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NEW SUBCLASS ANALYTIC FUNCTIONS ASSOCIATED WITH POLYLOGARITHM FUNCTION DEFINED BY A LINEAR DIFFERENTIAL OPERATOR

M. THIRUCHERAN, M. VINOTH KUMAR*, AND T. STALIN**

ABSTRACT. In this paper, we introduce and study a new subclass $S^n_{\beta,\lambda,\delta,b}(\alpha)$, involving polylogarithm functions which are associated with differential operator. we also obtain coefficient estimates, distortion theorem, radius of starlikeness and covex and extreme point of the class $S^n_{\beta,\lambda,\delta,b}(\alpha)$.

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1. Introduction

Let A represent the class of analytic function $\mathtt{f}(z)$ which is normalized by $\mathtt{f}(0) = \mathtt{f}'(0) - 1 = 0$ $\mathtt{f}(z)$ of the form

$$\mathtt{f}(z) = z + \sum_{k=2}^{\infty} a_k z^k, \tag{1.1}$$

which are analytic in the unit disk $U\{= z : \ |z| < 1\}$. The Hadamard product of

$$\mathtt{g}(z) = z + \sum_{k=2}^{\infty} b_k z^k, \tag{1.2}$$

and $\mathtt{f}(z)$, given in (1.1), is defined by

$$(\mathtt{f} * \mathtt{g})(z) = z + \sum_{k=2}^{\infty} a_k b_k z^k. \tag{1.3}$$

The standard polylogarithm function was studied by Leibniz and Bernoulli in 1969 [9]. For $\lambda \in \mathbf{N}$, with $\lambda \geq 2$, the polylogarithm function (which is a absolutely convergent series) is defined by

$$Li_\lambda(z) = \phi_\lambda(z) = \sum_{k=1}^{\infty} \frac{z^k}{(k)^\lambda}. \tag{1.4}$$

Recently, Auther in [10], considered various functional identities by using polylogarithm functions.

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** Corresponding Author.

For $\lambda \in \mathbf{N}$, Re$\lambda > 1$ and Re $c > -1$ the λ^{th} order polylogarithm function is defined by

$$\phi_\lambda(c;z) = \sum_{k=1}^{\infty} \frac{z^k}{(k+c)^\lambda} = \frac{1}{\Gamma(\lambda)} \int_0^1 z\left(log\left(\frac{1}{t}\right)\right)^{\lambda-1} \frac{t^c}{1-tz}dt. \tag{1.5}$$

For $\mathtt{f} \in$ A, Al-Shaqsi [2] introduced the following operator.

$$\psi_\lambda(c;z) = (1+c)^\lambda \phi_\lambda(c,z) * \mathtt{f}(z) = \frac{(1+c)^\lambda}{\Gamma(\lambda)} \int_0^1 t^{c-1}\left(log\left(\frac{1}{t}\right)\right)^{\lambda-1} \mathtt{f}(tz)dt, \tag{1.6}$$

where $\lambda \in \mathbf{N}$, Re $\lambda > 1$ and Re $c > 0$.

Now a days, the work using polylogarithm has been intensified brightly owing to its importance in several fields of mathematics, such as algebra, topology, geometry and quantum theory [6, 7].

Recntly Al-Shaqsi and Darus [2], Danyal Soybas Santosh B. Joshi and Haridas Pawar [8],S. Oi [11], Al-Shaqsi and Darus [12], T. Stalin et al. [13] and M. Thirucheran et al. [14] generalized Ruscheweyh and Salagean operators using polylogarithm functions on class $\mathcal{A}$ of analytic functions in the open unit disc $\mathbb{U} = \{z : |z| < 1\}$.

By motivated by the aforementioned work, we introduce the new subclass involving differential operator as below:

For $\mathfrak{f}(\xi) \in \mathcal{A}$, we now introduce the linear differential operator

$$\mathcal{L}^n_{\lambda,\delta} f(\xi) = \mathcal{L}_{\lambda,\delta}\left(\mathcal{L}^{n-1}_{\lambda,\delta}\mathfrak{f}(\xi)\right) = \xi + \sum_{k=2}^{\infty} [1+(k-1)\delta]^n \left(\frac{1+c}{k+c}\right)^\lambda a_k \xi^k, \tag{1.7}$$

which is a convolution of the well known operators of Al-Oboudi [1] and Al-Shaqsi [2].

Note that,

(i). $\mathcal{L}^n_{0,1} = \mathcal{D}^n$, Salagean [3]

(ii). $\mathcal{L}^n_{0,\delta} = \mathcal{D}^n_\delta$, Al-Oboudi [1]

(iii). $\mathcal{L}^0_{\lambda,\delta} = \psi_\lambda(c,\xi)$, Al-shaqsi [2].

(iv). $\mathcal{L}^0_{0,\delta} = \mathfrak{f}(\xi)$ and $\mathcal{L}^1_{0,1} = \xi\mathfrak{f}'(\xi)$, Ma.W and D.Minda [5].

Furthermore details about polylogarithm functions see Ponnusamy.S [**?**].

Hence,

$\mathcal{L}^0_{\lambda,\delta}\mathfrak{f}(\xi) = \xi + \sum_{k=2}^{\infty} \left(\frac{1+c}{k+c}\right)^\lambda a_k\xi^k$,

$\mathcal{L}^1_{\lambda,\delta}\mathfrak{f}(\xi) = (1-\delta)\psi_\lambda(c,\xi) + \delta\xi(\psi_\lambda(c,\xi))' = \xi + \sum_{k=2}^{\infty}[1+(k-1)\delta]\left(\frac{1+c}{k+c}\right)^\lambda a_k\xi^k$
$= \mathcal{L}_{\lambda,\delta}\mathfrak{f}(\xi)$, $\mathcal{L}^2_{\lambda,\delta}\mathfrak{f}(\xi) = \mathcal{L}_{\lambda,\delta}(\mathcal{L}_{\lambda,\delta}\mathfrak{f}(\xi))$,

similarly,

$\mathcal{L}^n_{\lambda,\delta} f(\xi) = \mathcal{L}_{\lambda,\delta}\left(\mathcal{L}^{n-1}_{\lambda,\delta}\mathfrak{f}(\xi)\right) = \xi + \sum_{k=2}^{\infty}[1+(k-1)\delta]^n\left(\frac{1+c}{k+c}\right)^\lambda a_k\xi^k$.

Let p be the class of functions of the form $p(\zeta) = 1 + p_1\zeta + p_2\zeta^2 + ...$,analytic in $\mathcal{U}$, which satisfy $Re\{p(\zeta)\} > 0$.

Definition 1.1. A function $\mathfrak{f}(\xi) \in \mathcal{A}$, given by (1.1), is said to be in the class $\mathcal{S}^n_{\beta,\gamma,\delta,b}(\phi(\xi))$, which satisfies

$$1+\frac{1}{b}\left(\frac{\xi\mathcal{L}^n_{\lambda,\delta}f(\xi))'}{\mathcal{L}^n_{\lambda,\delta}\mathfrak{f}(\xi)}-1\right)-\beta\left|\frac{\xi\mathcal{L}^n_{\lambda,\delta}f(\xi))'}{\mathcal{L}^n_{\lambda,\delta}\mathfrak{f}(\xi)}-1\right|\prec\phi(\xi), \tag{1.8}$$

where $n,\lambda\in N_0,\beta>0,\delta>0,b>0,\phi\in p$.

Remark 1.2. For, $\phi(\xi)=\frac{1+(1-2\alpha)\xi}{(1-\xi)}$, then $\mathcal{S}^n_{\beta,\gamma,\delta,b}(\phi(\xi))\equiv\mathcal{S}^n_{\beta,\gamma,\delta,b}(\alpha)$ be the class of $\mathfrak{f}\in\mathcal{A}$ satisfies

$$Re\left(1+\frac{1}{b}\left(\frac{\xi\mathcal{L}^n_{\lambda,\delta}f(\xi))'}{\mathcal{L}^n_{\lambda,\delta}\mathfrak{f}(\xi)}-1\right)-\beta\left|\frac{\xi\mathcal{L}^n_{\lambda,\delta}f(\xi))'}{\mathcal{L}^n_{\lambda,\delta}\mathfrak{f}(\xi)}-1\right|\right)>\alpha, \tag{1.9}$$

where $n,\lambda\in N_0,\beta>0,\delta>0,b>0,\phi\in p,0\le\alpha\le1$.

2. The Second Section

Theorem 2.1. *Let* $\mathtt{f}$ *be defined by* (1.9). *Then* $\mathtt{f}\in\mathcal{S}^n_{\beta,\gamma,\delta,b}(\alpha)$ *if and only if*

$$\sum_{k=2}^{\infty}(kb\beta-b\beta-k+1-b+b\alpha)[1+(k-1)\delta]^n\left(\frac{1+c}{k+c}\right)^{\lambda}|a_k|\le(1-\alpha)b \tag{2.1}$$

where $0\le\alpha<1,n\in N_0=N\cup\{0\},\delta>0,b>0,\lambda\ge0$.
Proof:
Suppose that the inequality (2.1) *is true and* $|z|<1$.
Then it shows that the value of
$1+\frac{1}{b}\left(\frac{z(\mathrm{D}^n_{\lambda,\delta}\mathtt{f}(z))'}{\mathrm{D}^n_{\lambda,\delta}\mathtt{f}(z)}-1\right)-\beta\left|\frac{z(\mathrm{D}^n_{\lambda,\delta}\mathtt{f}(z))'}{\mathrm{D}^n_{\lambda,\delta}\mathtt{f}(z)}-1\right|$ *lies in a circle with center* $|w|=1$ *and radius* $(1-\alpha)b$,
which gives $\sum_{k=2}^{\infty}(kb\beta-b\beta-k+1-b+b\alpha)[1+(k-1)\delta]^n\left(\frac{1+c}{k+c}\right)^{\lambda}|a_k|\le(1-\alpha)b$.
Hence $\mathtt{f}(z)$ *satisfies the condition* (2.1).
Conversely,
Let us assume that the function $\mathtt{f}$ *defined by* (1.9) *in the class* $\mathrm{S}^n_{\beta,\gamma,\delta,b}(\alpha)$, *then*
$Re\left(1+\frac{1}{b}\left(\frac{z(\mathrm{D}^n_{\lambda,\delta}\mathtt{f}(z))'}{\mathrm{D}^n_{\lambda,\delta}\mathtt{f}(z)}-1\right)-\beta\left|\frac{z(\mathrm{D}^n_{\lambda,\delta}\mathtt{f}(z))'}{\mathrm{D}^n_{\lambda,\delta}\mathtt{f}(z)}-1\right|\right)>\alpha$,
if we choose the value of z *on the real azs and let* $z\to1^-$ *through real values, we obtain*
$\sum_{k=2}^{\infty}(kb\beta-b\beta-k+1-b+b\alpha)[1+(k-1)\delta]^n\left(\frac{1+c}{k+c}\right)^{\lambda}|a_k|\le(1-\alpha)b$.
Hence the result is sharp.

3. Extreme Points

Theorem 3.1. *Let* $\mathtt{f}_1(z)=z,\mathtt{f}_k(z)=z+\sum_{k=2}^{\infty}\eta_k\frac{(1-\alpha)b}{\psi(\lambda)}z^k,k=2,3,....,$
where $\psi(\lambda)=\sum_{k=2}^{\infty}(kb\beta-b\beta-k+1-b+b\alpha)[1+(k-1)\delta]^n\left(\frac{1+c}{k+c}\right)^{\lambda}$. *Then* $\mathtt{f}\in\mathrm{S}^n_{\beta,\lambda,\delta,b}(\alpha)$ *if and only if it can be expressed in the form* $\mathtt{f}(z)=\sum_{k=1}^{\infty}\eta_k\mathtt{f}_k(z)$, *where* $\eta_k>0$ *and* $\sum_{k=1}^{\infty}\eta_k=1$.
Proof: *Let* $\mathtt{f}(z)=\sum_{k=1}^{\infty}\eta_k\mathtt{f}_k(z)$

$= z + \sum_{k=2}^{\infty} \eta_k \frac{(1-\alpha)b}{\psi(\lambda)} z^k$
$= \sum_{k=2}^{\infty} \eta_k \frac{(1-\alpha)b}{\psi(\lambda)} (\psi(\lambda)$
$= (1-\alpha)b \sum_{k=1}^{\infty} \eta_k$
$= (1-\alpha)b(1-\eta_1)$
$< (1-\alpha)b,$
which shows that $\mathtt{f} \in S^n_{\beta,\lambda,\delta,b}(\alpha)$.
Conversely,
Suppose that $\mathtt{f} \in S^n_{\beta,\lambda,\delta,b}(\alpha)$.
Since $|a_k| \leq \frac{(1-\alpha)b}{\psi(\lambda)}, k = 2, 3, ...$
Let $\eta_k \leq \frac{\psi(\lambda)}{(1-\alpha)b}, \eta_1 = 1 - \sum_{k=2}^{\infty} \eta_k$.
Then we obtain $\mathtt{f}(z) = \sum_{k=1}^{\infty} \eta_k \mathtt{f}_k(z)$.

4. Radius of Starlikeness and Convexity

Theorem 4.1. *The class* $S^n_{\beta,\lambda,\delta,b}(\alpha)$ *is convex .*
Proof:

Let the function $\mathtt{f}_j(z) = z + \sum_{k=2}^{\infty} a_{k,j} z^k, a_{k,j} \geq 0, j = 1, 2$ *lie in the class* $\mathtt{f} \in S^n_{\beta,\lambda,\delta,b}(\alpha)$, *it is sufficient to prove that* $h(z) = (\gamma+1)\mathtt{f}_1(z) - \gamma \mathtt{f}_2(z) \in S^n_{\beta,\lambda,\delta,b}(\alpha)$.
Since $h(z) = z + \sum_{k=2}^{\infty} [(1+\gamma)a_{k,1} - \gamma a_{k,2}] z^k$, *which implies that*

$$\sum_{k=2}^{\infty}(kb\beta - b\beta - k + 1 - b + b\alpha)[1+(k-1)\delta]^n \left(\frac{1+c}{k+c}\right)^{\lambda} (1+\gamma)a_{k,1}$$
$$-(kb\beta - b\beta - k + 1 - b + b\alpha)[1+(k-1)\delta]^n \left(\frac{1+c}{k+c}\right)^{\lambda} \gamma a_{k,2}$$
$$\leq (1+\gamma)(1-\alpha)b - \gamma(1-\alpha)b$$
$$\leq (1-\alpha)b.$$

Therefore $h \in S^n_{\beta,\lambda,\delta,b}(\alpha)$.
Hence $S^n_{\beta,\lambda,\delta,b}(\alpha)$ *is convex.*

Theorem 4.2. *Let* $\mathtt{f}(z) = z + \sum_{k=2}^{\infty} |a_k| z^k$, $\mathtt{f} \in S^n_{\beta,\lambda,\delta,b}(\alpha)$, *then* $\mathtt{f}$ *is close-to-convex of order* $\sigma (0 \leq \sigma < 1)$ *in the disc* $|z| < r_1$,
where $r_1 := \left(\frac{(1-\sigma)[(kb\beta - b\beta - k + 1 - b + b\alpha)[1+(k-1)\delta]^n \left(\frac{1+c}{k+c}\right)^{\lambda}]}{(k)(1-\alpha)b} \right)^{\frac{1}{k-1}}$.

Theorem 4.3. *Let* $\mathtt{f}(z) = z + \sum_{k=2}^{\infty} |a_k| z^k$, $\mathtt{f} \in S^n_{\beta,\lambda,\delta,b}(\alpha)$, *then* $\mathtt{f}$ *is starlike of order* σ *in the disc* $|z| < r_2$,
where $r_2 := inf \left(\frac{(1-\sigma)[(kb\beta - b\beta - k + 1 - b + b\alpha)[1+(k-1)\delta]^n \left(\frac{1+c}{k+c}\right)^{\lambda}]}{(k-\sigma)(1-\alpha)b} \right)^{\frac{1}{k-1}}, (k \geq 2)$.

Theorem 4.4. *Let* $\mathtt{f}(z) = z + \sum_{k=2}^{\infty} |a_k| z^k$, $\mathtt{f} \in S^n_{\beta,\lambda,\delta,b}(\alpha)$, *then* $\mathtt{f}$ *is convex of order* σ
in the disc $|z| < r_3$,
where $r_3 := inf \left(\frac{(1-\sigma)[(kb\beta - b\beta - k + 1 - b + b\alpha)[1+(k-1)\delta]^n \left(\frac{1+c}{k+c}\right)^{\lambda}]}{k(k-\sigma)(1-\alpha)b} \right)^{\frac{1}{k-1}}, (k \geq 2)$.

5. Distortion Theorem

Theorem 5.1. *Let* $\mathtt{f}(z) = z + \sum_{k=2}^{\infty} |a_k|\, z^k$, $\mathtt{f} \in \mathrm{S}^n_{\beta,\lambda,\delta,b}(\alpha)$,*then for* $|z| = r$, *we have*

$$r - \frac{(1-\alpha)b\left(\frac{1+c}{2+c}\right)^{\lambda}}{(\beta b+\alpha b-b-1)(1+\delta)^n}r^2 \le |\mathtt{f}(z)| \le r + \frac{(1-\alpha)b\left(\frac{1+c}{2+c}\right)^{\lambda}}{(\beta b+\alpha b-b-1)(1+\delta)^n}r^2, \quad (5.1)$$

$$and\ 1 - \frac{2(1-\alpha)b\left(\frac{1+c}{2+c}\right)^{\lambda}}{(\beta b+\alpha b-b-1)(1+\delta)^n}r \le |\mathtt{f}'(z)| \le 1 + \frac{2(1-\alpha)b\left(\frac{1+c}{2+c}\right)^{\lambda}}{(\beta b+\alpha b-b-1)(1+\delta)^n}r. \quad (5.2)$$

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M. Thirucheran: Post Graduate and Research Department of Mathematics, L N Government College, Ponneri, Chennai - 624 302, University of Madras, Tamil Nadu, INDIA.

Email address: drthirucheran@gmail.com

M. Vinoth Kumar: Post Graduate and Research Department of Mathematics,L N Government College, Ponneri, Chennai - 624 302, University of Madras, Tamil Nadu, INDIA.

Email address: vinnikil13@gmail.com

T. Stalin: Department of Mathematics, Vel Tech Ranga Sanku Arts College,, Avadi,Chennai-600 062, University of Madras,Tamil Nadu, INDIA.

Email address: goldstaleen@gmail.com