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TRANSMUTED WEIBULL GEOMETRIC DISTRIBUTION

SIMI SEBASTIAN, JANE A. LUKE, AND BINUMON JOSEPH*

Abstract. In this paper we developed a discrete distribution called transmuted Weibull geometric distribution, as a discrete analogue of transmuted Weibull distribution. The distribution is developed as a member of T-X family of distributions and various distributional properties of the model are studied.This distribution has increasing, decreasing and constant hazard rate and hence suitable for modelling different types of real data. Estimation of the parameters of the distributions are obtained using MLE method and relevance of the model is established by fitting to a real data set.

1. Introduction

Count data arises in many real life situations. Usually, reliability study assumes that the lifetime data are continuous and exponential and Weibull related distributions are used for modelling such data. But, sometimes it is impossible or inconvenient to measure the system life length in a continuous scale. For example,(i) an equipment operates in cycles and observation is the number of cycles prior to fail (ii) Machine is monitored only once per period and observation is the number of time periods successfully completed before failure. In many situations, life time data are truncated and hence system life time data is a discrete random variable. Usually standard discrete distributions like geometric and negative binomial are used to model lifetime data. But, more suitable discrete distributions have to be developed to model various types of real data.

Discretization of existing continuous distribution is a major research area for the last decade.Various techniques for generating families of discrete distribution have been proposed in literature. Many of the existing discrete life time distributions are derived from usual continuous distributions to handle different types of reliability data. These discretized distributions have similar properties to that of their continuous counterparts. Commonly used discrete distributions are geometric distribution, the discrete analogue of exponential distribution and negative binomial, the discrete analogue of gamma distribution. Nakagawa and Osaki (1975) proposed discrete Weibull as the discrete life time distribution analogous to the continuous Weibulll distribution. Discrete additive Weibull and discrete modified Weibull are used in life time modelling, which are discrete analogues of various modifications of Weibull distribution.Some basic results of discrete reliability distributions are studied by Salvia and Bollinger (1982) and Salvia (1996). Recently,

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Oseghale et.al (2022) developed exponentiated transmuted Weibull distribution which found application mostly in reliability analysis especially for data that are non-monotone and bimodal.

1.1. Transmuted Weibull Distribution.

A new generalization of Weibull distribution called , Transmuted Weibull distribution is introduced by Aryal and Tsokos (2011) using transmutation method. In this method, a rank transmutation map is used to link the cdf of one distribution with the quantile function of another (Shaw and Buckley, 2007).

A random variable X is said to have transmuted distribution if its cdf $G(x)$ is given by

$$G(x) = (1+\lambda)\, F(x) - \lambda[F(x)]^2, \;\; |\lambda| \leq 1 \tag{1.1}$$

where $F(x)$ is the cdf of the base distribution. When $\lambda = 0$, (**??**) becomes the cdf of the base distribution.

Adding parameters to a well established distribution gives more flexibility to model various types of data. Transmutation mapping is a novel technique for introducing skewness and kurtosis to the parametric family by adding a new shape parameter to the base distribution. Some simple transmutation mappings are applied on ranks to produce a modulation such as to introduce skewness and kurtosis in an existing model.Transmutation mapping can be applied to any base distribution without some of the difficulties that arise with the Gram-Charlier method. More positive features of transmutation mapping is given in Shaw and Buckley(2007). Some skewed transmuted distributions are skew-normal, skew - uniform and skew- exponential.

A random variable following Weibull distribution has cdf,

$$F(x) = 1 - e^{-\left(\frac{x}{\sigma}\right)^{\eta}} \text{ where } \eta > 0 \text{ and } \sigma > 0$$

Then the cdf of a transmuted Weibull distribution is obtained using (**??**) as,

$$G(x) = \left(1 - e^{-\left(\frac{x}{\sigma}\right)^{\eta}}\right)\left(1 + \lambda e^{-\left(\frac{x}{\sigma}\right)^{\eta}}\right), \;\; x > 0$$

Then the pdf of a transmuted Weibull distribution is,

$$f(x) = \left(\frac{\eta}{\sigma}\right)\left(\frac{x}{\sigma}\right)^{\eta-1} e^{-\left(\frac{x}{\sigma}\right)^{\eta}} \left(1 - \lambda + 2\lambda e^{-\left(\frac{x}{\sigma}\right)^{\eta}}\right)$$

$\eta > 0, \sigma > 0, |\lambda| \leq 1$ are the parameters of the model.

Transmuted Weibull distribution (TWD) is an extended model to analyse more complex data and it generalizes some of the widely used distributions. The Weibull distribution is clearly a special case for $\lambda = 0$. When $\lambda = \eta = 1$, the resulting distribution is exponential and when $\eta = 1$, it reduces to transmuted exponential

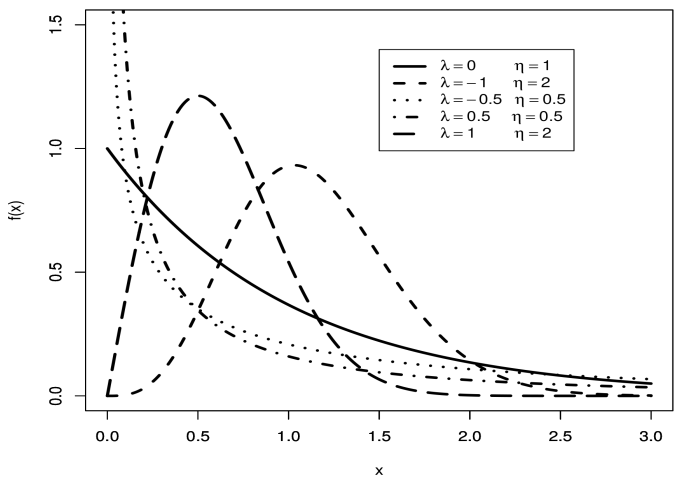


FIGURE 1. The graph of pdf of transmuted Weibull distribution for $\sigma = 1$ and different values of λ and η.

distribution.
The hazard rate function of TWD is,

$$h(t) = \left(\frac{\eta}{\sigma}\right)\left(\frac{t}{\sigma}\right)^{\eta-1}\left(\frac{1-\lambda+2\lambda e^{-\left(\frac{t}{\sigma}\right)^{\eta}}}{1-\lambda+\lambda e^{-\left(\frac{t}{\sigma}\right)^{\eta}}}\right) \text{ where } \eta > 0, \sigma > 0, |\lambda| \leq 1.$$

This distribution has an increasing, decreasing and constant hazard rate. Hence it is a suitable model to different types of life time data.

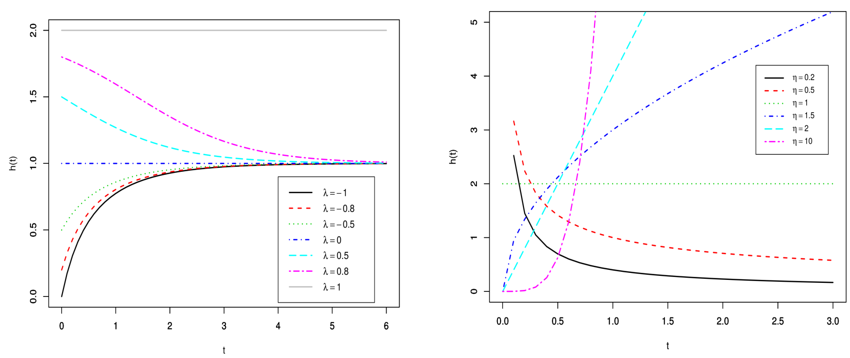
1.2. Discretization as T-geometric Family.

Alzaatreh et al. (2012) proposed a technique to generate new families of discrete distributions. Let $F(x)$ be the cdf of any discrete random variable X and let $r(t)$ be the pdf of a continuous random variable T, defined over $(0, \infty)$. Then the cdf of a family of discrete distributions called T-X family of distributions, derived from the non-negative continuous random variable T is given by

$$G(x) = \int_0^{-\log(1-F(x))} r(t)dt = R\left[-\log(1-F(x))\right]$$

where $R(t)$ is the cdf of the random variable T. Then, the pmf of the family of discrete distributions can be written as

$$g(x) = G(x) - G(x-1) = R\left[-\log(1-F(x))\right] - R\left[-\log(1-F(x-1))\right]$$



(a) (b)

FIGURE 2. Hazard rate function of transmuted Weibull distribution for different values of parameters

Particularly, if X follows a geometric distribution with parameter p (probability of failure), then a discrete family of distributions called T-geometric family of distributions corresponding to the discrete analogue of any continuous distribution defined over $(0, \infty)$ is given by the pmf

$$g(x) = R\left[c(x+1)\right] - R\left[c(x)\right], \ x = 0, 1, 2, \ldots \text{ where } c = -\log p > 0 \tag{1.2}$$

It can be shown that, if the distribution of a non negative continuous random variable T has a reversed J-shape, then the T-geometric distribution has a reversed J-shape (Alzaatreh et al. , 2012). Also, if T is unimodal, then the T-geometric family of distributions is also unimodal. Some of the discrete distributions analysed in this method are discrete Pareto, discrete Webull, discrete Burr and exponentiated exponential geometric distribution (Alzaatreh et al. , 2012) .

The present paper is organized as follows. In section 2, a discrete version of transmuted Weibull distribution is introduced. Important properties of the distribution are studied in section 3. Estimation of the parameters of the distribution using maximum likelihood method is addressed in section 4. In section 5, a real data set is used to explore the application of proposed distribution.

2. Transmuted Weibull Geometric Distribution

In this section we develop a discrete version of transmuted Weibull distribution as member of T-X family of distributions. T-geometric family generates the discrete analogue of any continuous random variable. The new distribution is developed as a member of T-geometric family and is called transmuted Weibull geometric distribution, denoted by TWG distribution.

Let f(x) be any measurable function of x and $W[f(x)]$ be a measurable function defined on (0,1). Alzaatreh et al. (2012) gave several choices of $f(x) = \lambda$ such that $W[\lambda]$ including $-log(1-\lambda)$, $\frac{\lambda}{1-\lambda}$, $log\frac{\lambda}{1-\lambda}$ and $log(-log\lambda)$.

Here we take $W[f(x)] = -\log(1-F(x))$, where $F(x)$ is the *cdf* of geometric distribution with pmf given by

$$f(x) = \begin{cases} p(1-p)^x & x = 0,1,2,\dots \\ 0 & \text{otherwise} \end{cases} \tag{2.1}$$

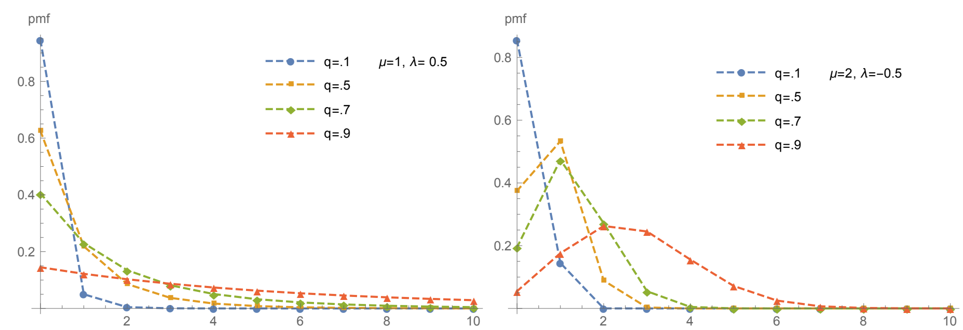
Then the cdf of TWG distribution is

$$G(x) = \begin{cases} \left(1-q^{(x+1)^\mu}\right)\left(1+\lambda q^{(x+1)^\mu}\right) & x = 0,1,2,\dots \\ 0 & \text{otherwise } |\lambda| \le 1, 0 < q < 1, \mu > 0 \end{cases} \tag{2.2}$$

The pmf of TWG distribution is obtained as

$$g(x) = \begin{cases} \left(1-q^{(x+1)^\mu}\right)\left(1+\lambda q^{(x+1)^\mu}\right) - \left(1-q^{x^\mu}\right)\left(1+\lambda q^{x^\mu}\right) & x = 0,1,2,\dots 0 < q < 1. \\ 0 & \text{otherwise} \end{cases} \tag{2.3}$$

The graph of pmf of TWG distribution for different values of parameters is given in Figure **??**.



(a) $\lambda > 0$ (b) $\lambda < 0$

FIGURE 3. The graph of pmf of TWG distribution for different values of parameters

This distribution generalizes some commonly used discrete distributions. When $\mu = 1$, (**??**) becomes pmf of discrete transmuted exponential distribution

$$g(x) = \left(1-q^{(x+1)}\right)\left(1+\lambda q^{(x+1)}\right) - (1-q^x)(1+\lambda q^x)$$

When $\lambda = 0$, (**??**) reduces to pmf of discrete Weibull distribution.

$$g(x) = q^{(x)^\mu} - q^{(x+1)^\mu} \quad x = 0,1,2,\dots$$

When $\lambda = \mu = 1$, (**??**) reduces to geometric distribution with pmf

$$g(x) = (1-q_1){q_1}^x, \ where\ q_1 = q^2, \ x = 0,1,2,\dots$$

Mean and variance of TWG distribution is obtained by

$$\begin{aligned}\text{Mean} &= E(X)\\ &= \sum_{x=0}^{\infty} x\left[\left(1-q^{(x+1)^{\mu}}\right)\left(1+\lambda q^{(x+1)^{\mu}}\right)-\left(1-q^{x^{\mu}}\right)\left(1+\lambda q^{x^{\mu}}\right)\right]\\ E(X^2) &= \sum_{x=0}^{\infty} x^2\left[\left(1-q^{(x+1)^{\mu}}\right)\left(1+\lambda q^{(x+1)^{\mu}}\right)-\left(1-q^{x^{\mu}}\right)\left(1+\lambda q^{x^{\mu}}\right)\right]\\ V(X) &= E(X^2)-(E(X))^2\end{aligned}$$

3. Reliability Properties

The reliability function of a random variable X is defined as, R(x)=1-F(x), which is the probability of an item not failing prior to some time x . For the TWG distribution,

$$R(x) = \left[\lambda\left(1+q^{(x+1)^{\mu}}\right)-1\right]q^{(x+1)^{\mu}} \text{ where } |\lambda|\leq 1, 0<q<1 \text{ and } \mu>0 \tag{3.1}$$

TWG distribution is a useful model in reliability analysis because of its analytical structure. TWG distribution retains the characteristics that corresponds to its continuous analogue in terms of reliability function and hazard rate.

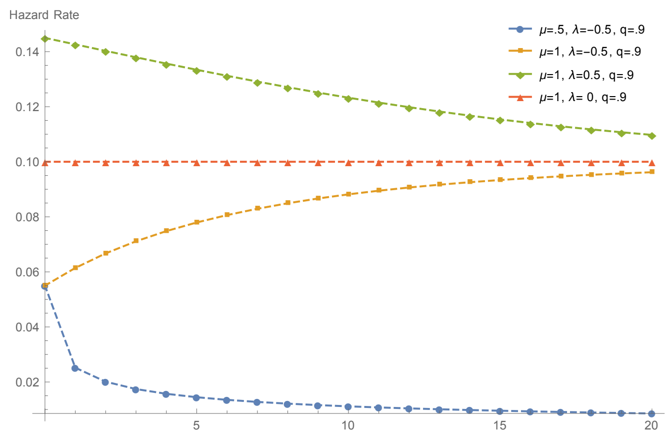
The hazard rate function h(x) of TWG distribution is obtained using $h(x) = \frac{P(x)}{R(x)}$. where R(x) is defined in (**??**). Figure **??** shows how the hazard rate function changes its shape when we vary the parameter μ, keeping λ and q constant. The graph of hazard rate function of TWG distribution for different values are plotted in Figure **??**. The following results are obtained on TWG distribution based on hazard rate function.

(1) If $\lambda = \mu = 1$, the failure rate is constant.
(2) If $\lambda = 0$ and $\mu = 1$, the failure rate is constant.
(3) If $\lambda = 1$, the failure rate is increasing for $\mu > 1$ and decreasing for $\mu < 1$.
(4) If $\mu = 1$, the failure rate is increasing for $\lambda < 0$ and decreasing for $\lambda > 0$.

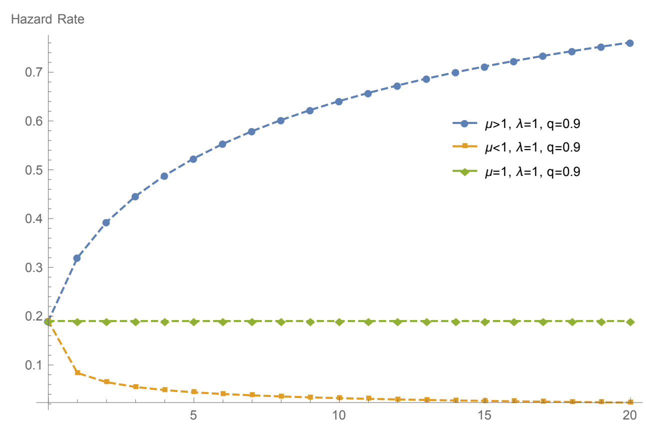
3.1. Random Variable Generation.

Let X be a random variable follows a continuous transmuted Weibull distribution. Then the cumulative distribution function of X is

$$F(x) = \left(1-e^{-\left(\frac{x}{\sigma}\right)^{\mu}}\right)\left(1+\lambda e^{-\left(\frac{x}{\sigma}\right)^{\mu}}\right), \text{ where } |\lambda|\leq 1, 0<q<1 \text{ and } \mu>0$$



(a)



(b)

FIGURE 4. The plot of hazard rate function of TWG distribution for different values of parameters

Let $U \sim U(0,1)$ uniform distribution of the continuous type. Then using inversion method, we get

$$x = \sigma \left[-\log \left(1 - \frac{\left(1 + \lambda - \sqrt{(1+\lambda)^2 - 4\lambda\mu}\right)}{2\lambda} \right) \right]^{\left(\frac{1}{\mu}\right)}$$

follows a transmuted Weibull distribution. Then a TWG random variable can obtained as $Y = [X]$, where $Y \sim TWG(q, \mu)$, and $[X]$ is the greatest integer value of X.

4. Estimation of Parameters in TWG distribution

In this section the parameters of $TWG(\lambda, q, \mu)$ is estimated using maximum likelihood method. Let $x_1, x_2, \ldots, x_n$ be the observations taken from a population following TWG distribution. Then the likelihood function of the sample is,

$$L = \prod_{i=1}^{n} \left[\left(1 - q^{(x_i+1)^\mu}\right) \left(1 + \lambda q^{(x_i+1)^\mu}\right) - \left(1 - q^{x_i^\mu}\right) \left(1 + \lambda q^{x_i^\mu}\right) \right]$$

$$\log L = \sum_{i=1}^{n} \log \left[\left(1 - q^{(x_i+1)^\mu}\right) \left(1 + \lambda q^{(x_i+1)^\mu}\right) - \left(1 - q^{x_i^\mu}\right) \left(1 + \lambda q^{x_i^\mu}\right) \right]$$

$$\frac{\partial \log L}{\partial q} = \sum_{i=1}^{n} \left[\frac{(\lambda - 1 - 2\lambda)(x+1)^\mu q^{(x+1)^\mu - 1} - (\lambda - 1 - 2\lambda) x^\mu q^{x^\mu - 1}}{\left(1 - q^{(x+1)^\mu}\right)\left(1 + \lambda q^{(x+1)^\mu}\right) - \left(1 - q^{x^\mu}\right)\left(1 + \lambda q^{x^\mu}\right)} \right]$$

$$\frac{\partial \log L}{\partial q} = \sum_{i=1}^{n} \left[\frac{(\lambda + 1)\left(x^\mu q^{x^\mu - 1} - (x+1)^\mu q^{(x+1)^\mu - 1}\right)}{\left(1 - q^{(x+1)^\mu}\right)\left(1 + \lambda q^{(x+1)^\mu}\right) - \left(1 - q^{x^\mu}\right)\left(1 + \lambda q^{x^\mu}\right)} \right]$$

$$\frac{\partial \log L}{\partial \mu} = \sum_{i=1}^{n} \left[\frac{\mu(1 + \lambda) \log q \left(q^{x^\mu} \log x + q^{(x+1)^\mu} \log(x+1)\right)}{\left(1 - q^{(x+1)^\mu}\right)\left(1 + \lambda q^{(x+1)^\mu}\right) - \left(1 - q^{x^\mu}\right)\left(1 + \lambda q^{x^\mu}\right)} \right]$$

$$\frac{\partial \log L}{\partial \lambda} = \sum_{i=1}^{n} \left[\frac{\left(1 - q^{(x+1)^\mu}\right) q^{(x+1)^\mu} - \left(1 - q^{x^\mu}\right) q^{x^\mu}}{\left(1 - q^{(x+1)^\mu}\right)\left(1 + \lambda q^{(x+1)^\mu}\right) - \left(1 - q^{x^\mu}\right)\left(1 + \lambda q^{x^\mu}\right)} \right]$$

On simplifying $\frac{\partial \log L}{\partial q} = 0$, $\frac{\partial \log L}{\partial \lambda} = 0$ and $\frac{\partial \log L}{\partial \mu} = 0$ simultaneously we get estimates of q, λ and μ.

5. Data Analysis

In this section, we will illustrate the usefulness of TWG distribution for modelling real life data. The data is given in Table **??** is the number of fires in Greece from 01.07.1998 to 31.08.1998, (Karlis and Xekalaki,2001). There are 123 observations in the data and the summary statistics of the data is as follows: Minimum=0; Maximum=43; Median=4; Mean=5.398; Variance=30.0449.

The goodness of fit of the TWG distribution is compared with three parameter Exponentiated Discrete Weibull distribution(Nekoukhou and Bidram,2015). The p.d.f of Exponentiated Discrete Weibull distribution (EDW)is

$$g(x) = \left(1 - p^{(x+1)^\alpha}\right)^\gamma - \left(1 - p^{x^\alpha}\right)^\gamma, \; x = 0, 1, 2, \ldots, 0 < p < 1, \alpha > 0, \gamma > 0$$

TABLE 1. Number of fires in forest district of Greece during 01.07.1998-31.08.1998.

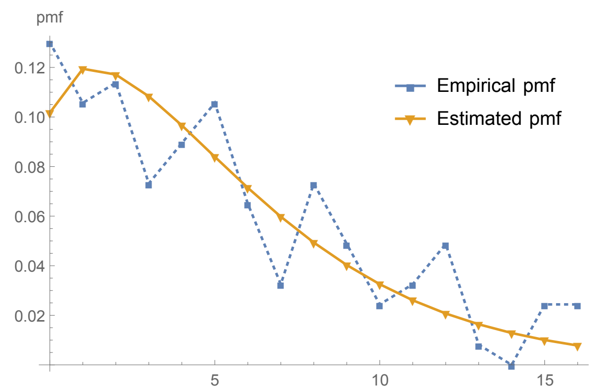
No. of fires in Greece	Observed frequency	Expected frequency(TWG) (MLE)	Expected frequency (EDW) (MLE)
0	16	12.5129	15.4535
1	13	14.6923	14.1958
2	14	14.4059	12.5393
3	9	13.3253	10.8567
4	11	11.8862	9.2826
5	13	10.3259	7.8673
6	8	8.7875	6.6241
7	4	7.3549	5.5486
8	9	6.0716	4.6283
9	6	4.9543	3.8471
10	3	4.0027	3.1883
11	4	3.2063	2.6355
12	6	2.5491	2.1736
13	0	2.0133	1.7891
14	0	1.5808	1.4699
15	4	1.2346	1.2057
≥ 15	3	3.9721	0.9876
Total	123	123	123
${\chi^2}_\alpha = 16.5$		$\chi^2 = 6.81$	$\chi^2 = 31.61$
		log L=-331.941 $\widehat{\mu}$= 1.1855 $\widehat{q}$= 0.8755 $\widehat{\lambda}$= -0.2020 AIC=669.882	log L=-339.7929 $\widehat{\alpha}$=1.0809 $\widehat{\gamma}$=1.0923 $\widehat{p}$=0.8599 AIC=685.5859

The estimates are obtained and fitted values of TWG distribution and EDW distribution are given in Table **??**. From the value of chi square and AIC, it seems that TWG distribution is a good fit to the given data compared to EDW distribution.

For the given data, TWG and EDW distributions have an increasing hazard rate and is shown in Figure 6.

6. Conclusion

In this paper we developed a discrete distribution suitable to model observations over non negative integers. Various distributional properties of the model are studied. Estimation of the parameters of the distributions are obtained using



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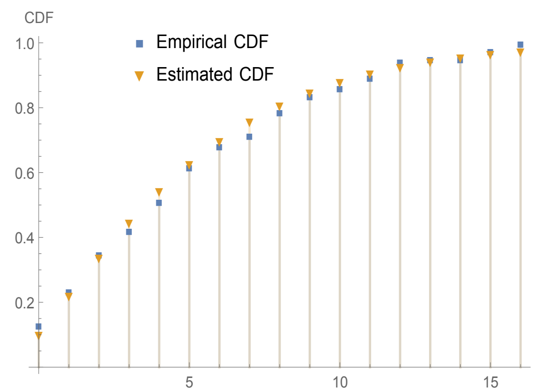

(b)

FIGURE 5. Empirical and fitted pdf and cdf of number of fires in Greece with TWG distribution

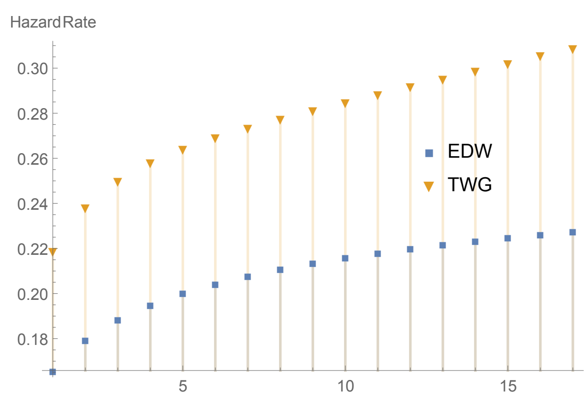

FIGURE 6. The graph of hazard rate function of TWG and EDW for the number of fires in Greece (Karlis and Xekalaki,2001).

MLE method and relevance of the model is established by fitting to a real data set.

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Simi Sebastian: Department of Statistics ,Goverment College Kattappana, Kerala, India.

E-mail address: simibinu01@gmail.com

Jane A. Luke: Department of Statistics ,Newman College Thodupuzha, Kerala, India.

E-mail address: janealuke@gmail.com

Binumon Joseph: Department of Mathematics ,Goverment Engineering College Idukki, Kerala, India

E-mail address: jbinumonjoseph@gmail.com