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STOCHASTICALLY RESETTING KAC-ORNSTEIN-UHLENBECK PROCESSES

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ABSTRACT. The Kac-Ornstein-Uhlenbeck processes with reset are studied in detail, including the joint distribution of the process and the corresponding counting Poisson process. The Laplace transform of the first passage time is also presented.

1. Introduction

Stochastically resetting diffusion processes with numerous applications to quantum mechanics, statistical physics, and financial modelling have recently become very popular in the physics literature, for a review see the special issue of *Journal of Physics A*: *Mathematical and Theoretical*, vol.55, No.46 (2022); see also [4, 12]. This model, inspired by many examples from nature, assumes a system of interacting (or independent) particles that, when reset, return to their initial position. Ornstein-Uhlenbeck processes with reset are already beginning to be studied, [6, 13].

Recently, these results have been generalised to a resetting systems of independent particles moving at finite velocities that change sequentially according to the underlying finite state Markov process $\varepsilon = \varepsilon(t)$, see [5, 2, 4]. Note that in the simplest case such processes describing *continuous* one-dimensional movement with a two-state basis, the so-called telegraph (or *flip-flop*, [3]) processes, are well studied, see for example [14, 7, 11].

Here, we present a model based on Ornstein-Uhlenbeck processes *of bounded variation*, see [9]. This process is defined as a solution to the Langevin equation based on telegraph process instead of Brownian motion. The process is called the Kac-Ornstein-Uhlenbeck process. In a continuous setting, the distributions of first passage times and the invariant measures have been recently studied in [9, 10].

In this article, we are adding jumps/resetting to this model. We will focus on two aspects: firstly, on explicit formulae for the joint distribution of the resetting Kac-Ornstein-Uhlenbeck process and the corresponding process $N(t)$, which counts the number of restarts, and, secondly, on the distribution of the time of the first crossing by this process of a given threshold.. Both aspects have good prospects for applications in financial modelling [11] or in mathematical physics [5, 2, 4].

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The analysis of such distributions for the Kac-Ornstein-Uhlenbeck processes turns out to be much more meaningful and rich than when it comes to the simple diffusion case. It essentially depends on the signs and values of the model parameters.

Section 2 contains the basic settings of the model. In Section 3, we analyse in detail the joint distribution of $\mathbb{X}(t)$ and $N(t)$, and Section 4 is devoted to the distributions of the first crossing moments.

2. Kac-Ornstein-Uhlenbeck processes with jumps

Let $\varepsilon = \varepsilon(t)$ be a continuous-time Markov chain with the finite state space E, $\varepsilon(t) \in E = \{0, 1, \ldots, d\}$, driven by switching intensities $\lambda_{ij} > 0,\ i, j \in E$, on the filtered probability space $(\Omega, \mathscr{F}, \mathscr{F}_t, \mathbb{P})$, $t \geq 0$. By $\mathbb{P}_i$ we denote the conditional probability measure under the given initial state $\varepsilon(0) = i,\ \mathbb{P}_i\{\cdot\} := \mathbb{P}\{\cdot \mid \varepsilon(0) = i\}$; $\mathbb{E}_i$ is the expectation with respect to $\mathbb{P}_i$, $i \in E$. Let $\Phi(t) = \exp(t\mathscr{L}) = (\Phi_{ij}(t))_{i,j\in E}$, $t \geq 0$, be the transition semigroup with the infinitesimal generator $\mathscr{L}$.

Let $\mathbb{A} = \mathbb{A}(t)$ be a continuous piecewise linear random process (telegraph process) based on ε:

$$\mathbb{A}(t) = \int_0^t a_{\varepsilon(s)} \mathrm{d}s,$$

where $a_0, a_1, \ldots, a_d \in \mathbb{R}$ are constants. Let $\mathbb{G} = \mathbb{G}(t)$, $t \geq 0$, be a piecewise linear process with jumps, defined with the same ε,

$$\mathbb{G}(t) = -\Gamma(t) + \sum_{n=1}^{N(t)} Y_n, \qquad \Gamma(t) = \int_0^t \gamma_{\varepsilon(s)} \mathrm{d}s,$$

where $\gamma_0, \gamma_1, \ldots, \gamma_d \in \mathbb{R}$. Here, the sequence of independent r.v. $\{Y_n\}_{n\geq 1}$ is independent on the driving process ε. Let $Y_n > -1$, *a.s.*

We study the random process $\mathbb{X} = \mathbb{X}(t)$ that defined by the stochastic equation

$$\mathbb{X}(t) = x + \mathbb{A}(t) + \int_0^t \mathbb{X}(s)\mathrm{d}\mathbb{G}(s), \qquad t \geq 0.$$

In differential form, this equation is equivalent to the Cauchy problem

$$\mathrm{d}\mathbb{X}(t) = \mathrm{d}\mathbb{A}(t) + \mathbb{X}(t-)\mathrm{d}\mathbb{G}(t), \qquad t > 0, \tag{2.1}$$

$$\mathbb{X}(0) = x.$$

The solution can be written explicitly:

$$\mathbb{X}(t) = \mathscr{E}_t(\mathbb{G}) \left[x + \int_0^t \mathscr{E}_s(\mathbb{G})^{-1} \mathrm{d}\mathbb{A}(s) \right]. \tag{2.2}$$

Here $\mathscr{E}_t(\mathbb{G}) = \mathrm{e}^{-\Gamma(t)}\kappa_t$, is the Doléans-Dade exponential, $\kappa_t = \prod_{n=1}^{N(t)}(1 + Y_n)$, $t \geq 0$.

In this paper, we study a particular case of $d = 1$. Therefore, $\Phi_{ij}(t)$, $t > 0$, is given by 2×2-matrix

$$\Phi(t) = (\Phi_{ij}(t))_{i,j\in\{0,1\}} = \frac{1}{2\lambda} \begin{pmatrix} \lambda_1 + \lambda_0 \mathrm{e}^{-2\lambda t} & \lambda_0(1 - \mathrm{e}^{-2\lambda t}) \\ \lambda_1(1 - \mathrm{e}^{-2\lambda t}) & \lambda_0 + \lambda_1 \mathrm{e}^{-2\lambda t} \end{pmatrix}, \qquad t \geq 0,$$

$2\lambda = \lambda_0 + \lambda_1$, see [11].

Note that the solution $\mathbb{X}$, (2.2), can only be represented in term of the process $\mathbb{G}$:

$$\mathbb{X}(t) = \mathrm{e}^{-\Gamma(t)}\kappa_t\left[x + k\cdot\int_0^t \kappa_s^{-1}\exp(\Gamma(s))\mathrm{d}\Gamma(s) + c\cdot\int_0^t \kappa_s^{-1}\exp(\Gamma(s))\mathrm{d}s\right],$$

with linking coefficients

$$k = \frac{a_0 - a_1}{\gamma_0 - \gamma_1} \text{ and } c = \frac{a_1\gamma_0 - a_0\gamma_1}{\gamma_0 - \gamma_1}.$$

The trajectories of the process $\mathbb{X} = \mathbb{X}(t)$ are formed by successive exchange between the two patterns

$$\phi_0(t,y) = \rho_0 + (y-\rho_0)\mathrm{e}^{-\gamma_0 t}, \qquad \phi_1(t,y) = \rho_1 + (y-\rho_1)\mathrm{e}^{-\gamma_1 t}, \tag{2.3}$$

which alternate in random moments τ_n and are accompanied by jumps occurring at the times of state switching. Here $\rho_0 = a_0/\gamma_0,\ \rho_1 = a_1/\gamma_1$.

Note that

$$\frac{\partial\phi_0(t,y)}{\partial t} - L_0(y)[\phi_0(t,y)] \equiv 0, \qquad \frac{\partial\phi_1(t,y)}{\partial t} - L_1(y)[\phi_1(t,y)] \equiv 0, \tag{2.4}$$

where

$$L_0(y)[\phi] = (a_0 - \gamma_0 y)\frac{\partial\phi}{\partial y}, \qquad L_1(y)[\phi] = (a_1 - \gamma_1 y)\frac{\partial\phi}{\partial y}. \tag{2.5}$$

Theorem 2.1. *The infinitesimal generator for the Markov process* $\langle\mathbb{X}(t),\ \varepsilon(t)\rangle$, $t \geq 0$, *has a form*

$$\mathscr{L} = \begin{pmatrix} -\lambda_0 + L_0 & \lambda_0\hat{Y}_0 \\ \lambda_1\hat{Y}_1 & -\lambda_1 + L_1 \end{pmatrix}, \tag{2.6}$$

where the differential operators L_0 *and* L_1 *are defined by* (2.5), *and the integral operators* $\hat{Y}_0,\ \hat{Y}_1$ *are defined by*

$$\hat{Y}_i[f](\cdot) = \int_{-1}^{\infty} f(\cdot(1+z))g_i(\mathrm{d}z), \qquad i \in \{0,1\}.$$

Here $g_0(\mathrm{d}y)$ *and* $g_1(\mathrm{d}y)$ *denote the distribution of the jump amplitude accompanying the switching from the state 0 and the state 1, respectively.*

Proof. The distribution of $\mathbb{X}(t)$ can be described by the following pair

$$P_0(t,\mathrm{d}x \mid y) = \mathbb{P}\{\mathbb{X}(t) \in \mathrm{d}x \mid \varepsilon(0) = 0,\ X(0) = y\},$$
$$P_1(t,\mathrm{d}x \mid y) = \mathbb{P}\{\mathbb{X}(t) \in \mathrm{d}x \mid \varepsilon(0) = 1,\ X(0) = y\}.$$

By definition of the process $\mathbb{X}$, conditioning on the first switching time, we obtain the system of integral equations that obey the measures $P_0(t,\mathrm{d}x \mid y)$ and $P_1(t,\mathrm{d}x \mid y)$,

$$\begin{cases} P_0(t,\cdot \mid y) = \mathrm{e}^{-\lambda_0 t}\delta_{\phi_0(t,y)}(\cdot) + \int_0^t \lambda_0\mathrm{e}^{-\lambda_0\tau}\mathscr{G}_0(t-\tau,\cdot;\tau,y)[P_1]\mathrm{d}\tau, \\ P_1(t,\cdot \mid y) = \mathrm{e}^{-\lambda_1 t}\delta_{\phi_1(t,y)}(\cdot) + \int_0^t \lambda_1\mathrm{e}^{-\lambda_1\tau}\mathscr{G}_1(t-\tau,\cdot;\ \tau,y)[P_0]\mathrm{d}\tau, \end{cases} \tag{2.7}$$

where the first terms with Dirac's δ-measure correspond to the ballistic movement without switching of the state, and $\mathscr{G}_0$, $\mathscr{G}_1$ are defined by

$$\begin{aligned}
\mathscr{G}_0(t-\tau,\cdot;\tau,y)[P_1] &= \int\limits_{y\in(-1,+\infty)} P_1(t-\tau,\cdot\mid(1+z)\phi_0(\tau,y))g_0(\mathrm{d}z),\\
\mathscr{G}_1(t-\tau,\cdot;\ \tau,y)[P_0] &= \int\limits_{y\in(-1,+\infty)} P_0(t-\tau,\cdot\mid(1+z)\phi_1(\tau,y))g_1(\mathrm{d}z).
\end{aligned} \tag{2.8}$$

Notice that due to (2.4), the following identities hold

$$\begin{aligned}
\left[\frac{\partial}{\partial t}-L_0(y)\right]\mathscr{G}_0(t-\tau,\cdot;\tau,y)[P_1] &\equiv -\frac{\partial}{\partial\tau}\mathscr{G}_0(t-\tau,\cdot;\tau,y)[P_1],\\
\left[\frac{\partial}{\partial t}-L_1(y)\right]\mathscr{G}_1(t-\tau,\cdot;\tau,y)[P_0] &\equiv -\frac{\partial}{\partial\tau}\mathscr{G}_1(t-\tau,\cdot;\tau,y)[P_0].
\end{aligned}$$

Therefore,

$$\begin{aligned}
\left[\frac{\partial}{\partial t}-L_0(y)\right]P_0(t,\mathrm{d}x\mid y) = &-\lambda_0\mathrm{e}^{-\lambda_0 t}\delta_{\phi_0(t,y)}(\cdot)+\lambda_0\mathrm{e}^{-\lambda_0 t}\mathscr{G}_0(0,\cdot;t,y)[P_1]\\
&-\int_0^t\lambda_0\mathrm{e}^{-\lambda_0\tau}\frac{\partial}{\partial\tau}\mathscr{G}_0(t-\tau,\cdot;\tau,y)[P_1]\mathrm{d}\tau.
\end{aligned}$$

After integration by parts, the latter equation turns into

$$\left[\frac{\partial}{\partial t}-L_0(y)\right]P_0(t,\mathrm{d}x\mid y) = -\lambda_0 P_0(t,\mathrm{d}x\mid y)+\lambda_0\mathscr{G}_0(t,\cdot;0,y)[P_1]. \tag{2.9}$$

Similarly, we obtain the equivalent form of the second equation of (2.7):

$$\left[\frac{\partial}{\partial t}-L_1(y)\right]P_1(t,\mathrm{d}x\mid y) = \lambda_1\mathscr{G}_1(t,\cdot;0,y)[P_0]-\lambda_1 P_1(t,\mathrm{d}x\mid y). \tag{2.10}$$

Since by definition (2.8)

$$\mathscr{G}_0(t,\mathrm{d}x;0,y)[P_1] = \int\limits_{y\in(-1,+\infty)} P_1(t,\cdot\mid(1+z)\phi_0(0,y))g_0(\mathrm{d}z) = \hat{Y}_0[P_1(t,\mathrm{d}x\mid\cdot)]$$

and

$$\mathscr{G}_1(t,\mathrm{d}x;0,y)[P_0] = \int\limits_{y\in(-1,+\infty)} P_0(t,\cdot\mid(1+z)\phi_1(0,y))g_1(\mathrm{d}z) = \hat{Y}_1[P_0(t,\mathrm{d}x\mid\cdot)],$$

integral system (2.7) due to (2.9)-(2.10) is equivalent to

$$\begin{cases}
\dfrac{\partial P_0}{\partial t}(t,\cdot\mid y) = (-\lambda_0+L_0)P_0(t,\cdot\mid y)+\lambda_0\cdot\hat{Y}_0[P_1](t,\cdot\mid y),\\
\dfrac{\partial P_1}{\partial t}(t,\cdot\mid y) = \lambda_1\cdot\hat{Y}_1[P_0](t,\cdot\mid y)+(-\lambda_1+L_0)P_1(t,\cdot\mid y),
\end{cases}$$

which gives (2.6).

In what follows, we will assume that at each regime change, the particle returns completely to its original position, i.e., $Y_n\equiv -1$. In this case, the solution $\mathbb{X}$ of the equation

(2.1) is formed by a successive reset to the origin, occurring at times τ_n between the patterns ϕ_0 and ϕ_1,

$$\phi_0(t) = \rho_0\left(1-\mathrm{e}^{-\gamma_0 t}\right), \quad \phi_1(t) = \rho_1\left(1-\mathrm{e}^{-\gamma_1 t}\right), \qquad t>0, \tag{2.11}$$

cf (2.3). Assume for definiteness that $\rho_0 \le \rho_1$.

Note that the function $\phi = \phi(t) = \rho(1-\mathrm{e}^{-\gamma t})$ follows a certain behaviour depending on the signs of the parameters. For example, if $\gamma < 0$, then $\phi(t)$ monotonically increases up to $+\infty$ ($\rho < 0$) or decreases up to $-\infty$ ($\rho > 0$). If $\gamma > 0$, then $\phi(t)$ tends to ρ (increases if $\rho > 0$ or decreases if $\rho < 0$). Hence, if $\gamma \ne 0$, then there exists an inverse function ϕ^{-1},

$$\phi^{-1}(x) = -\gamma^{-1}\log(1-x/\rho),$$

and its domain depends on the signs of ρ and γ.

3. The joint distribution of $\mathbb{X}(t)$ and $N(t)$.

Let us first assume that the process $\mathbb{X} = \mathbb{X}(t)$ starts at the origin, $\mathbb{X}(0) = 0$.

The joint distribution of the particle's position $\mathbb{X}(t)$ and the number of state switches $N(t)$ is defined by the probability density functions $p_i(t,x;n)$,

$$p_i(t,x;n)\mathrm{d}x = \mathbb{P}_i\{\mathbb{X}(t) \in \mathrm{d}x,\ N(t) = n\}, \qquad i \in \{0,1\}, \quad n \ge 1.$$

The probability density functions $p_0(t,x;n)$ and $p_1(t,x;n)$, $n \ge 1$, can be presented separately for even and odd number n.

When there are no switching, the distribution is by definition singular,

$$\begin{aligned} p_0(t,\mathrm{d}x;\,0) = \mathbb{P}_0(\mathbb{X}(t) \in \mathrm{d}x,\ N(t) = 0) =& \exp(-\lambda_0 t)\delta_{\phi_0(t)}(\mathrm{d}x),\\ p_1(t,\mathrm{d}x;\,0) = \mathbb{P}_1(\mathbb{X}(t) \in \mathrm{d}x,\ N(t) = 0) =& \exp(-\lambda_1 t)\delta_{\phi_1(t)}(\mathrm{d}x), \end{aligned} \tag{3.1}$$

see (2.11), and the distribution of $\mathbb{X}(t)$ can be written by summing up:

$$\begin{aligned} \mathbb{P}_0\{\mathbb{X}(t) \in \mathrm{d}x\} =& \exp(-\lambda_0 t)\delta_{\phi_0(t)}(\mathrm{d}x) + \mathscr{P}_0(t,x)\mathrm{d}x,\\ \mathbb{P}_1\{\mathbb{X}(t) \in \mathrm{d}x\} =& \exp(-\lambda_1 t)\delta_{\phi_1(t)}(\mathrm{d}x) + \mathscr{P}_1(t,x)\mathrm{d}x, \end{aligned}$$

where the regular part of the distribution corresponds to the sums

$$\mathscr{P}_0(t,x) = \sum_{n=1}^{\infty} p_0(t,x;n), \qquad \mathscr{P}_1(t,x) = \sum_{n=1}^{\infty} p_1(t,x;n). \tag{3.2}$$

When the process $\mathbb{X} = \mathbb{X}(t)$ starts at an arbitrary point y, the starting point is lost after the first state switch. Therefore, only the singular part changes: the distribution P_t of $\mathbb{X}(t)$ is given by the probability density functions

$$p_i(t,x \mid y) = \mathbb{P}_i\{\mathbb{X}(t) \in \mathrm{d}x \mid \mathbb{X}(0) = y\}/\mathrm{d}x, \qquad i \in \{0,1\},$$

where

$$\begin{aligned} p_0(t,x \mid y) =& \mathrm{e}^{-\lambda_0 t}\delta(x-\phi_0(t,y) + \mathscr{P}_0(t,x),\\ p_1(t,x \mid y) =& \mathrm{e}^{-\lambda_1 t}\delta(x-\phi_1(t,y)) + \mathscr{P}_1(t,x), \end{aligned} \tag{3.3}$$

see (2.3).

Since the particle returns to the origin after any state switch and then continues deterministically until the next reset, the distribution of the point $\mathbb{X}(t)$ is determined by the distribution of the time of the *last* switch. Exact formulae are given by the following theorem.

Theorem 3.1. *The joint distribution of $\mathbb{X}(t)$ and $N(t)$ is determined as follows.*
If $\varepsilon(0) = 0$, then

$$p_0(t,x;2n+1) = \frac{\lambda_0^{n+1}\lambda_1^n \mathrm{e}^{-\lambda_0 t}\left(t-\phi_1^{-1}(x)\right)^{2n}}{|a_1|(2n)!}\left[1-\frac{x}{\rho_1}\right]^{-1+\gamma_1^{-1}(\lambda_1-\lambda_0)} \times\Phi(n,2n+1;(\lambda_0-\lambda_1)(t-\phi_1^{-1}(x)))\cdot\mathbb{1}_{\{0<\phi_1^{-1}(x)<t\}}, \tag{3.4}$$

$$p_0(t,x;2n) = \frac{\lambda_0^n\lambda_1^n \mathrm{e}^{-\lambda_0 t}\left(t-\phi_0^{-1}(x)\right)^{2n-1}}{|a_0|(2n-1)!}\left[1-\frac{x}{\rho_0}\right]^{-1} \times\Phi(n,2n;(\lambda_0-\lambda_1)(t-\phi_0^{-1}(x)))\cdot\mathbb{1}_{\{0<\phi_0^{-1}(x)<t\}}, \tag{3.5}$$

If $\varepsilon(0) = 1$, then

$$p_1(t,x;2n) = \frac{\lambda_0^n\lambda_1^n \mathrm{e}^{-\lambda_1 t}\left(t-\phi_1^{-1}(x)\right)^{2n-1}}{|a_1|(2n-1)!}\left[1-\frac{x}{\rho_1}\right]^{-1} \times\Phi(n,2n;(\lambda_1-\lambda_0)(t-\phi_1^{-1}(x)))\cdot\mathbb{1}_{\{0<\phi_1^{-1}(x)<t\}}, \tag{3.6}$$

$$p_1(t,x;2n+1) = \frac{\lambda_0^n\lambda_1^{n+1} \mathrm{e}^{-\lambda_1 t}\left(t-\phi_0^{-1}(x)\right)^{2n}}{|a_0|(2n)!}\left[1-\frac{x}{\rho_0}\right]^{-1+\gamma_0^{-1}(\lambda_0-\lambda_1)} \times\Phi(n,2n+1;(\lambda_1-\lambda_0)(t-\phi_0^{-1}(x)))\cdot\mathbb{1}_{\{0<\phi_0^{-1}(x)<t\}}. \tag{3.7}$$

Here $\phi_0^{-1}(x)$ and $\phi_1^{-1}(x)$ are the inverse functions,

$$\phi_0^{-1}(x) = -\gamma_0^{-1}\log(1-x/\rho_0), \qquad \phi_1^{-1}(x) = -\gamma_1^{-1}\log(1-x/\rho_1),$$

and Φ is a confluent hypergeometric function.

Proof. Let $\{\tau_n\}_{n\geq 1}$ be a sequence of the successive reset times. Recall that the n-th switching time τ_n has a generalised Erlang distribution given by

$$\begin{aligned}\mathbb{P}_0\{\tau_n\in\mathrm{d}t\} =& \lambda_0^{(\times,n)}\frac{t^{n-1}}{(n-1)!}\mathrm{e}^{-\lambda_0 t}\Phi([n/2],n;(\lambda_0-\lambda_1)t)\mathrm{d}t,\\ \mathbb{P}_1\{\tau_n\in\mathrm{d}t\} =& \lambda_1^{(\times,n)}\frac{t^{n-1}}{(n-1)!}\mathrm{e}^{-\lambda_1 t}\Phi([n/2],n;(\lambda_1-\lambda_0)t)\mathrm{d}t,\end{aligned} \tag{3.8}$$

see [11, Remark 1.3] and [8]. Here $\lambda_0^{(\times,n)}$ is the product of successively alternating λ_0 and λ_1 (beginning with λ_0).

We present here a detailed proof of (3.4)-(3.5). The remaining formulae are proved similarly.

By conditioning on the *last* switching, using (3.1), we obtain the following integral relations,

$$p_0(t,x;2n+1) = \int_0^t \mathrm{e}^{-\lambda_1(t-s)}\delta(x-\phi_1(t-s))\mathbb{P}_0\{\tau_{2n+1}\in\mathrm{d}s\}, \tag{3.9}$$

$x>0$, $n\geq 0$, see Fig. 1.

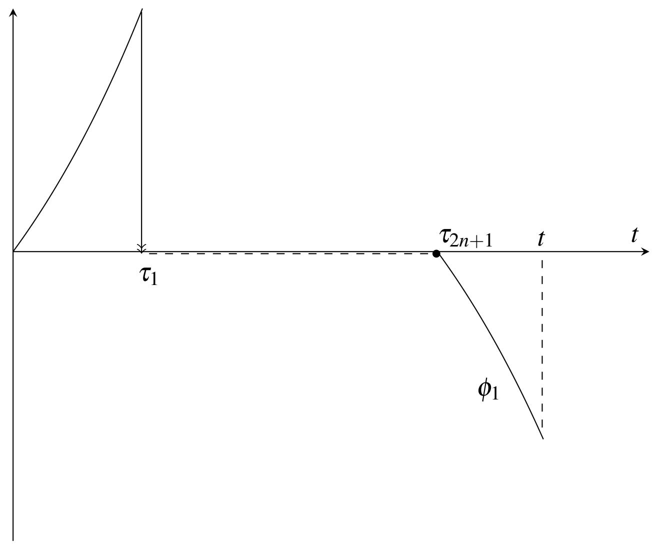

FIGURE 1. To formula (3.9).

By (3.9) and (3.8), we have

$$p_0(t,x;2n+1)=\frac{\lambda_0^{n+1}\lambda_1^n}{(2n)!}\int_0^t \mathrm{e}^{-\lambda_0 s-\lambda_1(t-s)}s^{2n}\Phi(n,2n+1;(\lambda_0-\lambda_1)s)\delta(x-\phi_1(t-s))\mathrm{d}s. \tag{3.10}$$

Apply the change of variables $y=x-\phi_1(t-s)$ instead of s, $0<s<t$, such that $s=t-\phi_1^{-1}(x-y)=t+\gamma_1^{-1}\log\left(1-\frac{x-y}{\rho_1}\right)$. Hence, $\mathrm{e}^{-\lambda_0 s-\lambda_1(t-s)}=\left(1-\frac{x-y}{\rho_1}\right)^{(\lambda_1-\lambda_0)\gamma_1^{-1}}\mathrm{e}^{-\lambda_0 t}$ and $\mathrm{d}s=a_1^{-1}\left[1-\dfrac{x-y}{\rho_1}\right]^{-1}\mathrm{d}y$. Since the integral (3.10) does not vanish only if $0<\phi_1^{-1}(x)<t$, we obtain

$$\begin{aligned}p_0(t,x;2n+1)=&\frac{\lambda_0^{n+1}\lambda_1^n\mathrm{e}^{-\lambda_0 t}\left(t-\phi_1^{-1}(x)\right)^{2n}}{|a_1|(2n)!}\times\\ &\times\left[1-\frac{x}{\rho_1}\right]^{-1+(\lambda_1-\lambda_0)\gamma_1^{-1}}\Phi(n,2n+1;(\lambda_0-\lambda_1)(t-\phi_1^{-1}(x)))\mathbb{1}_{\{0<\phi_1^{-1}(x)<t\}},\end{aligned}$$

which coincide with (3.4).

Similarly,

$$p_0(t,x;2n)=\frac{\lambda_0^n\lambda_1^n\mathrm{e}^{-\lambda_1 t}}{(2n-1)!}\int_0^t s^{2n-1}\Phi(n,2n;(\lambda_0-\lambda_1)s)\delta(x-\phi_0(t-s))\mathrm{d}s,$$

which, after the same change of variable, gives (3.5). □

The regular parts $\mathscr{P}_0(t,x)$, $\mathscr{P}_1(t,x)$ of the distribution follow by summing up, see (3.2). Notice that in the symmetric case these functions can be written in the closed form.

Corollary 3.2. *Let $\lambda_0=\lambda_1=\lambda$. Therefore,*

$$\begin{aligned}\mathscr{P}_0(t,x)=&\frac{\lambda}{2|a_0|}(1-x/\rho_0)^{-1+\lambda/\gamma_0}\left[1-\mathrm{e}^{-2\lambda t}(1-x/\rho_0)^{-2\lambda/\gamma_0}\right]\mathbb{1}_{\{0<\phi_0^{-1}(x)<t\}}\\&+\frac{\lambda}{2|a_1|}(1-x/\rho_1)^{-1+\lambda/\gamma_1}\left[1+\mathrm{e}^{-2\lambda t}(1-x/\rho_1)^{-2\lambda/\gamma_1}\right]\mathbb{1}_{\{0<\phi_1^{-1}(x)<t\}},\\\mathscr{P}_1(t,x)=&\frac{\lambda}{2|a_0|}(1-x/\rho_0)^{-1+\lambda/\gamma_0}\left[1+\mathrm{e}^{-2\lambda t}(1-x/\rho_0)^{-2\lambda/\gamma_0}\right]\mathbb{1}_{\{0<\phi_0^{-1}(x)<t\}}\\&+\frac{\lambda}{2|a_1|}(1-x/\rho_1)^{-1+\lambda/\gamma_1}\left[1-\mathrm{e}^{-2\lambda t}(1-x/\rho_1)^{-2\lambda/\gamma_1}\right]\mathbb{1}_{\{0<\phi_1^{-1}(x)<t\}}.\end{aligned}\tag{3.11}$$

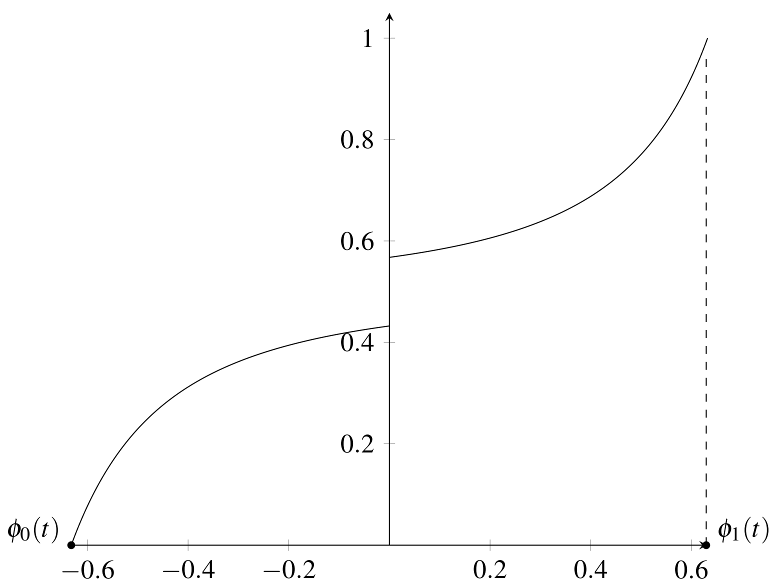

FIGURE 2. Pure attraction: the probability density function $\mathscr{P}_0(t,x)$, $\phi_0(t)<x<\phi_1(t)$, with $\lambda=1$, $\gamma_0=\gamma_1=+1$; $\rho_0=-1,\rho_1=1$; $t=1$; see (3.11).

Formulae (3.4)-(3.6) as well as (3.11) have different meanings with different signs of the parameters.

Namely, if both velocities γ are positive (the case of pure attraction), then (after the first switching of state) the distribution of $\mathbb{X}(t)$ is supported on the compact $[\phi_0(t),\ \phi_1(t)]\subset[\rho_0,\ \rho_1]$. More precisely, in this case the probability density functions $p_0(t,x;2n+1)$ and $p_1(t,x;2n)$ are supported on the positive half-interval, $[0,\phi_1(t))$, and the functions $p_0(t,x;2n)$ and $p_1(t,x;2n+1)$ are supported on the negative half-interval, $(\phi_0(t),0]$.

Compare also Fig. 2 and Fig. 3, which presents two different cases of pure attraction.

Conversely, if both γ are negative, then the distribution of $\mathbb{X}(t)$ is located in the interval $[\phi_1(t),\ \phi_0(t)]$, $t>0$, which increases with t logarithmically. In this case, $p_0(t,x;2n+1)$ and $p_1(t,x;2n)$ are supported on $(\phi_1(t),0]$, while $p_0(t,x;2n)$ and $p_1(t,x;2n+1)$ are defined for $x\in[0,\phi_0(t))$, see Fig. 4.

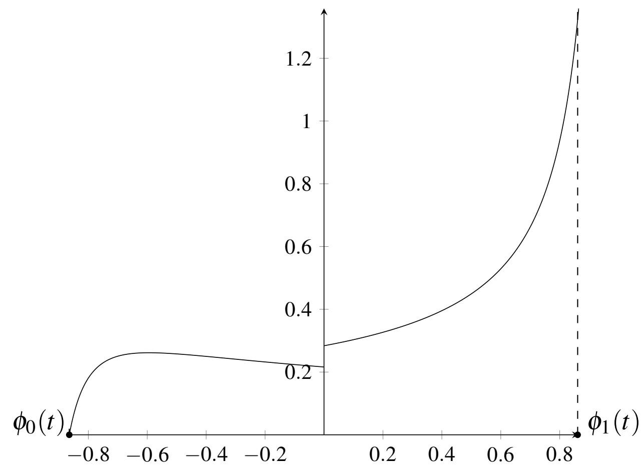

FIGURE 3. Pure attraction: the probability density function $\mathscr{P}_0(t,x),\ \phi_0(t) < x < \phi_1(t)$, with $\lambda = 1,\ \gamma_0 = \gamma_1 = 2;\ \rho_0 = -1, \rho_1 = 1;\ t = 1$.

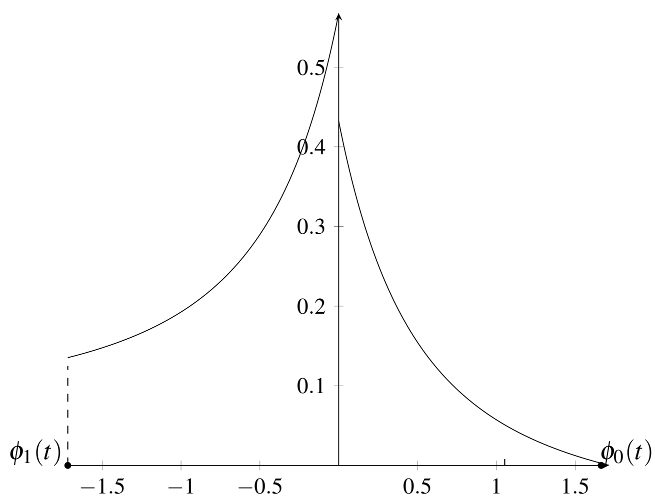

FIGURE 4. Pure reflection: the probability density function $\mathscr{P}_0(t,x),\ \phi_0(t) < x < \phi_1(t)$, with $\lambda = 1,\ \gamma_0 = \gamma_1 = -1;\ \rho_0 = -1, \rho_1 = 1;\ t = 1$;

In a mixed case, say if $\gamma_0 < 0 < \gamma_1$, then $\operatorname{supp}\mathscr{P}_0 \subset [0,\infty)$ and $\operatorname{supp}\mathscr{P}_1 \subset [0,\infty)$; if $\gamma_0 > 0 > \gamma_1$, then the distribution is supported on negative semi-line. Fig. 5 and Fig. 6 correspond to two different probability density functions in the mixed case.

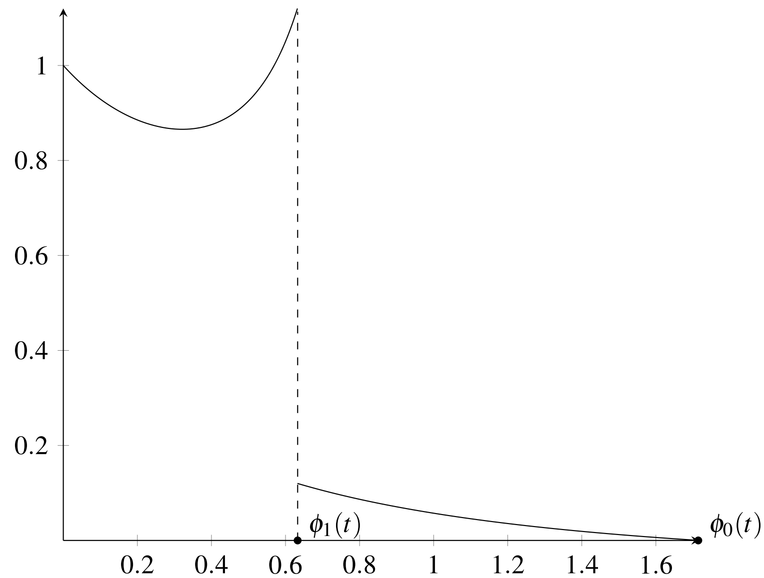

FIGURE 5. The probability density function $\mathscr{P}_0(t,x),\ 0<x<\phi_0(t)$, with $\lambda=1,\ \rho_0=-1, \rho_1=1$; $t=1$, in the mixed case $\gamma_0=-1,\gamma_1=1$; see (3.11).

The integration in (3.11) confirms that the distribution of $\mathbb{X}(t)$ is proper. Indeed,

$$\int_{-\infty}^{\infty} p_0(t,x\mid y)\mathrm{d}x = \mathrm{e}^{-\lambda t} + \int_{-\infty}^{\infty} \mathscr{P}_0(t,x)\mathrm{d}x$$

$$= \mathrm{e}^{-\lambda t} + \frac{\lambda}{2|a_0|}\int\limits_{x:\,0<\phi_0^{-1}(x)<t}\left[(1-x/\rho_0)^{-1+\lambda/\gamma_0} - \mathrm{e}^{-2\lambda t}(1-x/\rho_0)^{-1-\lambda/\gamma_0}\right]\mathrm{d}x$$

$$+ \frac{\lambda}{2|a_1|}\int\limits_{x:\,0<\phi_1^{-1}(x)<t}\left[(1-x/\rho_1)^{-1+\lambda/\gamma_1} + \mathrm{e}^{-2\lambda t}(1-x/\rho_1)^{-1-\lambda/\gamma_1}\right]\mathrm{d}x.$$

By applying the changes of variables, $\tau=\phi_0^{-1}(x)$ (in the first integral) and $\tau=\phi_1^{-1}(x)$ (in the second integral), we obtain

$$\int_{-\infty}^{\infty} p_0(t,x\mid y)\mathrm{d}x$$

$$= \mathrm{e}^{-\lambda t} + \frac{1}{2}\left[(1-\mathrm{e}^{-\lambda t}) - (\mathrm{e}^{-\lambda t}-\mathrm{e}^{-2\lambda t})\right] + \frac{1}{2}\left[(1-\mathrm{e}^{-\lambda t}) + (\mathrm{e}^{-\lambda t}-\mathrm{e}^{-2\lambda t})\right] \equiv 1.$$

Similarly, $\int_{-\infty}^{\infty} p_1(t,x\mid y)\mathrm{d}x \equiv 1$.

In the case of pure attraction, i.e. if $\gamma_0,\gamma_1>0$, the support of the distribution of $\mathbb{X}(t)$ leis in the compact interval $[\rho_0,\ \rho_1]$, and $\mathbb{X}(t)$ converges in distribution as $t\to\infty$. Let us give a detailed proof in the symmetric case.

Theorem 3.3. *Let* $\gamma_0,\gamma_1>0$*.*

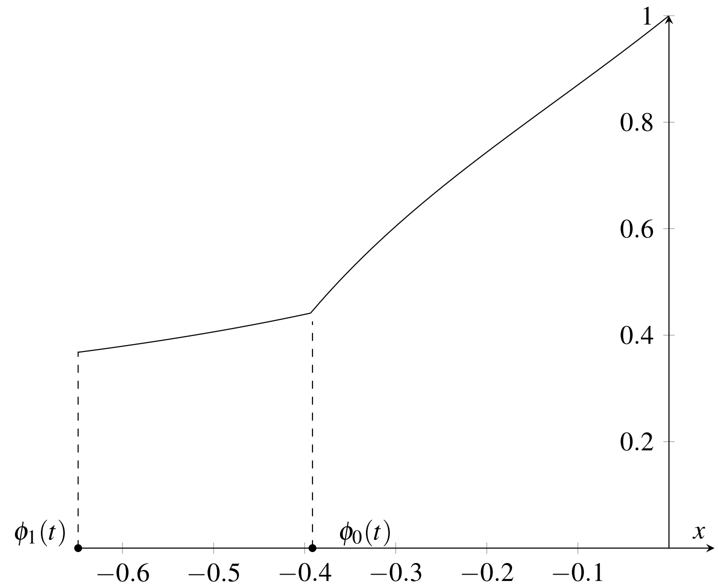

FIGURE 6. The probability density function $\mathscr{P}_0(t,x)$, $\phi_1(t) < x < 0$, with $\lambda = 1$, $\rho_0 = -1, \rho_1 = 1$; $t = 0.5$, in the mixed case $\gamma_0 = 1, \gamma_1 = -1$; see (3.11).

In the symmetric case, $\lambda_0 = \lambda_1 = \lambda$, *the process* $\mathbb{X}(t)$ *weakly converges*,

$$\mathbb{X}(t) \xrightarrow{D} \mathbb{X}^*, \qquad t \to \infty,$$

where the distribution of $\mathbb{X}^*$ *is determined by the probability density function* $\mathscr{P}^*(x) = \mathbb{P}\{\mathbb{X}^* \in \mathrm{d}x\}/\mathrm{d}x$, *which is given by*

$$\mathscr{P}^*(x) = \frac{\lambda}{2|a_1|}(1 - x/\rho_1)^{-1+\lambda/\gamma_1}\mathbb{1}_{\{\phi_1^{-1}(x)>0\}} + \frac{\lambda}{2|a_0|}(1 - x/\rho_0)^{-1+\lambda/\gamma_0}\mathbb{1}_{\{\phi_0^{-1}(x)>0\}}. \tag{3.12}$$

Proof. Note that the pointwise convergence to the distribution (3.12) follows directly from (3.11) and (3.3). We need to check if the family of measures $\{P_t\}_{t\geq 0}$ is tight.

As before, the sense and the proofs are different for different combinations of parameter signs. Let first $\rho_0 < 0 < \rho_1$. Hence $a_0 < 0 < a_1$. Note that due to (2.3), in this case we have

$$\lim_{t\to\infty} \phi_0(t) = \rho_0, \qquad \lim_{t\to\infty} \phi_1(t) = \rho_1.$$

In this case, by (3.11) the distribution $\{P_t\}$ are supported on the interval $[\phi_0(t), \phi_1(t)] \subset (\rho_0,\ \rho_1)$, see Fig. 2 and Fig. 3, and the probability density functions $\mathscr{P}_i(t,x)$ converge:

$$\begin{aligned}\mathscr{P}^*(x) &= \lim_{t\to\infty} \mathscr{P}_0(t,x) = \lim_{t\to\infty} \mathscr{P}_1(t,x)\\ &= \frac{\lambda}{2a_1}(1 - x/\rho_1)^{-1+\lambda/\gamma_1}\mathbb{1}_{\{0<x<\rho_1\}} - \frac{\lambda}{2a_0}(1 - x/\rho_0)^{-1+\lambda/\gamma_0}\mathbb{1}_{\{\rho_0<x<0\}},\end{aligned}$$

$a_0 < 0 < a_1$.

Further, for any $\delta > 0,\ \delta < 1$, by virtue of explicit representation (3.11) for $t \geq -\max(1/\gamma_0,\ 1/\gamma_1)\cdot\log\delta$, we have:

$$\begin{aligned}&\mathbb{P}_i\{\rho_0(1+\delta) < \mathbb{X}(t) < \rho_1(1-\delta)\}\\ &= \frac{\lambda}{2a_1}\int_0^{\rho_1(1-\delta)}(1-x/\rho_1)^{-1+\lambda/\rho_1}\mathrm{d}x - \frac{\lambda}{2a_0}\int_{\rho_0(1-\delta)}^0(1-x/\rho_0)^{-1+\lambda/\rho_0}\mathrm{d}x\\ &= \frac{1}{2}\left[(1-\delta^{\lambda/\gamma_1})+(1-\delta^{\lambda/\gamma_0})\right] = 1-\frac{1}{2}(\delta^{\lambda/\gamma_0}+\delta^{\lambda/\gamma_1}).\end{aligned}$$

Therefore, the family of distributions of $\mathbb{X}(t),\ t>0$ is compact on $\mathscr{D}[\rho_0,\ \rho_1]$ that is, for any $\varepsilon>0$, exists $\delta=\delta(\varepsilon)$ such that for $t\geq -\max(1/\gamma_0,\ 1/\gamma_1)\cdot\log\delta$

$$\mathbb{P}_0\{\rho_0+\delta < \mathbb{X}(t) < \rho_1-\delta\} > 1-\varepsilon$$

uniformly in $t, t\geq -\max(1/\gamma_0,\ 1/\gamma_1)\cdot\log\delta$.

The result follows from Prohorov's theorem, see [1, Theorem 13.3]. □

4. First crossing time

Let

$$\mathscr{T}(x) = \inf\{t>0:\ X(t)=x\}$$

be the moment when the process $X=X(t)$ first crosses the threshold x. Define the Laplace transforms (moment generating functions of $\mathscr{T}(x)$) as

$$\ell_0(q,x) = \mathbb{E}_0[\exp(-q\mathscr{T}(x))], \qquad \ell_1(q,x) = \mathbb{E}_1[\exp(-q\mathscr{T}(x))]$$

We represent $\ell_0(q,x)$ and $\ell_1(q,x)$ explicitly using the following notations. Let

$$\kappa = \kappa(q) = \frac{\lambda_0\lambda_1}{(q+\lambda_0)(q+\lambda_1)}$$

and

$$\begin{aligned}G_0(q,w_0,w_1) &= \frac{w_0+\frac{\lambda_0}{q+\lambda_0}(1-w_0)w_1}{1-\kappa(q)(1-w_0)(1-w_1)},\\ G_1(q,w_0,w_1) &= \frac{w_1+\frac{\lambda_1}{q+\lambda_1}w_0(1-w_1)}{1-\kappa(q)(1-w_0)(1-w_1)},\\ F(q,w) = \frac{w}{1-\kappa(q)(1-w)} &= \frac{(q+\lambda_0)(q+\lambda_1)w}{q(q+\lambda_0+\lambda_1)+\lambda_0\lambda_1 w},\\ &0<w_0,w_1,w<1.\end{aligned} \tag{4.1}$$

Everywhere below we use the following expressions instead of w:

$$\begin{aligned}w_0 = w_0(q,x) &= \exp\left(-(q+\lambda_0)\phi_0^{-1}(x)\right) = (1-x/\rho_0)^{(q+\lambda_0)/\gamma_0}\,\mathbb{1}_{\{x/\rho_0<1\}},\\ w_1 = w_1(q,x) &= \exp\left(-(q+\lambda_1)\phi_1^{-1}(x)\right) = (1-x/\rho_1)^{(q+\lambda_1)/\gamma_1}\,\mathbb{1}_{\{x/\rho_1<1\}}.\end{aligned} \tag{4.2}$$

4.1. Only attraction or only reflection: $\gamma_0 \cdot \gamma_1 > 0$. Let both γ be positive first.

Theorem 4.1. *Let $\gamma_0, \gamma_1 > 0$.*
Functions $\ell_0(q,x)$ and $\ell_1(q,x)$ are given explicitly.

- *Let $\mathbb{X}(0) = 0 \in [\rho_0,\ \rho_1]$.*
  The following formulae hold:
$$\begin{aligned}\ell_0(q,x) =& F(q,w_0)\mathbb{1}_{\{\rho_0<x<0\}} + \frac{\lambda_0}{q+\lambda_0}F(q,w_1)\mathbb{1}_{\{0<x<\rho_1\}}\\ &\text{and}\\ \ell_1(q,x) =& \frac{\lambda_1}{q+\lambda_1}F(q,w_0)\mathbb{1}_{\{\rho_0<x<0\}} + F(q,w_1)\mathbb{1}_{\{0<x<\rho_1\}}.\end{aligned} \tag{4.3}$$
- *Let $\mathbb{X}(0) = 0 < \rho_0 < \rho_1$.The following formulae hold*:
$$\begin{aligned}\ell_0(q,x) =& G_0(q,w_0,w_1)\mathbb{1}_{\{0<x<\rho_0\}} + \frac{\lambda_0}{q+\lambda_0}F(q,w_1)\mathbb{1}_{\{\rho_0<x<\rho_1\}}\\ &\text{and}\\ \ell_1(q,x) =& G_1(q,w_0,w_1)\mathbb{1}_{\{0<x<\rho_0\}} + F(q,w_1)\mathbb{1}_{\{\rho_0<x<\rho_1\}}.\end{aligned} \tag{4.4}$$
- *Let $\rho_0 < \rho_1 < 0 = \mathbb{X}(0)$. The following formulae hold*:
$$\begin{aligned}\ell_0(q,x) =& G_0(q,w_0,w_1)\mathbb{1}_{\{\rho_1<x<0\}} + F(q,w_1)\mathbb{1}_{\{\rho_0<x<\rho_1\}}\\ &\text{and}\\ \ell_1(q,x) =& G_1(q,w_0,w_1)\mathbb{1}_{\{\rho_1<x<0\}} + \frac{\lambda_1}{q+\lambda_1}F(q,w_1)\mathbb{1}_{\{\rho_0<x<\rho_1\}}.\end{aligned} \tag{4.5}$$

Here $w_0 = w_0(q,x)$ and $w_1 = w_1(q,x)$ are defined by (4.2).

Proof. The proof is based on coupled integral equations which are derived by conditioning on the *first* switching of the underlying process $\varepsilon = \varepsilon(t)$.

Let first $0 \in (\rho_0, \rho_1)$. In this case, if $x \notin (\rho_0,\ \rho_1)$, then the threshold x is never reached.

Let $\rho_0 < \mathbb{X}(0) = 0 < x < \rho_1$. Starting with the state 0 (the downwards movement), the path of $\mathbb{X}(t)$ does not reach the threshold $x, x > 0$, without resetting. Hence
$$\ell_0(q,x) = \int_0^\infty \lambda_0 \mathrm{e}^{-(q+\lambda_0)s}\ell_1(q,x)\mathrm{d}s = \frac{\lambda_0}{q+\lambda_0}\ell_1(q,x).$$
Otherwise, if the initial state is 1, then the threshold x can be reached without a reset, or, alternatively, after a switch that occurred before. Therefore,
$$\begin{aligned}\ell_1(q,x) &= \mathrm{e}^{-(q+\lambda_1)\phi_1^{-1}(x)} + \int_0^{\phi_1^{-1}(x)} \lambda_1 \mathrm{e}^{-(q+\lambda_1)s}\ell_0(q,x)\mathrm{d}s\\ &= w_1(q,x) + \frac{\lambda_1}{q+\lambda_1}\ell_0(q,x)(1-w_1(q,x)),\end{aligned}$$
where $w_1(q,x) = \exp(-(q+\lambda_1)\phi_1^{-1}(x)) = (1-x/\rho_1)^{(q+\lambda_1)/\gamma_1}$. Hence,
$$\ell_0(q,x) = \frac{\frac{\lambda_0}{q+\lambda_0}w_1(q,x)}{1-\kappa(q)(1-w_1(q,x))}, \qquad \ell_1(q,x) = \frac{w_1(q,x)}{1-\kappa(q)(1-w_1(q,x))},$$
which coincide with formulae (4.3) (with $0 < x < \rho_1$). The variant when the threshold is located below the starting point, $\rho_0 < x < 0 = \mathbb{X}(0) < \rho_1$, is analysed similarly.

Let the process $\mathbb{X}$ start below the interval $[\rho_0,\ \rho_1]$, and let the threshold x be between the initial point and both attracting levels: $0=\mathbb{X}(0)<x<\rho_0<\rho_1$. In this case, in the same manner as before, we obtain coupled equations of the form

$$\begin{aligned}\ell_0(q,x)=&w_0(q,x)+\frac{\lambda_0}{q+\lambda_0}\ell_1(q,x)(1-w_0(q,x)),\\ \ell_1(q,x)=&w_1(q,x)+\frac{\lambda_1}{q+\lambda_1}\ell_0(q,x)(1-w_1(q,x)),\end{aligned}$$

which gives the first option of (4.4).

Other variants of (4.4) and (4.5) are proved in a similar way. □

The expected first crossing time, $\mathbb{E}[\mathscr{T}(x)]$, can be obtained differentiating in (4.3), (4.4) and (4.5). Since $\kappa(0)=1$, by virtue of (4.1) we obtain

$$\begin{aligned}G_0(q,w_0(q,x),w_1(q,x))|_{q=0}\equiv&1,\quad G_1(q,w_0(q,x),w_1(q,x))|_{q=0}\equiv 1,\\ F(q,w(q,x))|_{q=0}\equiv&1,\quad w(q,x)|_{q=0}=(1-x/\rho)^{\lambda/\gamma};\\ \frac{\mathrm{d}}{\mathrm{d}q}F(q,w(q,x))|_{q=0}=&-\frac{\lambda_0+\lambda_1}{\lambda_0\lambda_1}\left[w(0,x)^{-1}-1\right],\\ \frac{\mathrm{d}}{\mathrm{d}q}G_0(q,w_0(q,x),w_1(q,x))|_{q=0}=&-Z^{-1}(1-w_0(0,x))\left[\lambda_0^{-1}+\lambda_1^{-1}(1-w_1(0,x))\right],\\ \frac{\mathrm{d}}{\mathrm{d}q}G_1(q,w_0(q,x),w_1(q,x))|_{q=0}=&-Z^{-1}(1-w_1(0,x))\left[\lambda_1^{-1}+\lambda_0^{-1}(1-w_0(0,x))\right]\end{aligned}\tag{4.6}$$

with $w(q,x)=(1-x/\rho)^{(q+\lambda)/\gamma}$ and $Z=w_0(0,x)+w_1(0,x)-w_0(0,x)w_1(0,x)$.

The explicit formulae for the expected crossing time, $\mathbb{E}[\mathscr{T}(x)]$, are given below.

Corollary 4.2. *Let $\gamma_0,\gamma_1>0$ and $\lambda_0,\lambda_1>0$.*

The expected values of $\mathscr{T}(x)$ are given by the following explicit formulae.

- *Let $\mathbb{X}(0)=0\in[\rho_0,\rho_1]$. In this case*:

$$\begin{aligned}&\mathbb{E}[\mathscr{T}(x)\mid\varepsilon(0)=0]\\ &=\frac{\lambda_0+\lambda_1}{\lambda_0\lambda_1}\left[\left((1-x/\rho_0)^{-\lambda_0/\gamma_0}-1\right)\mathbb{1}_{\{\rho_0<x<0\}}\right.\\ &\qquad\left.+\left((1-x/\rho_1)^{-\lambda_1/\gamma_1}+\frac{\lambda_0}{\lambda_0+\lambda_1}\right)\mathbb{1}_{\{0<x<\rho_1\}}\right]\end{aligned}$$

and

$$\begin{aligned}&\mathbb{E}[\mathscr{T}(x)\mid\varepsilon(0)=1]\\ &=\frac{\lambda_0+\lambda_1}{\lambda_0\lambda_1}\left[\left((1-x/\rho_0)^{-\lambda_0/\gamma_0}+\frac{\lambda_1}{\lambda_0+\lambda_1}\right)\mathbb{1}_{\{\rho_0<x<0\}}\right.\\ &\qquad\left.+\left((1-x/\rho_1)^{-\lambda_1/\gamma_1}-1\right)\mathbb{1}_{\{0<x<\rho_1\}}\right].\end{aligned}\tag{4.7}$$

- *Let* $\mathbb{X}(0)=0<\rho_0<\rho_1$. *In this case*:

$$\begin{aligned}
&\mathbb{E}[\mathscr{T}(x)\mid\varepsilon(0)=0]\\
&\qquad=Z^{-1}(1-w_0(0,x))\left(\lambda_0^{-1}+\lambda_1^{-1}(1-w_1(0,x))\right)\mathbb{1}_{\{0<x<\rho_0\}}\\
&\qquad+\lambda_0^{-1}\left(1+(1+\lambda_0/\lambda_1)(w_1(0,x)^{-1}-1)\right)\mathbb{1}_{\{\rho_0<x<\rho_1\}}\\
&\quad\textit{and}\\
&\mathbb{E}[\mathscr{T}(x)\mid\varepsilon(0)=1]\\
&\qquad=Z^{-1}(1-w_1(0,x))\left(\lambda_1^{-1}+\lambda_0^{-1}(1-w_0(0,x))\right)\mathbb{1}_{\{0<x<\rho_0\}}\\
&\qquad+\frac{\lambda_0+\lambda_1}{\lambda_0\lambda_1}\left[w_1(0,x)^{-1}-1\right]\mathbb{1}_{\{\rho_0<x<\rho_1\}}.
\end{aligned}\tag{4.8}$$

- *Let* $\rho_0<\rho_1<0=\mathbb{X}(0)$. *In this case*:

$$\begin{aligned}
&\mathbb{E}[\mathscr{T}(x)\mid\varepsilon(0)=0]\\
&\qquad=Z^{-1}(1-w_0(0,x))\left(\lambda_0^{-1}+\lambda_1^{-1}(1-w_1(0,x))\right)\mathbb{1}_{\{\rho_1<x<0\}}\\
&\qquad+\frac{\lambda_0+\lambda_1}{\lambda_0\lambda_1}\left[w_1(0,x)^{-1}-1\right]\mathbb{1}_{\{\rho_0<x<\rho_1\}}\\
&\quad\textit{and}\\
&\mathbb{E}[\mathscr{T}(x)\mid\varepsilon(0)=1]\\
&\qquad=Z^{-1}(1-w_1(0,x))\left(\lambda_1^{-1}+\lambda_0^{-1}(1-w_0(0,x))\right)\mathbb{1}_{\{\rho_1<x<0\}}\\
&\qquad+\lambda_1^{-1}\left(1+(1+\lambda_0/\lambda_1)(w_1(0,x)^{-1}-1)\right)\mathbb{1}_{\{\rho_0<x<\rho_1\}}.
\end{aligned}\tag{4.9}$$

Proof. To find $\mathbb{E}[\mathscr{T}(x)\mid\varepsilon(0)=0]$ and $\mathbb{E}[\mathscr{T}(x)\mid\varepsilon(0)=1]$ in the case $\rho_0<0<\rho_1$ we use (4.3),

$$\begin{aligned}
&\mathbb{E}[\mathscr{T}(x)\mid\varepsilon(0)=0]\\
&=-\frac{\mathrm{d}}{\mathrm{d}q}F(q,w_0(q))|_{q=0}\mathbb{1}_{\{\rho_0<x<0\}}+\left(\lambda_0^{-1}-\frac{\mathrm{d}}{\mathrm{d}q}F(q,w_1(q))|_{q=0}\right)\mathbb{1}_{\{0<x<\rho_1\}}.
\end{aligned}$$

By virtue of (4.6), this gives the formula (4.3) for $\mathbb{E}[\mathscr{T}(x)\mid\varepsilon(0)=0]$.

Next, by virtue of (4.4)

$$\begin{aligned}
&\mathbb{E}[\mathscr{T}(x)\mid\varepsilon(0)=0]\\
&=-\frac{\mathrm{d}}{\mathrm{d}q}G_0(q,w_0(q,x),w_1(q,x))|_{q=0}\mathbb{1}_{\{0<x<\rho_0\}}\\
&\qquad+\left(\lambda_0^{-1}-\frac{\mathrm{d}}{\mathrm{d}q}F(q,w_1(q,x))|_{q=0}\right)\mathbb{1}_{\{\rho_0<x<\rho_1\}},
\end{aligned}$$

which gives the first option of (4.8).

Other variants are proved similarly. □

Remark 4.3. Formulae (4.6) are verified with $\lambda_0,\lambda_1>0$.

If $\lambda_0=0$, then, by definition (4.1), $G_0(q,w_0,w_1)=w_0$. Therefore

$$\frac{\mathrm{d}}{\mathrm{d}q}G_0(q,w_0(q,x),w_1(q,x))|_{q=0}=\frac{\mathrm{d}}{\mathrm{d}q}w_0(q,x)|_{q=0}=\log(1-x/\rho_0)\mathbb{1}_{\{x/\rho_0<1\}}.\tag{4.10}$$

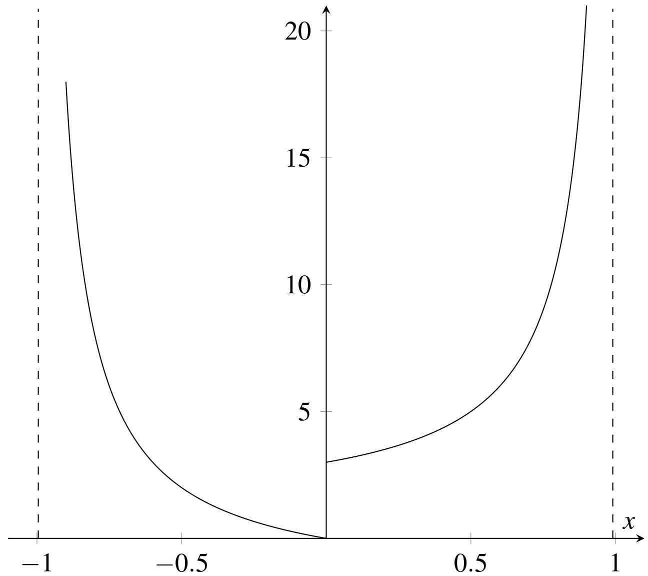

FIGURE 7. The expectation with $\mathbb{E}[\mathscr{T}(x) \mid \varepsilon(0)=0]$, $\rho_0 < x < \rho_1$, with $\lambda = 1$, $\rho_0 = -1, \rho_1 = 1$ in the attractive case $\gamma_0 = 1, \gamma_1 = 1$; see (4.7).

Similarly, if $\lambda_1 = 0$, then $G_1(q, w_0, w_1)|_{q=0} = w_1$, and

$$\frac{\mathrm{d}}{\mathrm{d}q} G_1(q, w_0(q,x), w_1(q,x))|_{q=0} = \frac{\mathrm{d}}{\mathrm{d}q} w_1(q,x)|_{q=0} = \log(1 - x/\rho_1)\mathbb{1}_{\{x/\rho_1<1\}}. \quad (4.11)$$

Further, if $\lambda_0 \cdot \lambda_1 = 0$, then $\kappa = 0$, $F(q, w(q,x)) = w(q,x)$ and

$$\frac{\mathrm{d}}{\mathrm{d}q} F(q, w(q,x))|_{q=0} = \frac{\mathrm{d}}{\mathrm{d}q} w(q,x)|_{q=0} = w(0,x)\log(1 - x/\rho). \quad (4.12)$$

Formulae (4.10)-(4.12) can be used to simplify previous results for the case of zero switching intensities.

The case of both negative γ looks simpler.

Theorem 4.4. *Let $\gamma_0, \gamma_1 < 0$. In this case functions ℓ_0 and ℓ_1 are defined as follows.*

- *Let $\rho_0 < 0 = \mathbb{X}(0) < \rho_1$. Then*

$$\begin{aligned} \ell_0(q,x) =& \frac{\lambda_0}{q+\lambda_0} F(q, w_1)\mathbb{1}_{\{x<0\}} + F(q, w_0)\mathbb{1}_{\{x>0\}}, \\ &\textit{and} \\ \ell_1(q,x) =& F(q, w_1)\mathbb{1}_{\{x<0\}} + \frac{\lambda_1}{q+\lambda_1} F(q, w_0)\mathbb{1}_{\{x>0\}}. \end{aligned} \quad (4.13)$$

- *Let $\mathbb{X}(0) = 0 < \rho_0 < \rho_1$. Therefore,*

$$\ell_0(q,x) = G_0(q, w_0, w_1)\mathbb{1}_{\{x<0\}} \quad \textit{and} \quad \ell_1(q,x) = G_1(q, w_0, w_1)\mathbb{1}_{\{x<0\}}. \quad (4.14)$$

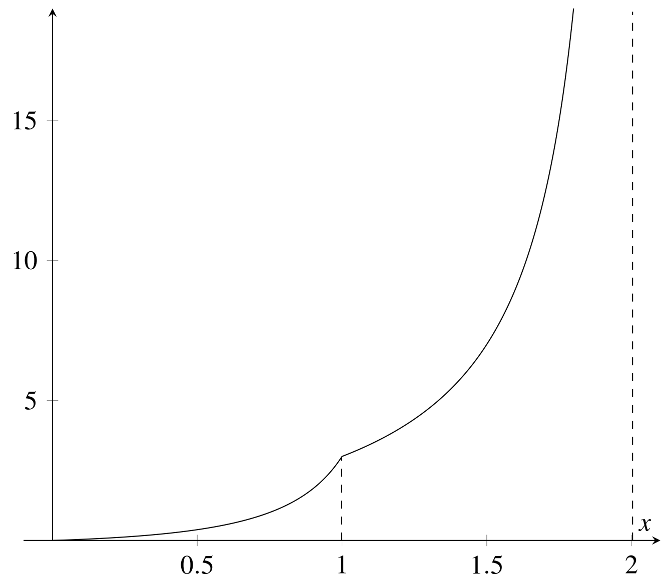

FIGURE 8. The expectation with $\mathbb{E}[\mathscr{T}(x) \mid \varepsilon(0) = 0]$, $0 < x < \rho_1$, with $\lambda = 1,\ \rho_0 = 1, \rho_1 = 2$, in the attractive case $\gamma_0 = 1, \gamma_1 = 1$; see (4.8).

- *Let* $\rho_0 < \rho_1 < 0 = \mathbb{X}(0)$. *Therefore,*

$$\ell_0(q,x) = G_0(q,w_0,w_1)\mathbb{1}_{\{x>0\}} \qquad \text{and} \qquad \ell_1(q,x) = G_1(q,w_0,w_1)\mathbb{1}_{\{x>0\}}. \tag{4.15}$$

Here $w_0 = w_0(q,x)$ *and* $w_1 = w_1(q,x)$ *are defined by* (4.2).

Proof. To prove (4.13)-(4.15) it is sufficient to repeat main steps of the proof of Theorem 4.1. □

The expectations $\mathbb{E}[\mathscr{T}(x)]$ can be written explicitly in the following form.

Corollary 4.5. • *Let* $\rho_0 < 0 = \mathbb{X}(0) < \rho_1$. *Then*

$$\mathbb{E}[\mathscr{T}(x) \mid \varepsilon(0) = 0] = \left[\lambda_0^{-1} + \frac{\lambda_0+\lambda_1}{\lambda_0\lambda_1}\left(w_1(0,x)^{-1} - 1\right)\right]\mathbb{1}_{\{x<0\}} + \frac{\lambda_0+\lambda_1}{\lambda_0\lambda_1}\left(w_0(0,x)^{-1} - 1\right)\mathbb{1}_{\{x>0\}}$$

and

$$\mathbb{E}[\mathscr{T}(x) \mid \varepsilon(0) = 1] = \frac{\lambda_0+\lambda_1}{\lambda_0\lambda_1}\left(w_1(0,x)^{-1} - 1\right)\mathbb{1}_{\{x<0\}} + \left[\lambda_1^{-1} + \frac{\lambda_0+\lambda_1}{\lambda_0\lambda_1}\left(w_0(0,x)^{-1} - 1\right)\right]\mathbb{1}_{\{x>0\}}. \tag{4.16}$$

- *Let* $\mathbb{X}(0) = 0 < \rho_0 < \rho_1$. *Therefore,*

$$\begin{aligned} \mathbb{E}[\mathscr{T}(x) \mid \varepsilon(0) = 0] &= Z^{-1}(1 - w_0(0,x))\left(\lambda_0^{-1} + \lambda_1^{-1}(1 - w_1(0,x))\right)\mathbb{1}_{\{x<0\}}, \\ \mathbb{E}[\mathscr{T}(x) \mid \varepsilon(0) = 1] &= Z^{-1}(1 - w_1(0,x))\left(\lambda_1^{-1} + \lambda_0^{-1}(1 - w_0(0,x))\right)\mathbb{1}_{\{x<0\}}. \end{aligned} \tag{4.17}$$

- *Let* $\rho_0 < \rho_1 < 0 = \mathbb{X}(0)$. *Therefore,*

$$\begin{aligned}\mathbb{E}[\mathscr{T}(x) \mid \varepsilon(0)=0] &= Z^{-1}(1-w_0(0,x))\left(\lambda_0^{-1}+\lambda_1^{-1}(1-w_1(0,x))\right)\mathbb{1}_{\{x>0\}},\\ \mathbb{E}[\mathscr{T}(x) \mid \varepsilon(0)=1] &= Z^{-1}(1-w_1(0,x))\left(\lambda_1^{-1}+\lambda_0^{-1}(1-w_0(0,x))\right)\mathbb{1}_{\{x>0\}}.\end{aligned} \tag{4.18}$$

4.2. Mixed case: attraction/reflection, $\gamma_0 \cdot \gamma_1 < 0$. Let γ_0 and γ_1 be of opposite signs. Assume for definiteness that $\gamma_0 > 0 > \gamma_1$.

Theorem 4.6. *Let $\gamma_0 > 0 > \gamma_1$. We have the following formulae.*

- *Let* $\rho_0 < 0 < \rho_1$.

$$\begin{aligned}\ell_0(q,x) =& \frac{\lambda_0}{q+\lambda_0}F(w_1)\mathbb{1}_{\{x<\rho_0\}} + G_0(w_0,w_1)\mathbb{1}_{\{\rho_0<x<0\}},\\ &\textit{and}\\ \ell_1(q,x) =& F(w_1)\mathbb{1}_{\{x<\rho_0\}} + G_1(w_0,w_1)\mathbb{1}_{\{\rho_0<x<0\}}.\end{aligned} \tag{4.19}$$

- *Let* $0 < \rho_0 < \rho_1$. *Therefore,*

$$\ell_0(q,x) = G_0(w_0,w_1)\mathbb{1}_{\{x<0\}} \qquad \textit{and} \qquad \ell_1(q,x) = G_1(w_0,w_1)\mathbb{1}_{\{x<0\}}. \tag{4.20}$$

- *Let* $\rho_0 < \rho_1 < 0$. *Therefore,*

$$\ell_0(q,x) = G_0(w_0,w_1)\mathbb{1}_{\{x>0\}} \qquad \textit{and} \qquad \ell_1(q,x) = G_1(w_0,w_1)\mathbb{1}_{\{x>0\}}. \tag{4.21}$$

Here $w_0 = w_0(q,x)$ and $w_1 = w_1(q,x)$ are defined by (4.2)*.*

The proof of this theorem is similar to previous proofs.

In the case $\gamma_0 < 0 < \gamma_1$ the formulae for ℓ_0 and ℓ_1 are symmetric to (4.19)-(4.21).

The case of zero switching intensities can be analysed using corresponding simplification, see Remark 4.3.

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