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BAYESIAN ESTIMATION IN M/D/1 QUEUE RELATIVE TO LINEX LOSS FUNCTION.

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ABSTRACT: $X_1, X_2, \ldots, X_n, \ldots$ is a sequence of i.i.d random variables in the Imbedded Markov chain analysis of the M/G/1 queue.In particular ,the common random variable has the Poisson distribution with mean ρ ,the traffic intensity for the M/D/1 queue .Utilizing this the Bayes estimator of traffic intensity relative to LINEX loss and conjugate prior are derived.Comparison of estimators relative to LINEX loss and Squared Error Loss(SEL) have been obtained in order to check the inadmissibility of the estimators.

1. Introduction and Preliminaries

Many applications of queueing theory are discussed in articles on operations research, industrial engineering, probability, and management science. Queuing analysis is primarily investigated as a branch of operations research, since the outcomes are frequently utilised to make decisions. Waiting lines in hospitals, coffee shops, banks, airports, and other places are exemplars of the queuing theory. Queuing theory is used to analyze computers, telecommunications systems, logistics and manufacturing systems. Clarke (1957) was one among the early researchers on classical inference in queues, followed by research work in Bayesian approach and in this field was carried out by Armero and Bayyari (1999),Choudhury and Borthakur (2008). In M/M/1 queues, Chowdhury and Mukherjee (2013) investigated Bayesian inference for traffic intensity. A study of M/D/1 queuing system related to UMVUE and Maximum likelihood estimation is discussed by Srinivas and Kale (2016), followed by Chandrasekhar, et.al. (2020).

In M/D/1 queue, $X_1, X_2, \ldots X_n$ be a random sample of n observations from well-known Poisson distribution with pmfas defined by

$$P(X=x)=\frac{e^{-\rho}\rho^{x}}{x!},\ x=0,1,2,\ldots \quad (1.1)$$

where X_j is the number of entity arrivals during the j^{th} client's service time.

The positive and negative errors of estimation are given equal weight in the symmetrical standard squared error loss function (SEL), which is frequently used in Bayesian analysis and is given by $L\ (\hat{\rho},\rho)=(\hat{\rho}-\rho)^2$ (1.2)

and is inappropriate in many situations. Underestimation is more dangerous than overestimation in estimation problems, and conversely. A dam which is built with an underestimation of peak water level is considerably more dangerous than one that is built with an overestimation. (Zellner (1986)). Overestimation is more dangerous than underestimation when determining the average life of the space shuttle. (Harris (1992)). Varian (1975) proposed the LINEX loss function to deal with situations when either overestimation or underestimation is more serious than the other. The LINEX loss function is represented as follows,

$L(\Delta)=b(e^{a\Delta}-a\Delta-1), b>0, a\neq 0$ where $\Delta=\hat{\rho}-\rho$ (1.3)

where the constant 'a' is the shape parameter. The direction of asymmetry is indicated by sign of the constant, whereas the shape parameter's magnitude represents the degree of asymmetry. Underestimation is much dangerous than overestimation when a>0, and vice versa. When |a|is small, LINEX behaves like SEL. Hence SEL is an

Key words: Bayes estimator, The M/D/1 queue, LINEX loss, SEL, inadmissible.

approximation to LINEX when |a|is small. Since $L(\Delta)$ rises linearly or exponentially it is called LINEX loss function.

The application of the LINEX loss function has been justified by Basu and Ebrahimi (1991) in the perspective of exponential distribution reliability. The Linex loss function was used by Pandey and Rai (1992) to estimate the mean of a normal distribution, Jaising (1993) and several authors have considered this loss in variety of situations of interest.

The article summary is as follows. In second section, Bayes estimator of traffic intensity is derived relative to LINEX loss by minimization of risk. In Section 3 and 4,comparison of estimators relative to SEL and LINEX loss is made.

2. Bayes Estimator of traffic intensity based on LINEX loss function.

Theorem 2.1:

Assuming LINEX loss, the Bayesian estimator of traffic intensity, is of form

$$\hat{\rho}_L = \frac{1}{a}\ln\left(\frac{a+\alpha+n}{n+\alpha}\right)^{\sum x_I+\beta} \quad (2.1)$$

Proof :

The prior density of ρ is conjugate prior which is given as

Gamma prior: $g_1(\rho) = \frac{1}{\Gamma\beta} \propto^{\beta} e^{-\propto\rho}\rho^{\beta-1}, 0 < \rho < \infty; \propto, \beta > 0$

(2.2)

The Bayes theorem is utilized to combine the likelihood function of pmfin (1.1) with the conjugate prior to generate the so called posterior density,

$$\Pi_2(\rho/x) = \frac{(n+\alpha)^{\sum xi+\beta} e^{-(\alpha+n)\rho} \rho^{\sum x_i+\beta-1}}{\Gamma\sum xi+\beta} \quad (2.3)$$

Based on LINEX loss, the frequentist risk is given by the formula,

$$R(\hat{\rho},\rho) = \int_0^\infty (e^{a(\hat{\rho}-\rho)} - a(\hat{\rho}-\rho)-1).\frac{(n+\alpha)^{\sum xi+\beta}}{\sum x_i+\beta} e^{-(\alpha+n)\rho}\rho^{\sum x_i+\beta-1}d\rho$$

$$R(\hat{\rho},\rho) = \frac{(n+\alpha)^{\sum xi+\beta}}{\sum x_i+\beta}\left[e^{a\hat{\rho}}\int_0^\infty \rho^{\sum x_i+\beta-1}e^{-\rho(a+\alpha+n)}d\rho - a\hat{\rho}\int_0^\infty e^{-(\alpha+n)\rho}\rho^{\sum x_i+\beta}d\rho - \int_0^\infty e^{-(\alpha+n)\rho}\rho^{\sum x_i+\beta-1}d\rho\right]$$

$$R(\hat{\rho},\rho) = \left(\frac{n+\alpha}{a+\alpha+n}\right)^{\sum x_i+\beta} e^{a\hat{\rho}} - \hat{\rho}a + a(\sum x_i+\beta) - 1$$

(2.4)

The optimal estimator is one that minimizes the risk with respect to $\hat{\rho}$ and is given as

$$\hat{\rho}_L = \frac{1}{a}\ln\left(\frac{a+n+\alpha}{n+\alpha}\right)^{\sum x_i+\beta}$$

Theorem 2.2:

Assuming SEL, the Bayesian estimator of traffic intensity, is of form

$$\hat{\rho}_S = \frac{\sum xi+\beta}{\alpha+n} \quad (2.5)$$

Proof:

By using the posterior density (2.3) and Squared Error Loss(SEL), the frequentist risk is obtained as

$$R(\hat{\rho},\rho) = \hat{\rho}^2 + \frac{(\sum xi+\beta+1)(\sum x_i+\beta)}{(\alpha+n)^2} - \frac{2\hat{\rho}\ (\sum xi+\beta)}{(\alpha+n)} \qquad (2.6)$$

The optimal estimator is one that minimizes the risk with respect to $\hat{\rho}$ and is given as

$$\hat{\rho}_S = \frac{\sum xi+\beta}{\alpha+n}$$

3. Comparison of estimators relative to SEL function.

The frequentist risk of the estimator $\hat{\rho}_s$ relative to SEL is given by

$$R_s(\hat{\rho}_s,\rho) = E\left[L(\hat{\rho}_s,\rho)\right]$$

$$= \int_0^\infty \frac{(n+\alpha)^{\sum x_i+\beta}}{\Gamma\sum x_i+\beta}(\hat{\rho}_s-\rho)^2 e^{-(\alpha+n)\rho}\rho^{\sum x_i+\beta-1}d\rho$$

$$= \hat{\rho}_s{}^2 + \frac{(\sum xi+\beta+1)(\sum x_i+\beta)}{(\alpha+n)^2} - \frac{2\hat{\rho}_s(\sum xi+\beta)}{(\alpha+n)}$$

(3.1)

Substituting the value of $\hat{\rho}_s$ in (3.1) gives $R_s(\rho,\hat{\rho}_s)$ as

$$R_s(\hat{\rho}_s,\rho) = \frac{\sum x_i+\beta}{(\alpha+n)^2} \qquad (3.2)$$

The risk function of the estimator $\hat{\rho}_L$ relative to SEL is given by

$$R_s(\hat{\rho}_L,\rho) = E\left[L(\hat{\rho}_L,\rho)\right]$$

$$= \int_0^\infty \frac{(n+\alpha)^{\sum x_i+\beta}}{\Gamma\sum x_i+\beta}(\hat{\rho}_L-\rho)^2 e^{-(\alpha+n)\rho}\rho^{\sum x_i+\beta-1}d\rho$$

and the risk function is obtained as

$$R_s(\hat{\rho}_L,\rho)=\frac{1}{a^2}\ln^2\left(\frac{a+\alpha+n}{n+\alpha}\right)^{\sum x_i+\beta}+\frac{(\sum x_i+\beta)+1)(\sum x_i+\beta)}{(n+\alpha)^2}-2\frac{1}{a}\log\left(\frac{a+\alpha+n}{n+\alpha}\right)^{\sum x_i+\beta}\left(\frac{\sum x_i+\beta}{(n+\alpha)}\right)$$

The difference of risk is obtained as

$R_s(\hat{\rho}_L,\rho)$ - $R_s(\hat{\rho}_S,\rho)$ =

$$\frac{1}{a^2}\ln^2\left(\frac{a+\alpha+n}{n+\alpha}\right)^{\sum x_i+\beta}-2\frac{1}{a}\log\left(\frac{a+\alpha+n}{n+\alpha}\right)^{\sum x_i+\beta}\left(\frac{\sum x_i+\beta}{(\alpha+n)}\right)+\frac{(\sum x_i+\beta)^2}{(\alpha+n)^2}$$

(3.3)

>0 for a> 0 and α+n>0

Therefore $R_s(\hat{\rho}_s,\rho)<R_s(\hat{\rho}_L,\rho)$

$\hat{\rho}_s$ is R better decision rule (estimator) compared to $\hat{\rho}_{LINEX}$

Thus $\hat{\rho}_{LINEX}$ in terms of Squared Error Loss is inadmissible.

4. **Comparison of competing estimators under LINEX loss function.**

The frequentist risk of the estimator $\hat{\rho}_s$ relative to LINEX loss is

$R_L(\hat{\rho}_s,\rho)=E\left[L(\hat{\rho}_s,\rho)\right]$ and is obtained as

$$R_L(\hat{\rho}_s,\rho)=e^{\frac{a\sum x_{ii}+\beta}{n+\alpha}}\left(\frac{n+\alpha}{a+\alpha+n}\right)^{\sum x_{ii}+\beta}-a\frac{\sum x_i+\beta}{n+\alpha}+a\left(\sum x_i+\beta\right)-1$$

(4.1)

The frequentist risk of the estimator $\hat{\rho}_L$ based on LINEX loss is obtained as

$$R_L(\hat{\rho}_L,\rho)=E\left[L(\hat{\rho}_L,\rho)\right]$$

$$=\left(\frac{n+\alpha}{a+\alpha+n}\right)^{\sum x_i+\beta}e^{a\hat{\rho}_L}-a\hat{\rho}_L+a\left(\sum x_i+\beta\right)-1$$

(4.2)

Substituting $\hat{\rho}_L$ in (4.2),we get

$$R_L(\hat{\rho}_L,\rho)=a\left(\sum x_i+\beta\right)-\left(\sum x_i+\beta\right)\ln\left(\frac{a+\alpha+n}{n+\alpha}\right) \qquad (4.3)$$

Now consider the risk difference,

$$R_L(\hat{\rho}_L,\rho) - R_L(\hat{\rho}_S,\rho) =$$

$$(\sum x_i+\beta)\left\{\frac{a}{n+\alpha}-\ln\left(\frac{a+\alpha+n}{n+\alpha}\right)\right\}-e^{a\frac{\sum x+\beta_i}{n+\alpha}}\left(\frac{n+\alpha}{a+\alpha+n}\right)^{\sum x_i+\beta}$$

(4.4)

<0 for a>0 and (α+n) not equal to−a

That is $R_L(\rho,\hat{\rho}_L) < R_L(\rho,\hat{\rho}_s)\,\hat{\rho}_L$ is R better decision rule (estimator) compared to $\hat{\rho}_S$. Thus $\hat{\rho}_S$ in terms of LINEX loss is inadmissible.

5. CONCLUSION

The Bayesian analysis of ρ in M/D/1 queue using LINEX loss is presented in this article. Comparison of estimators relative to SEL and LINEX loss is done in order to examine the inadmissibility. It is observed that $\hat{\rho}_{LINEX}$ in terms of squared error loss is inadmissible and $\hat{\rho}_S$ in terms of LINEX loss is inadmissible. Bayes estimation in relation to other queuing systems is being considered and may be included in a future communication.

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