi}{2}$. The constants m, l, k_0, and g_0 are, respectively, the mass, the length of the suspension, the coefficient of friction, and the free fall acceleration. Let $a_1 = 1$ and $k = 4$. The gain $k = 4$ is chosen so that

$$\left|\frac{a_1x_2 + h\,(x)}{g\,(x)}\right| = \left|l^2\,(m-k_0)\,x_2 - mg_0l\cos(x_1)\right| \le l^2\,|m-k_0|\,(2\pi) - mg_0l \le 3,68,$$

where the estimate is calculated on the set $\{|x_1| \le \pi,\ \ |x_1+x_2| \le \pi\}$ for $0.05 \le m \le 0,2$; $0.9 \le l \le 1.1$ and $k_0 = 0.02$.

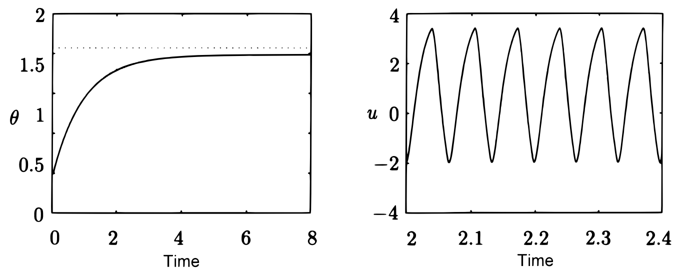

FIGURE 3. Sliding mode control with unmodeled actuator dynamics

Two approaches will be proposed below to reduce the impact of chatter or completely eliminate it. The first approach involves the division of control into two components – continuous and discontinuous, corresponding to switching. The amplitude of the latter must be reduced. Let $\hat{h}(x)$ and $\hat{g}(x)$ be the nominal models of the functions $h(x)$ and $g(x)$, respectively. Taking

$$u = -\frac{\left[a_1 x_2 + \hat{h}\,(x)\right]}{\hat{g}\,(x)} + \vartheta,$$

results in

$$\dot{s} = a_1\left[1 - \frac{g\,(x)}{\hat{g}\,(x)}\right] x_2 + h\,(x) - \frac{g\,(x)}{\hat{g}\,(x)}\hat{h}\,(x) + g\,(x)\,\vartheta = \delta\,(x) + g\,(x)\,\vartheta.$$

If the perturbation term $\delta(x)$ satisfies the inequality

$$\left|\frac{\delta(x)}{g(x)}\right| \leq \varrho(x)$$

as a control law, we can take $v = -\beta(x) sign(s)$, where $\beta(x) \geq \varrho(x) + \beta_0$. Since ϱ is an upper estimate of the perturbation term, it is very likely that this value must be less than the boundary of the entire function. Therefore, when using this approach, it can be expected that the amplitude of the discontinuous component will be smaller. For example, returning to the pendulum equation and taking $\hat{m} = 0,125$, $\hat{l} = 1$, $\hat{k_0} = 0,025$ to be nominal values of m, l, k_0, we have Then the modified control law in the sliding mode takes the form

$$u = -0,1x_2 + 1,2263\cos x_1 - 2sign\,(s)\,.$$

Is easy to see that the amplitude of the switching component decreased and became equal not to 4, as before, but to 2. The figure 4 shows the results of computer simulation of the system using modified control under conditions of unsimulated drive dynamics. It is clearly seen that the amplitude of the chattering has decreased.

The second approach used to eliminate chattering from the system dynamics is to replace the sign function with a saturation function with a large switching line slope. In this case, the control law has the form $u = -\beta(x) sat\ (\frac{s}{\varepsilon})$, where $sat(\cdot)$

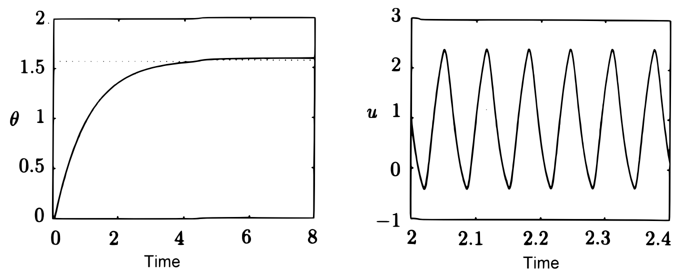

Figure 4. Modified sliding mode control with unmodeled actuator dynamics

is the saturation function defined by the equality

$$sat(y) = \begin{cases} y, \quad if \quad |y| \le 1, \\ sign(y), \quad if \quad |y| > 1, \end{cases}$$

and ε is a positive constant. Graphs of the sign and saturation functions are shown in Figure 5. Slope of line switching function $sat(\frac{s}{\varepsilon})$ is equal to $\frac{1}{\varepsilon}$.

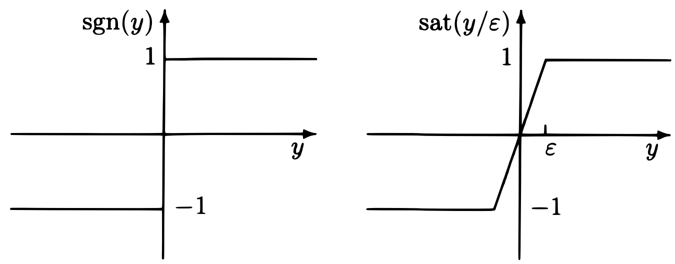

Figure 5. The signum nonlinearity and its saturation function approximation

A good approximation requires that the constant ε be small. In the limiting case $\varepsilon \to 0$, the saturation function $sat(\frac{s}{\varepsilon})$ tends to the sign function $sign(s)$. To analyze the performance of the "continuous" sliding mode controller, we examine the reaching phase by using the function $V = \frac{1}{2}s^2$ whose derivative satisfies the inequality $\dot{V} \le -g_0\beta_0 |s|$ when $|s| \ge \varepsilon$, i.e. outside the boundary layer $\{|s| \le \varepsilon\}$. Therefore, for $|s(0)| > \varepsilon$ the function $|s(t)|$ will be strictly decreasing until the trajectory reaches the set $\{|s| \le \varepsilon\}$ in finite time. After this moment, the trajectory will move within the specified boundary layer at all future times. In this mode, the system dynamics is described by the equation $\dot{x_1} = -a_1x_1 + s$, where $|s| \le \varepsilon$ and the derivative of the function $V_1 = \frac{1}{2}x_1^2$ satisfies the inequality

$$\dot{V_1} = -a_1x_1^2 + x_1s \le -a_1x_1^2 + |x_1|\,\varepsilon \le -(1-\theta_1)\,a_1x_1^2, \forall\,|x_1| \ge \frac{\varepsilon}{a_1\theta_1},$$

where $0 < \theta_1 < 1$. Thus, the trajectory reaches the set $\Omega_\varepsilon = \left\{|x_1| \le \frac{\varepsilon}{a_1\theta_1},\ |s| \le \varepsilon\right\}$ in finite time. In the general case, stabilization of the origin is not guaranteed, but boundedness is guaranteed with a limiting bound, the value of which can be reduced by decreasing ε. The nature of the processes inside depends on the specifics of the problem under consideration. As an example, consider the pendulum equation. Inside the strip $\{|s| \leq \varepsilon\}$ the control law is reduced to the linear control $u = -\frac{ks}{\varepsilon}$, as a result of which we obtain a closed system

$$\begin{cases} \dot{x}_1 = x_2, \\ \dot{x}_2 = -\dfrac{g_0}{l}\sin(x_1+\delta_1) - \dfrac{k_0}{m}x_2 + \dfrac{k}{ml^2\varepsilon}(a_1x_1+x_2), \end{cases}$$

This system has a single equilibrium point $(\bar{x}_1, 0)$, where the quantity $\bar{x}_1$ satisfies the equation

$$\varepsilon m g_0 l \sin(\bar{x}_1+\delta_1) + k a_1 \bar{x}_1 = 0$$

and can be approximated for small e as follows: $\bar{x}_1 \approx -\frac{\varepsilon m g_0 l}{k a_1}$. We move the equilibrium point to the origin using the change of variables $y_1 = x_1 - \bar{x}_1$, $y_2 = x_2$.

As a result, we get the system

$$\begin{cases} \dot{y_1} = y_2, \\ \dot{y_2} = -\sigma(y_1) - \left(\dfrac{k_0}{m} + \dfrac{k}{ml^2\varepsilon}\right) y_2, \end{cases}$$

where $\sigma(y_1) = -\frac{g_0}{l}\left[sin(y_1+\bar{x}_1+\delta_1) - \sin(\bar{x}_1+\delta_1)\right] + \frac{ka_1}{ml^2\varepsilon}y_1$. Using $\widetilde{V} = \int_0^{y_1}\sigma(s)\ ds + \frac{1}{2}y_2^2$ as the Lyapunov function, one can show that the function $\widetilde{V} \geq -\frac{g_0}{2l}y_1^2 + \frac{ka_1}{ml^2\varepsilon}y_1^2 + \frac{1}{2}y_2^2$ is positive definite for $\frac{k}{\varepsilon} > \frac{mlg_0}{a_1}$ and its derivative satisfies the equality

$$\dot{\widetilde{V}} = -\left(\frac{k_0}{m} + \frac{k}{ml^2\varepsilon}\right) y_2^2.$$

It follows from the invariance principle that the equilibrium point $(\bar{x}_{1,}0)$ is asymptotically stable and attracts any trajectory to Ω_ε.

In order to provide higher control accuracy, we need to choose the constant ε as small as possible. However, be aware that if ε is set too low and if there are delays or unsimulated dynamics in the system, chattering can occur. Figure 6 shows the results of the pendulum motion simulation for the case of "continuous" control in sliding mode for two different values of ε. Figure 7 shows the simulation results of this system in the presence of unsimulated drive dynamics with the transfer function $\frac{1}{(0.01s+1)^2}$. It is interesting to note that for an ideal controller, a decrease in ε leads to an increase in the control accuracy, but in the presence of delays, this effect is not observed due to the appearance of a chattering.

In some cases it is possible to ensure the stabilization of the origin of coordinates without a strong decrease in ε. A similar situation arises, for example, at $h(0) = 0$, when the behavior of the system inside the band is described by the equation of state

$$\begin{cases} \dot{x_1} = x_2, \\ \dot{x_2} = h(x) - \left[\dfrac{g(x)k}{\varepsilon}\right](a_1x_1+x_2), \end{cases}$$

and the origin is the equilibrium point. It is necessary to choose ε small enough to stabilize the origin and make Ω_ε a subset of its attraction domain. In the case

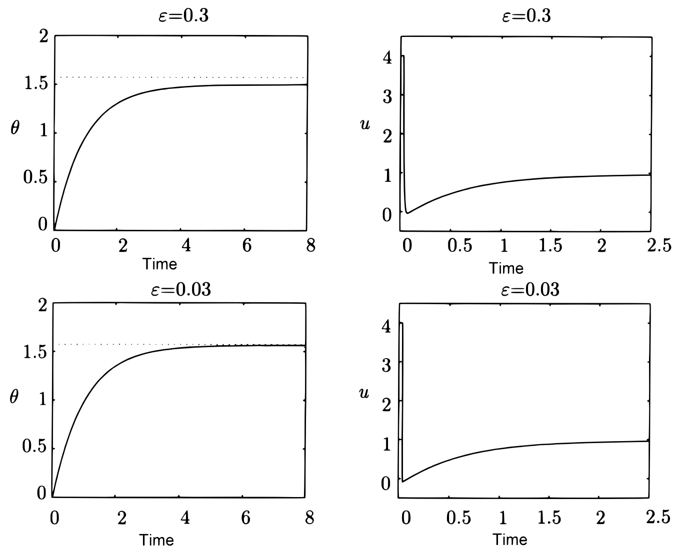

FIGURE 6. Continuous sliding mode control

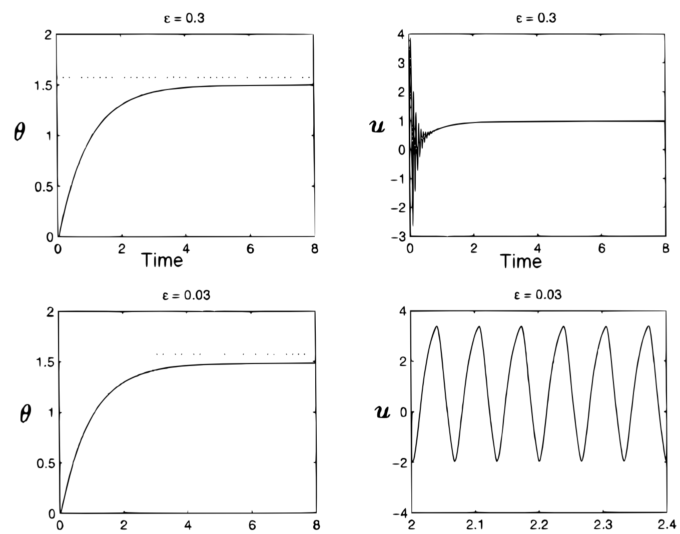

FIGURE 7. Continuous sliding mode control with unmodeled actuator dynamics

of a pendulum equation with $\delta_1 = \pi$, applying the above approach to analyzing the stability of the system leads to the conclusion that this goal can be achieved if $\frac{k}{\varepsilon} > \frac{mlg_0}{a_1}$. For $l \leq 1,1$, $m \leq 0,2$, $k = 4$ and $a_1 = 1$, choose $\varepsilon < 1.8534$. Figure 14.10 shows the results of a computer simulation for the case $\varepsilon = 1$.

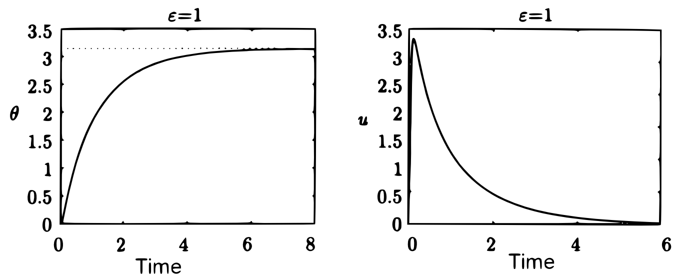

FIGURE 8. Continuous sliding mode control when $\delta_1 = \pi$

If the δ_1 is any angle other than 0 or π (the open-loop equilibrium points), the system stabilizes at an equilibrium point different from the origin with an error in the steady state, the approximate value of which was obtained earlier: $\frac{\varepsilon m g_0 l}{k a_1} \sin \delta_1$. In order to obtain a zero error in the steady state, one should use the control law with an integral action. Let $x_0 = \int x_1$. Then the augmented system has the form

$$\begin{cases} \dot{x_0} = x_1, \\ \dot{x_1} = x_2, \\ \dot{x_2} = -\dfrac{g_0}{l} \sin(x_1 + \delta_1) - \dfrac{k_0}{m} x_2 + \dfrac{1}{ml^2} u. \end{cases}$$

Let $s = a_0 x_0 + a_1 x_1 + x_2$, where the coefficients are such that the matrix

$$A_0 = \begin{bmatrix} 0 & 1 \\ -a_0 & -a_1 \end{bmatrix}$$

is Hurwitz. If in the area under consideration

$$ml^2 \left| a_0 x_0 + a_1 x_1 - \frac{g_0}{l} \sin(x_1 + \delta_1) - \frac{k_0}{m} x_2 \right| \le k_1,$$

it is possible to apply continuous control in the sliding mode of the form $u = -k\ sat\left(\frac{s}{\varepsilon}\right)$, $k > k_1$. This control law ensures that s will reach the boundary layer $\{|s| \le \varepsilon\}$ in a finite time, because $s\dot{s} \le -(k - k_1)|s|$ for $|s| \ge \varepsilon$.

Inside the boundary layer, the system is described by the equation

$$\dot{\eta} = A_0 \eta + B_0 s,$$

where $\eta = \begin{bmatrix} x_0 \\ x_1 \end{bmatrix}$, $B_0 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$. Setting $\dot{V}_1 = \eta^T P_0 \eta$, where P_0 is the solution of the Lyapunov equation $P_0 A_0 + A_0 P_0^T = -I$, we can be verified that

$$\dot{V}_1 = -\eta^T \eta + 2\eta^T P_0 B_0 \le -(1 - \theta_1) \|\eta\|_2^2, \quad \forall \|\eta\|_2^2 \ge 2 \|P_0 B_0\|_2 \frac{\varepsilon}{\theta_1},$$

where $0 < \theta_1 < 1$.

Thus, all trajectories reach the set

$$\Omega_\varepsilon = \left\{ V_1(\eta) \le \frac{4 \|P_0 B_0\|_2^2 \varepsilon \|P_0\|_2}{\theta_1^2}, |s| \le \varepsilon \right\}$$

for the end time. Inside the Ω_ε system

$$\begin{cases} \dot{x_0} = x_1, \\ \dot{x_1} = x_2, \\ \dot{x_2} = -\dfrac{g_0}{l}\sin(x_1+\delta_1) - \dfrac{k_0}{m}x_2 - \dfrac{k}{ml^2}(a_0x_0 + a_1x_1 + x_2). \end{cases}$$

has a single equilibrium point at $\bar{x} = \left[-\frac{\varepsilon m l g_0}{k a_1}\sin\delta_1,\ 0,\ 0\right]^T$. By repeating the above analysis of the stability of the system, we can show that for sufficiently small ε the equilibrium point $\bar{x}$ is asymptotically stable and every trajectory in Ω_ε tends to ε as $t \to \infty$. Therefore, the angle θ tends to the desired position δ_1. The results of computer simulation for $m = 0.1$, $l = 1$, $k_0 = 0.002$, $\delta_1 = \frac{\pi}{2}$, $a_0 = a_1 = 1$, $k = 4$ and $\varepsilon = 1$ are shown in figure 9.

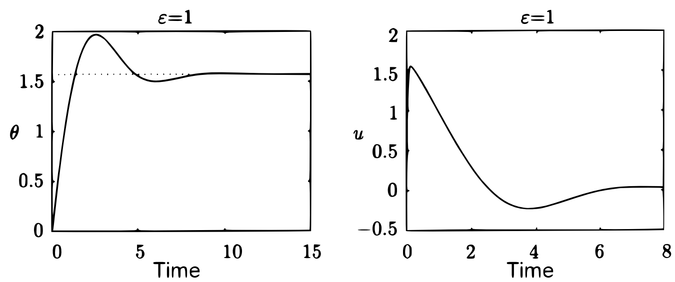

FIGURE 9. Continuous sliding mode control when $\delta_1 = \frac{\pi}{2}$

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