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ON HENSTOCK - KURZWEIL SUMUDU TRANSFORM

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Abstract. In the present research paper, the Sumudu Transform is treated as a Henstock - Kurzweil Integral. Some existential conditions are obtained. Some properties are considered. It is given that Sumudu Transform exists as Henstock - Kurzweil Integral and finally the Inversion Theorem is established.

Keywords. Henstock - Kurzweil Integral, Sumudu Transform.

AMS Classification 26A39, 45A05.

1. Introduction

If $f : [0, \infty] \to R$ then its Sumudu Transform at $1/v \in C$ is defined as the integral

$$1/v \int_0^\infty f(t)e^{-t/v}dt.$$

In previous decades many authors are interested to study the Sumudu Transform in a classical way on the real line. Due to its simple formulation and consequent special and useful properties, the Sumudu Transform has already much promised [2, 3, 4].

In above equation the function $e^{-t/v}, 1/v \in C$ is not in $Bv[0, \infty)$ but it is in $Bv(I)$ for any compact $I \subset [0, \infty)$. Because of this we can not consider Sumudu Transform as Henstock Kurzweil Integral directly. Hence we can not get any special condition for the existence of Sumudu Transform as a Henstock - Kurzweil Integral. In present work we give some different existence conditions. All elementary properties are discussed by this method. The Inverse Sumudu Transform is obtained by method of post generalization method [6]. Also the present work somewhere essential to reverse the order of repeated integrals which is justified in [8].

In this paper we consider, Sumudu Transform as a Henstock - Kurzweil Integral. The Henstock - Kurzweil Integral is a generalisation of Reimann, Lebesgue, Denjoy and Perron's, but we consider here it in terms of Riemann sums.

The rest of the paper is arranged as under: In section 2, basic notations and results are given. Existential conditions and properties are obtained in section 3. Henstock - Kurzweil Sumudu transform of derivative of some functions are obtained as an application in section 4. In section 5, inversion result is proved.

Key words and phrases. Henstock - Kurzweil Integral, Sumudu Transform.

2. Preliminaries

Following notations and results are used through out paper.
Let $0 \leq p \leq q \leq \infty$ and denote the space of all Henstock - Kurzweil integrable functions on $[p,q]$ as $\mathcal{HK}([p,q])$

$$\mathcal{HK}_{loc} = \{f \in \mathcal{HK}(I), I \subset [0,\infty) \ is\ compact\}$$
$$\mathbf{BV}(I) = \{fis\ bounded\ variationonI\}$$
$$\mathbf{BV}(\infty) = \{f \in \mathbf{BV}(a,\infty]\ for\ some\ p \in [0,\infty]\}$$

Definition 2.1. [5] Let $E \subseteq I$ and $f : I \to \mathbb{R}$ belongs to $\mathcal{AC}^*{}_\delta(E)$ if for each $\epsilon > 0$, there exist $\omega > 0$ and a Gauge δ on E such that $\sum_{i=1}^{n}|f(d_i) - f(c_i)| < \epsilon$ whenever $\mathcal{P} = (x_i, [c_i, d_i])_{i=1}^{n}$ is a δ fine subpartition of E and $\sum_{i=1}^{n}(d_i - c_i) < \omega$. Let f belongs to $\mathcal{ACG}^*{}_\delta(I)$ if I be written as a countable union of sets on each of which the function $f$is $\mathcal{AC}^*{}_\delta$.

Definition 2.2. [10]Let g_n be functions on $[p,q]$, we say that g_n is of uniform bounded variation on $[p,q]$, if there is constant K such that $|g_n| \leq K, V(g_n) \leq K, \forall\ n.$ ∎

Theorem 2.3. [10] *Let $\{g_n\}$ be a sequence of uniform bounded variation on $[p,q]$ and if $f : [a,b] \to \mathbb{R}$ is $\mathcal{HK}$ Integrable, such that $g_n \to g$, then $\int_p^q fg_n \to \int_p^q fg$ as $n \to \infty$.*

Theorem 2.4. *(Chartier-Dirichlet's Test)* [1] *Let f and g be functions defined on $[p,\infty)$. Suppose that*
i) $g \in \mathcal{HK}[p,c]$ for every $c \geq p$, and G defined by $G(x) = \int\limits_p^x g(x)dx$ is bounded on $[p,\infty)$;
ii) f is of bounded variation on $[p,\infty)$ and $\lim\limits_{x\to\infty} = 0$,
then $fg \in \mathcal{HK}([p,\infty))$.

Theorem 2.5. *(Dubois-Reymond's Test)* [1] *Let f and ψ be functions defined on $[p,\infty)$ and suppose that:*
i) $f \in \mathcal{HK}([p,c])$ for all $c \geq a$ and $F(x) = \int\limits_p^x f$ is bounded on $[p,\infty)$
ii) ψ is differentiable on $[p,\infty)$ and $\psi^{'}$ is Lebesgue integrable on $[p,\infty)$
iii) $\lim\limits_{x\to\infty} F(x)\psi(x)$ exists
then $f\psi \in \mathcal{HK}([p,\infty))$.

Theorem 2.6. [12] *Let $0 < \omega < q-p$ and $\psi(x) \in \mathcal{C}^2$, $(p \leq x \leq p+\omega)$, $\psi^{'}(p) = 0$, $\psi^{''}(p) < 0$, $\psi(x)$ is non increasing on $(p,q]$. If $f(x) \in \mathcal{HK}([p,q])$. Then*

$$\int_p^q f(x)e^{k\psi(x)}dx \to f(p)e^{k\psi(p)}(\tfrac{-\pi}{2k\psi^{"}(p)})^{\frac{1}{2}}\ as\ k \to \infty.$$

Theorem 2.7. [12] *If $p < q$, $\gamma > 0$ then $\int_p^q e^{-k\gamma(x-p)^2}dx \to \frac{1}{2}\sqrt{\frac{\pi}{k\gamma}}$ as $k \to \infty$.*

3. Properties of Existence

We consider the problem of existence of Sumudu transform as a Henstock-Kurzweil integral.If function f be locally integrable i.e. f $\in \mathcal{S}_{Loc}(o,\infty)$ and f is of exponential order,that is, for some constants M; $t_0 > 0$ and real k, f satisfies $|f(t)| \leq Me^{kt}$ for all $t \geq t_0$[13]. However we give different existential conditions in the section.

Theorem 3.1. *If $f : [0,\infty) \to R$ be continuous function such that $F(x) = \int_0^x f; 0 \leq x < \infty$ is bounded on $[0,\infty)$, then Sumudu Transform $\mathcal{S}\{f(t)\}(v)$ of $f(t)$ exists. That is,*

$\mathcal{S}\{f(t)\}(v) = 1/v \int_0^\infty f(t)e^{-t/v}dt; \quad Re(\frac{1}{v} > 0.$

Proof. Since the function $e^{-t/v} : [0,\infty) \to \mathbb{R}$ is continuous on $[0,\infty)$ with $\lim_{t\to\infty} e^{-t/v} = 0$, $Re(\dfrac{1}{v}) > 0$, and $(e^{-t/v})'$ is absolutely integrable over $[0,\infty)$. The result follows from Du Bois - Reymond Test [7].

Theorem 3.2. *If $\mathcal{S}\{f(t)\}(v) = 1/v \int_0^\infty e^{-t/v}dt$ exists for $Re(\frac{1}{v}) > 0$, then $f(t) \in \mathcal{HK}_{Loc}$.*

Proof. Note that the function $t \in e^{-t/v}$, $\frac{1}{v} \in \mathbb{C}$, as function of real variable t is not of bounded variation on $[0,\infty)$ but is of bounded variation on any compact interval $[a,b]$.

Theorem 3.3. *Let $f \in \mathcal{HK}_{LOC}$ then the Sumudu Transform of $f(x)$ exists, if $f \in \mathcal{HK}([0,\infty)) \cap \mathcal{BV}(\infty)$.*

Proof. Proof follows from [9] and [11].

4. Applications

The usual algebraic properties of linearity, translation, dilation, modulation etc. familiar from the Henstock - Kurzweil Sumudu transform, proof of which are easy. Also some other reasons which are analogous to the classical way with some changes in the hypothesis which we discuss below. In this section we find the Henstock - Kurzweil Sumudu transform of derivative of functions.

4.1. Henstock - Kurzweil Sumudu Transform of Derivative.

Theorem 4.1. *If $f : [0,\infty) \to (\mathbb{R})$ be a function and $f(t) \in \mathcal{ACG^*}_\delta(\mathbb{R})$ such that $\lim_{t\to\infty} \dfrac{f(t)}{e^{t/v}} = 0$, then $\frac{1}{v} \in \mathbb{C}$, both $S\{f(t)\}(v)$ and $S\{f^{'}(t)\}(v)$ fail to exist or $S\{f^{'}(t)\}\frac{1}{v} = \frac{S\{f(t)\}-f(0)}{v}$.*

Proof.

$$G_1(V) = F_1(1/v)$$
$$G_1(V) = \frac{G(v) - f(0)}{v}$$
$$G_1(V) = \frac{1}{v}G(v) - f(v)$$
$$G_1(V) = \mathcal{S}\{f(t)\} - \frac{f(0)}{v}$$
$$\mathcal{S}\{f^{'}(t)\}\frac{1}{v} = \frac{\mathcal{S}\{f(t)\} - f(0)}{v}$$

by fact that derivative of $\mathcal{ACG^*}_\delta$ function is always Henstock - Kurzweil integrable and Hakes theorem [1].

generally considered condition on f as for $k = 0, 1, ..., (n-1)$, $f^{(k)}$ are $\mathcal{ACG^*}_\delta(\mathbb{R})$ and $\lim_{t\to\infty} \frac{f^{(k)}(t)}{e^{t/v}} = 0$.

4.2. Henstock - Kurzweil Sumudu Transform of $tf(t)$.

Theorem 4.2. *Let $f : [0, \infty) \to \mathbb{R}$ be function such that $\mathcal{S}\{f(t)\}(v) = \bar{f}(1/v)$ defined on a compact interval that is $[\xi, \zeta], \xi, \zeta \in \mathbb{R}$. Define $g(t) = tf(t)$. If $g(t) \in \mathcal{HK}(\mathbb{R})^+$. then $\mathcal{S}\{g(t)\}(v)$ exists and*

$$\mathcal{S}\{g(t)\}(v) = [\mathcal{S}\{f(t)\}(1/v)]' a.e., \frac{1}{v} \in \mathbb{R}.$$

Proof.
Suppose that

$$\mathcal{S}\{f(t)\}(1/v) = \bar{f}(1/v) = \int_0^\infty e^{-t/v} f(t)dt$$

exists on $[\xi, \zeta]$, $\xi, \zeta \in \mathbb{R}$. Define $g(t) = tf(t)$, for differentiating under the Henstock - Kurzweil integral we have the necessary and sufficient condition

$$\int_{t=0}^\infty \int_{v=p}^q e^{-t/v} tf(t)dv\, dt = \int_{v=p}^q \int_{t=0}^\infty e^{-t/v} f(t)dt\, dv$$

for all $[p, q] \in [\xi, \zeta]$ [10]. Observe that the L.H.S. of above equation exists therefore by Lemma 25(a)[9] above equation hold. Hence we differentiate under the Henstock - Kurzweil integral, we get

$$\int_{v=a}^q \int_{t=0}^\infty e^{-t/v} tf(t)dt\, dv = \int_p^q (\bar{f}(1/v))' ds$$

for all $[p, q] \in [\xi, \zeta]$. Therefore

$$\mathcal{S}\{tf(t)\}(v) = v[\mathcal{S}\{f(t)\}(1/v)]' \ a.e.$$

on $[\xi, \zeta]$. Hence the derivative of the Sumudu Transform of $f(t)$ exists a.e. on $[\xi, \zeta]$.

We can generalize this result as follows: If for $n \in \mathbb{N}, t^n f(t)$ is a Henstock - Kurzweil integrable then

$$\mathcal{S}\{t^n f(t)\}$$

exists and

$$\mathcal{S}\{t^n f(t)\} = v^n\{\mathcal{S}(f(t))\}^n$$

. Hence derivative of any order of $\mathcal{S}\{f(t)\}$ exists a.e. So $\mathcal{S}\{f(t)\}$ is analytic a.e.

4.3. Henstock - Kurzweil Sumudu Transform of $\frac{1}{t}$.

Theorem 4.3. *Let $f : [0, \infty) \in \mathbb{R}$, if $\mathcal{S}\{f(t)\}(v) = \bar{f}(1/v)$ exists and $\frac{f(t)}{t} \in \mathcal{HK}(\mathbb{R})^+$ then $\mathcal{S}\{f(t)\}v$ exists and*

$$\int_0^\infty e^{-t/v} \frac{f(t)}{t} dt = \int_v^\infty \bar{f}(\tau)d\tau, \quad \frac{1}{v} \in \mathbb{R}.$$

Proof.
Since $\frac{f(t)}{t}$ is Henstock-Kurzweil integral and $e^{-t/v}$ is of bounded variation on compact interval then $\mathcal{S}\{f(t)\}v$ exists and by Lemma 25(a) in [8], we have

$$\int_0^\infty e^{-t/v}\frac{f(t)}{t}dt = \int_{1/v}^\infty \bar{f}(\tau)d(\tau).$$

4.4. Henstock - Kurzweil Transform of Integration.

Theorem 4.4. *Let $f : [0,\infty) \in \mathbb{R}$ be Henstock - Kurzweil integrable function and $\mathcal{S}\{f(t)\}(v)$ exists, then*
$\mathcal{S}\{\int_0^t f(\tau)d\tau\}(v)$ exists and

$$\mathcal{S}\{\int_0^t f(\tau)d\tau\}(v) = vG(v)\{v\mathcal{S}[f(t)]\}.$$

Proof.
Let $f(t) = \int_0^t f(\tau)d\tau,\ t \in \mathbb{R}$.
and if f is Henstock-Kurzweil integrable, then we have, $\lim\limits_{t\to\infty} F(t)$
bounded a.e. on $\mathbb{R}$ and by using integration by parts we get

$$\mathcal{S}\{\int_0^t f(\tau)d\tau\}(v) = vG(v)\{v\mathcal{S}[f(t)]\}.$$

5. Inversion

In order to obtain inverse of Henstock - Kurzweil Sumudu Transform we use Post's generalized differentiation method [6].

Theorem 5.1. *Let $0 < \omega < q - p$ and $\psi(x) \in \mathcal{C}^2$, $(p \le x \le p+\omega)$, $\psi^{'}(p) = 0$, $\psi^{''}(p) < 0$, $\psi(x)$ is non increasing on $(p,q]$. If $f(x) \in \mathcal{HK}([p,q])$, then*

$$\int_p^q f(x)e^{k\psi(x)}dx \to f(p)e^{k\psi(p)}(\frac{-\pi}{2k\psi"(p)})^{\frac{1}{2}}$$

as $k \to \infty$.

Proof. Consider the integral

$$I_k = \int\limits_p^q [f(x) - f(p)]e^{k[\psi(x)-\psi(p)]}dx.$$

We show that $I_k \to 0$ as $n \to \infty$

$$g_k(x) = \begin{cases} e^{k[\psi(x)-\psi(p)} & \text{if} \quad x \in (p,q] \\ o & \text{if} \quad x = p \end{cases}$$

Then $g_k(x) = e^{k(\psi(x)-\psi(p)} \to 0$, $\forall\ x\ \epsilon\ [p,q]$ as $k \to \infty$ is of uniform bounded variation on $\mathcal{R}$ with $g_k(x) \to 0$ as $k \to \infty$. Now by Theorem 2.3, we say that $I_k \to 0$ as $k \to \infty$ and the result follows by Theorem 2.6.

Theorem 5.2. *Suppose $0 < \omega < q-p$, and $\psi(x) \in \mathcal{C}^2$, $(q-\omega \leq x \leq q)$, $\psi^{'}(q) = 0, \psi^{"}(q) < 0, \psi(x)$ non decreasing on $[p,q)$. If $f(x) \in \mathcal{HK}([p,q])$, then*

$$\int_p^q f(x)e^{k\psi(x)}dx \to f(q)e^{k\psi(q)}(\frac{-\pi}{2k\psi^{"}(q)})^{\frac{1}{2}}$$

as $k \to \infty$.

Similarly this result can be proved.

Theorem 5.3. ***Inversion Theorem***

If $f(x) \to \mathcal{HK}((0,\infty))$ and if the Sumudu Transform of $f(t)$, $\mathcal{S}\{f(t)\}(v) = \frac{1}{v}\int_0^\infty e^{-\frac{t}{v}}f(t)dt$ exists , then

$$\lim_{k\to\infty}\frac{1}{k!}(\frac{k}{t})^{k+1}\int_0^\infty \frac{1}{z}e^{\frac{-kz}{t}}z^k f(z)dz = f(t)$$

Proof. The Inversion Theorem follows immediately, if we prove the following,

(1) If
$f(t) \to \mathcal{HK}([t,\infty))$, $0 < t \leq x < \infty$,
then

$$\lim_{k\to\infty}(\frac{k}{t})^{k+1}\int_t^\infty \frac{1}{z}e^{\frac{-kz}{t}}z^k f(z)dz = \frac{f(t)}{2}$$

(2) If
$f(t) \to \mathcal{HK}((0,t])$, $(0 < x \leq t)$,
then

$$\lim_{k\to\infty}(\frac{k}{t})^{k+1}\int_0^t \frac{1}{z}e^{\frac{-kz}{t}}z^k f(z)dz = \frac{f(t)}{2}.$$

We have the relation

$$\int_p^q f(t)e^{k\psi(x)}dx = f(p)e^{k\psi(p)}\left(\frac{-\pi}{2k\psi^{"}(p)}\right)^{\frac{1}{2}}$$

as $k \to \infty$. Take $p = t$ so then

$$\int_t^q f(t)e^{k\psi(x)}dx = f(t)e^{k\psi(t)}\left(\frac{-\pi}{2k\psi^{"}(t)}\right)^{\frac{1}{2}}.$$

By using Stirling's Formula and taking $\psi(x) = Inx - \frac{x}{t}$ we get

$$\lim_{k\to\infty}\frac{1}{k!}\left(\frac{k}{t}\right)^{k+1}\int_t^q f(x)e^{\frac{-kx}{t}}x^k dx = \frac{f(t)}{2}.$$

Applying Hake's theorem we get

$$\lim_{k\to\infty}\frac{1}{k!}\left(\frac{k}{t}\right)^{k+1}\int_t^\infty f(x)e^{\frac{-kx}{t}}x^k dx = \frac{f(t)}{2}.$$

Similarly we prove the second result. Hence the result.

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References

[1] Bartle R. G.: *A Modern Theory of Integration,* Grad. Studies in Math, Vol. 32 Amer, Math Soc. Providence; (2001).

[2] Belgacen Fethi bin muhammed, Karaballi Ahmed Abdullatif, Kalla Shyam L.: *Analytical Investigations of the Sumudu Transform and Applications to Integral Production Equations,* Mathematical Problems in Engineering. (2003).

[3] Choudhary M. S., Tikare Sanket A.: *On Gauge Laplace Transform,* Int. Journal of Math. Analysis Vol. 5,(2011).

[4] Erdlyi A.: *Tables of Integral Transforms,*Vol. 1, McGraw Hill Book Co., Inc,(1954).

[5] Gordon R. A.: *The Integrals of Lebsgue, Denjoy, Perron and Henstock,* Grad. Studies in Math, Vol. 4 Amer. Math. Soc. Providence,(1994).

[6] Post E. L.: *Generalized Diferentiation,*Trans. Of the Amer. Math. Soc., 32(1930), 723 – 781,(1930).

[7] Swartz C.: *Introduction to Gauge Integrals,* World Scientific, Singapore;(2001).

[8] Talvila E.: *Necessary and sufficient conditions for differentiating under the Integral sign,* Amer. Math. Monthly, 108.(2001).

[9] Talvila E.: *Henstock – Kurzweil Fourier Transform,* Illinois J. of Math., 46, 4(2002), 1207 – 1226

[10] Talvila E.: *Limits and Henstock Integrals of Products,* Real Anal. Exchange, 25(1999 – 2000), 907 – 918.

[11] Torres F.J.M.,Reyna J.A.E.,Cardenas MA G.R.: About an existence theorem of the Henstock-Kurzewil Fourier transform, Proyecciones, 27, 3,(2008), 307-318.

[12] Widder D. V.: *The Laplace Transform,*; Princeton University Press, Princeton(1946).

[13] Zayed A. I.: *Hand Book of Function and Generalized Function Transformations,*CRC Press, Inc,(1996).

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