Set-Valued Mathematics and Applications
Vol. 1 (2017), ISSN: 0973-7375

ON THE INTERMEDIATE MULTIVALUED FUNCTIONS

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Abstract. For topological spaces X and Y we consider conditions, by which for arbitrary multivalued maps $G : X \to Y$ and $H : X \to Y$, such, that $G(x) \subseteq H(x)$ for each $x \in X$ and G and H are respectively upper and lower semicontinuous, there is $F : X \to Y$ continuous, such, that $G(x) \subseteq F(x) \subseteq H(x)$. We also consider conditions on topological spaces, by which the Hahn's theorem on the intermediate function has a multivalued analog.

1. History

Suppose X is a set, Y is a partially ordered set, $g : X \to Y$ and $h : X \to Y$ — maps, such, that $g(x) \leq h(x)$ for all $x \in X$. Map $f : X \to Y$ is called intermediate / strictly intermediate / for pair (g, h) if $g(x) \leq f(x) \leq h(x)$ on X /$g(x) < f(x) < h(x)$, if $g(x) < h(x)$, and $g(x) = f(x) = h(x)$, if $g(x) = h(x)$/. If X is a topological space, $Y = \mathbb{R}$, then we say that maps $g : X \to \mathbb{R}$ and $h : X \to \mathbb{R}$ form a Hahn's pair / strict Hahn's pair/, if g is upper semicontinuous, h — lower semicontinuous and $g(x) \leq h(x)$ /g(x)¡h(x)/ on X. H. Hahn [5] proved, that every Hahn's pair (g, h) on metric space X has continuous intermediate function $f : X \to \mathbb{R}$. J. Dieudonne [2] proved it for case of X being paracompact, H. Tong [10,11] and M. Katetov [6,7] compiled Hahn's theorem for the normal spaces, noting, that the existance of intermediate continuous function f for every Hahn's pair on T_1-space X implies normality of X.

These results were developed in papers by K. Dowker [3] and E. Michael [8]. First one together with M. Katetow [6] established, that in class of T_1-spaces X existance of strictly intermediate continuous function $f : X \to \mathbb{R}$ for each strict Hahn pair (g, h) is equivalent to normality and paracompactness of X, second one established, that in class of T_1-spaces X existance of strictly intermediate continuous function $f : X \to \mathbb{R}$ for each Hahn pair (g, h) is equivalent to perfect normality of X. New approach to the proof of these results has been presented by C. Good and I. Stars [4]. K. Yamazaki [13], developing his previous investigations [14] and results by J.M. Borwein, M. Thera [1], proved theorems about the intermediate map $f : X \to \mathbb{R}$ for Hahn's pairs analogies (g, h) with values in Banach lattices.

Recently there have appeared new versions of Hahn theorem. In paper [15] it was proved that for each Hahn pair (g, h) on segment $[a, b]$, where g and h — are increasing functions, there is an intermediate increasing continuous function $f : [a, b] \to \mathbb{R}$. Then in [16] for strict Hahn pairs (g, h) on the segment X of $\mathbb{R}$ there were constructed piecewise linear or infinitely differentiable functions $f : X \to \mathbb{R}$

Date of Submission January 11, 2017 ; Date of Acceptance May 15, 2017 , Communicated by Zelinskyi Yurii Borisovich

such that satisfy some additional conditions. These results were compiled in [18,19] for the Frechet differentiable maps using partitions of unity.

2. Multivalued maps and staging of the problem

It is well known, that the concept of upper and lower semicontinuity can be applied to the multivalued maps $F : X \to Y$, that correlate nonempty subset of Y for each point $x \in X$, i.e. are maps $F : X \to \mathcal{P}(Y)$ with values in set $\mathcal{P}(Y) = 2^Y \setminus \{\emptyset\}$ of all nonempty subset of space Y. Lets recall, that multivalued map $F : X \to Y$ from topological space X to the topological space Y is called upper /lower/ semicontinuous in point $x_0 \in X$, if for each open set U in Y, such that $F(x_0) \subseteq V$ $/F(x_0) \cap V \neq \emptyset/$ there is such neighborhood U of the point $x_0 \in X$, that $F(x) \subseteq V$ $/F(x) \cap V \neq \emptyset/$ for each $x \in U$ (here we use the terminology from [9]). We say, that F is continuous at x_0, if it is upper and lower semicontinuous at this point. Map $F : X \to Y$ is called continuous, upper or lower semicontinuous, if it is so at every point of the space X.

The set $\mathcal{P}(Y)$is equipped with natural partial order, which is the relation of inclusion $\subseteq$ of Y subsets , that allows to transfer the concept of Hahn's pair to the case of multivalued maps. We say that a multivalued maps form the Hahn /Hahn strict/ pair, if G is upper semicontinuous, H is lower semicontinuous and $G(x) \subseteq H(x)$ $/G(x) \subset H(x)/$ for each $x \in X$.

The following problems arise naturally:

Problem 1. In which conditions on spaces X and Y each Hahn pair (G, H) from multivalued maps $G, H : X \to Y$ has continuous intermediate multivalued map $F : X \to Y$?

Problem 2. In which conditions each strict Hahn's pair (G, H) from X to Y has strict intermediate continuous multivalued map $F : X \to Y$?

Problem 3. In which conditions each Hahn's pair (G, H) from X to Y has strict intermediate continuous multivalued map $F : X \to Y$?

In this paper we begin to investigate this problems. Our results relate to the problem 1. We show that for the normal T_1-space X and any Hahn's pair (G, H) of multivalued maps $G : X \to Y$ and $H : X \to Y$, which values are segments, there is intermediate continuous map $F : X \to Y$, that has segments as values in $\mathbb{R}$. Then we show, that the existence of intermediate continuous map $F : X \to Y$ for Hahn's pair (G, H) of maps $G : X \to \mathbb{R}$, $G(x) = (-\infty, g(x)]$, and $H : X \to \mathbb{R}$, $H(X) = (-\infty, h(x)]$, implies that (g, h) is Hahn's pair on X and has continuous intermediate function $f : X \to \mathbb{R}$ on X.

3. Existence of continuous intermediate multivalued function.

Let X be a topological space and $F : X \to \mathbb{R}$ a multivalued map, with values as segments $F(x) = [f_1(x), f_2(x)]$, where $f_1 : X \to \mathbb{R}$ are functions, for which $f_1(x) \leq f_2(x)$ on X.

Lemma 3.1. *Map $F : X \to \mathbb{R}$, $F(x) = [f_1(x), f_2(x)]$, is upper semicontinuous if and only if function $f_1 : X \to \mathbb{R}$ is lower semicontinuous, and $f_2 : X \to \mathbb{R}$ is upper semicontinuous.*

Proof. Let F be upper semicontinuous in x_0 and $\varepsilon > 0$. Open set $V = (f_1(x_0) - \varepsilon, f_2(x_0) + \varepsilon)$ contains segment $F(x_0)$. This implies that there exists U neighborhood of x_0 such that $F(x) \subseteq V$ when $x \in U$. Since $f_1(x) \in F(x)$ and $f_2(x) \in F(x)$ for any x , then $\{f_1(x), f_2(x)\} \subseteq V$ for each $x \in U$, so,

$$f_1(x_0) - \varepsilon < f_1(x) \leq f_2(x) < f_2(x_0) + \varepsilon$$

for each $x \in U$, so f_1 is lower semicontinuous, and f_2 is upper semicontinuous in x_0.

Let f_1 be lower semicontinuous, f_2 upper semicontinuous in x_0 and $\varepsilon > 0$. Then there are such a neighborhoods U_1 and U_2 of point x_0, that

$$f_1(x) > f_1(x_0) - \varepsilon \text{ on } U_1 \text{ and } f_2(x) < f_2(x_0) + \varepsilon \text{ on } U_2$$

Intersection $U = U_1 \cap U_2$ is also a neighborhood of the point x_0 and for $x \in U$:

$$f_1(x_0) - \varepsilon < f_1(x) \leq f_2(x) < f_2(x) + \varepsilon.$$

Let V be an arbitrary subset of $\mathbb{R}$, that contains $F(x_0) = [f_1(x_0), f_2(x_0)]$. Since $f_i(x_0) \in V$ for $i = 1, 2$ there exist such $\varepsilon_i > 0$ for $i = 1, 2$ that:

$$(f_i(x_0) - \varepsilon_i, f_i(x_0) + \varepsilon_i) \subseteq V, \text{ where } i = 1, 2.$$

Lets put $\varepsilon = \min\{\varepsilon_1, \varepsilon_2\}$. Obviously, for this number:

$$V_0 = (f_1(x_0) - \varepsilon_1, f_2(x_0) + \varepsilon_2) \subseteq V.$$

From the proven before, there exists neighborhood U of the point x_0 in X, such, that

$$F(x) \subseteq V_0,$$

as soon as $x \in U$. Then also $F(x) \subseteq V$ for each $x \in U$, so, multivalued function F is upper semicontinuous in x_0. □

Lemma 3.2. *Map $F : X \to \mathbb{R}$, $F(x) = [f_1(x), f_2(x)]$, is lower semicontinuous if and only if function $f_1 : X \to \mathbb{R}$ is upper semicontinuous, and $f_2 : X \to \mathbb{R}$ is upper semicontinuous.*

Proof. Let F be lower semicontinuous in x_0 and $\varepsilon > 0$. Consider open set $V = (-\infty, f_1(x_0) + \varepsilon)$, for which, obviously $V \cup F(x_0) \neq \emptyset$, that implies there is neighborhood U of point x_0 in X, that $V \cup F(x) \neq \emptyset$, as soon as $x \in U$ Lets take for $x \in U$ a point $y_1 \in V \cup F(x)$. Then $y_1 \in V$, so, $y_1 < f_1(x_0) + \varepsilon$, and $y_1 \in F(x)$, so $y_1 \geq f_1(x)$. From that we have $f_1(x) < f_1(x_0) + \varepsilon$ on U, and f is upper semicontinuous in x_0.

Similarly the lower semicontinuousness of function f_2 in x_0 is proven, for this it is only required to consider open set $V = (f_2(x_0) - \varepsilon, +\infty)$.

Vice versa, let function f_1 be upper semicontinuous, f_2 lower and point x_0 and V is open set in $\mathbb{R}$, for which $V \cup F(x_0) \neq \emptyset$. Lets consider any point $y \in V \cup F(x_0)$. Then $f_1(x_0) \leq y \leq f_2(x_0)$ and $y \in V$. The fact that set V is open implies that there is $\varepsilon > 0$, such that $(y - \varepsilon, y + \varepsilon) \subseteq V$. Also, the fact that f_1 and f_2 are semicontinuous upper and lower respectively at the point x_0 implies that that there is neighborhood U of point x_0 in X, such that for $x \in U$:

$$f_1(x) < y + \varepsilon \text{ and } f_2(x) > y - \varepsilon.$$

Then for $x \in U$ we have: $F(x) \cup V \neq \emptyset$. Indeed, for points $y_1 = f_1(x)$ and $y_2 = f_2(x)$ we have that $y_1 < y + \varepsilon$ and $y_2 > y - \varepsilon$. If $y_2 < y + \varepsilon$, then $y_2 \in (y - \varepsilon, y + \varepsilon) \cup F(x) \subseteq V \cup F(x)$, and if $y_1 > y - \varepsilon$, then $y_1 \in V \cup F(x)$. Lets now $y_2 \geq y + \varepsilon$ and $y_1 \leq y - \varepsilon$. Then $(y - \varepsilon, y + \varepsilon) \subseteq [y_1, y_2] = F(x)$ and from that $y \in V \cup F(x)$. In any case $F(x) \cup V \neq \emptyset$. □

Theorem 3.3. *Let X be a normal space and (G, H) be Hahn's pair such that $G(x) = [g_1(x), g_2(x)]$ and $H(x) = [h_1(x), h_2(x)]$ for each $x \in X$. Then there is intermediate for (G, H) continuous $F : X \to \mathbb{R}$ such that $F(x) = [f_1(x), f_2(x)]$ on X, where functions f_1 and f_2 are continuous.*

Proof. From the condition $G(x) \subseteq H(x)$ on X we get that

$$h_1(x) \leq g_1(x) \leq g_2(x) \leq h_2(x)$$

for each $x \in X$. From lemmas 1 and 2 we obtain, that (h_1, g_1) and (g_2, h_2) are Hahn's pairs on X. The Hahn theorem implies that there exist continuous functions $f_i : X \to \mathbb{R}$ for $i = 1, 2$ such,that:

$$h_1(x) \leq f_1(x) \leq g_1(x) \text{ and } h_1(x) \leq f_1(x) \leq g_1(x)$$

for each $x \in X$. Since $g_1(x) \leq g_2(x)$, then $f_1(x) \leq f_2(x)$ on X. Lemma 3 implies that the map

$$F : X \to \mathbb{R}, F(x) = [f_1(x), f_2(x)],$$

is continuous, and $G(x) \subseteq F(x) \subseteq H(x)$ for each $x \in X$. □

4. Hahn's theorem as a corollary of it's multivalued version.

Lets $f : X \to \mathbb{R}$ be a function, defined on the topological space X and $F(x) = (-\infty, f(x)]$.The following lemmas can be proven similarly to the proofs of lemmas 1-3:

Lemma 4.1. *Function f is upper semicontinuous if and only if multivalued function F is upper semicontinuous.*

Lemma 4.2. *Function f is lower semicontinuous if and only if multivalued function F is lower semicontinuous.*

Lemma 4.3. *Function f is continuous if and only if multivalued function F is continuous.*

Lemmas 1 and 2 immediately imply:

Lemma 4.4. *Lets X be topological space, (g, h) be a Hahn's pair on X, $G(x) = (-\infty, g(x)]$ and $H(x) = (-\infty, g(x)]$. Then (G, H) is also Hahn's pair on X.*

Theorem 4.5. *Lets X be topological space, (g, h) be a Hahn's space on X, $G(x) = (-\infty, g(x)]$ and $H(x) = (-\infty, g(x)]$, and Hahn's pair (G, H) has intermediate continuous function $F : X \to \mathbb{R}$. Then Hahn's pair (g, h) also has intermediate continuous function $f : X \to \mathbb{R}$*

Proof. From condition $G(x) \subseteq F(x) \subseteq H(x)$ on X. Lets put

$$f(x) = \sup F(x).$$

Since $g(x) \in G(x)$, then $g(x) \in F(x)$, and $g(x) \leq f(x)$. Here, the inclusion $F(x) \subseteq H(x)$ implies, that

$$f(x) = \sup F(x) \leq \sup H(x) = h(x).$$

Since that, we have, that $g(x) \leq f(x) \leq h(x)$ on X.

Lets $\varepsilon > 0$. Consider open set $V_1 = (-\infty, f(x_0)+\varepsilon)$. It is clear, that $F(x_0) \subseteq V_1$. The fact that F is upper semicontinuous at x_0 implies that there is neighborhood U_1 of point x_0 in X, such, that $F(x) \subseteq V_1$ as soon as $x \in U$. In that case:

$$f(x) = \sup F(x) \leq \sup V_1 = f(x_0) + \varepsilon$$

on U_1, so, function f is upper semicontinuous at x_0.

For open set $V_2 = (f(x_0) - \varepsilon, +\infty)$ we have, that $V_2 \cup F(x_0) \neq \emptyset$. Indeed, since $f(x_0) = \sup F(x_0)$, we have, that there is $y \in F(x_0)$, such, that $y > f(x_0) - \varepsilon$. It is clear that $y \in V_2 \cup F(x_0)$. The fact that F is lower semicontinuous at x_0 implies, that there is neighborhood U_2 of the point x_0m such, that $F(x) \cup V_2 \neq \emptyset$ when $x \in U_2$. Then for each $x \in U$ there is $y_x \in F(x) \cup V_2$, for which, obviously, $f(x_0) - \varepsilon < y_x \leq f(x)$, and then:

$$f(x) > f(x_0) - \varepsilon$$

on U_2 and f is lower semicontinuous at x_0.

On the neighborhood $U = U_1 \cup U_2$ of the point x_0 in X the following inequalities are true:

$$f(x_0) - \varepsilon < f(x) < f(x_0) + \varepsilon,$$

and that implies the continuity of function f at point x_0.

□

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