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STRUCTURES OF TERNARY SEMINEAR RING

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Abstract. The notion of regular ternary seminear rings, normal ternary seminear rings, idempotents in ternary seminear rings and idempotent pairs in ternary seminear rings are defined and their characterization are derived. Commutative ternary seminear rings are also defined and their properties are obtained.

1. Introduction

The regular ternary semirings was fathered by Dutta.T.K. and Kar.S.[2]. Furthermore, various great research was done and is being done by many authors[1],[3],[10] in the area of ternary semirings. Additively regular seminear rings was initiated by Sardar.S.K. and Mukherjee.R.,[7][8]. Radha.D.[5][6] came up with the notion of commutative seminear rings. Perumal.R., Arulprakasam.R and Radhakrishnan.M.,[4] introduced a normal seminear rings.The ternary seminear ring was first fathered by us [11], where the existence of one distributive law proves which is very useful in coordinating certain important classes of geometric planes (ie., Descarte's method of coordinating the usual plane by the field of real numbers was one of the most successful steps in geometry). The cancellative ternary seminear ring was discussed by us [12].

In this paper we introduces structures of ternary seminear rings like idempotent in ternary seminear rings, idempotent pairs in ternary seminear rings, normal ternary seminear rings and regular ternary seminear rings and derive their properties.Also commutative ternary seminear rings, quasi commutative ternary seminear rings and Pseudo commutative ternary seminear rings are studied and their characterisations are obtained.

2. Preliminaries

This section recalls some basic definitions which are used in this paper.

Definition 2.1. A seminear ring R will be called regular if for every $a \in R$ there exists $b \in R$ such that $a = aba$.

Definition 2.2. An element a of ternary S is completely regular if there exists x in S satisfying the following conditions: i) a=a+x+a ii) aa(a+x)=a+x.

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Definition 2.3. An additively completely regular seminearring S, for $a \in S$, an element $x \in S$ satisfying a+x+a=a and a+x = x+a.

Definition 2.4. An element a in a seminear ring R is said to be idempotent if $a^2 = a$.

Definition 2.5. A seminear ring R is called a left(right) normal seminear ring if $a \in Ra(aR)$ for each $a \in R$. The seminear ring R is normal if it is both left normal and right normal.

Definition 2.6. A seminear ring N is called Pseudo commutative seminear ring if xyz=zyx for every $x, y, z \in N$.

Definition 2.7. A ternary seminear ring is a nonempty set T together with a binary operation called addition '+' and a ternary operation called ternary multiplication denoted by juxtaposition, such that $(i)(T, +)$ is a semigroup. (ii)T is a ternary semigroup under ternary multiplication.
$(iii)xy(z + u) = xyz + xyu$ for all $x, y, z, u \in T$.

Definition 2.8. A ternary seminear ring T is said to have an absorbing zero if there exists an element $0 \in T$ such that $i)x + 0 = 0 + x = x$ for all $x \in T$. $ii)xy0 = x0y = 0xy = 0$ for all $x, y \in T$.

Remark 2.9. Throughout this paper T will always stand for the ternary seminear ring and it will always mean that ternary seminear ring with an absorbing zero.

Definition 2.10. Let T and U be two ternary seminear rings and $\phi : T \to U$ be a mapping. ϕ is said to be a homomorphism of T into U if (i) $\phi(x+y) = \phi(x)+\phi(y)$ and (ii) $\phi(xyz) = \phi(x)\phi(y)\phi(z)$ for all x,y,z $\in$ T.

Definition 2.11. Let T and U be two ternary seminear rings and $\phi : T \to U$ be a homomorphism If $T = U$ then ϕ is said to be an endomorphism. If ϕ is also onto then it is called an epimorphism. If ϕ is also one to one then it is called monomorphism. If ϕ is epimorphism as well as monomorphism then it is said to be an isomorphism. An isomorphism ϕ is said to be an automorphism if T=U.

Definition 2.12. Let T and U be two ternary seminear rings. If $\phi : T \to U$ is an isomorphism then T is said to be isomorphic to U and is denoted as T $\cong$ U.

3. Structures of Ternary Seminear Rings

In this section some structures of ternary seminear rings are defined and some of their properties are derived.

Definition 3.1. An element x in a ternary seminear ring T is said to be an Idempotent element if $x^3 = x$. A ternary seminear ring T is said to be an idempotent ternary seminear ring if every element of T is idempotent, whereas the set of all idempotents of T is denoted by E.

Definition 3.2. A ternary seminear ring T is said to be a zero divisor if for $x, y, z \in T, xyz = 0$ which implies that $x = 0$ or $y = 0$ or $z = 0$.

Definition 3.3. Let T be a ternary seminear ring, then $C(T) = \{a \in T/xya = axy = xay \ \forall x, y \in T\}$ is called as center of T.

Definition 3.4. An idempotent $e \in E$ is called a central idempotent of a ternary seminear ring if $e \in C(T)$. If every e in E is a central idempotent of a ternary seminear ring T, then we can say $E \subseteq C(T)$.

Definition 3.5. An element $e \in T$ is called a multiplicative identity if $eex = exe = xee = x$ for every $x \in T$

Definition 3.6. A ternary seminear ring T is called Boolean if $x^3 = x$ for all $x \in T$ (i.e. Every element of T is idempotent).

Definition 3.7. Let T be a ternary seminear ring. A pair (x,y) of elements x,y in T is said to be an idempotent pair if $xy(xyz) = xyz$ and $(zxy)xy = zxy$, for all $z \in T$.

Definition 3.8. A pair (x,y) in T is said to be a natural idempotent pair if $xyx = x$ and $yxy = y$.

Definition 3.9. Let (x,y) and (u,v) be two idempotent pairs in T. They are said to be equivalent ie., $(x, y)\rho(u, v)$ if $xyz = uvz$ and $zxy = zuv$ for all $z \in T$.

Remark 3.10. An idempotent pair (x,y) containing an equivalence class is denoted as ($\overline{x, y}$)
Let $E(T)$ be the set of all equivalence classes of an idempotent pairs in T.

Theorem 3.11. *Let T be a ternary seminear ring. Then each idempotent pair (x,y) in T is equivalent to a natural idempotent pair (u,v) in T.*

Proof. Let T be a ternary seminear ring and (x,y) be an idempotent pair in T. If $u = xyx$ and $v = yxy$, then $uvu = xy(xyx)yxyx = xyxyxyx = xy(xyx)yx = xyxyx = xy(xyx) = xyx = u$. And, $vuv = yx(yxy)xyxy = yxyxyxy = yx(yxy)xy = yxyxy = yx(yxy) = yxy = v$.

Again, $uvz = xyxyxyz = xy(xyx)yz = xyxyz = xy(xyz) = xyz$. Also, $zuv = zxyxyxy = (zxy)xyxy = zxyxy = (zxy)xy = zxy$. Therefore every idempotent pair (x,y) in T is equivalent to a natural idempotent pair (u,v) in T. □

Remark 3.12. Multiplicatively cancellative (MC) means it is enough to show that the ternary cancellative operations in the cancellative ternary seminear ring[12] (ie,. we can omit the binary addition).

Theorem 3.13. *Let T be a ternary seminear ring. T is M.C if one of the following criteria holds*
i) T is (left and right) M.C.
ii) T is laterally M.C
iii) $uxu = uyu \Rightarrow x = y$*, for all* $u \in T$

Proof. To prove the theorem it is enough to show that the given criteria are equivalent. First $(i) \Rightarrow (ii)$,

$$\begin{aligned} Let\ uxv &= uyv \\ u(uxv)v &= u(yv)v \\ uu(xvv) &= uu(yvv) \\ xvv &= yvv \\ x &= y \end{aligned}$$

Obviously, $(ii) \Rightarrow (iii)$
$(iii) \Rightarrow (i)$,

$$\begin{aligned} Let\ uvx &= uvy \\ (uvx)vu &= (uvy)vu \\ u(vxv)u &= u(vyv)u \\ vxv &= vyv \\ x &= y \end{aligned}$$

This shows that the Left M.C holds in T . Similarly , we can show that the Right M.C also holds in T. □

Definition 3.14. Let e be an element in a ternary seminear ring T. e is said to be a bi-unital element if for all $x \in T$, then xee=eex=x.

Theorem 3.15. *Let T be a MC ternary seminear ring. In T any idempotent element is a bi-unital element.*

Proof. Let T be a ternary seminear ring. If e is an idempotent element in T, then $(xee)ee = xe(eee) = xee$. Therefore $(xee)ee = xee$ for all $x \in T$ $xee = x$. Similarly, we can show $eex = x$. Therefore $xee = eex = x$, which implies that e is a bi-unital element. □

Theorem 3.16. *Let T be a multiplicatively cancellative ternary seminear ring. In T idempotent pair of elements are all equivalent.*

Proof. Let T be a M.C ternary seminear ring. In T idempotent pair (x,y). By definition 3.11, we have $xy(xyz) = xyz \Rightarrow xyz = z$. Again for any idempotent pairs ((u,v), we have for all $z \in T$, then xyz=z=uvz. Similarly, zxy=z=zuv, for all $z \in T$. Therefore $(x, y)\rho(u, v)$. □

Definition 3.17. A ternary seminear ring T is called Normal if xyT=Txy for all $x, y \in T$.

Theorem 3.18. *Homomorphic images of a normal ternary seminear ring are also so.*

Proof. Let T be a normal ternary seminear ring for all $x, y \in T$ if $xyT = Txy$. Let $\phi : T \to T'$ be a homomorphism of a ternary seminear ring, where $T^{'}$ be a ternary seminear ring and if ϕ is onto, then ϕ is an epimorphism. Now, we have to prove T' is also normal, ie., $uvT' = T'uv$. Let $uvx \in uvT'$, for all $u, v \in T'$,

then $\phi(a) = u$, $\phi(b) = v$ and $\phi(c) = x$.
Therefore

$$\begin{aligned} uvx &= \phi(a)\phi(b)\phi(c) \\ &= \phi(abc) \\ &= \phi(cab) \\ &= \phi(c)\phi(a)\phi(b) \\ uvx &= xuv \in T'uv. \end{aligned}$$

where $uvx \in uvT'$ and $xuv \in T'uv$, then $uvT' \subset T'uv$. Similarly $T'uv \subset uvT'$. Thus $uvT' = T'uv$. Therefore T' is normal ternary seminear ring. Hence the homomorphic images of a normal ternary seminear ring are also normal ternary seminear ring. □

Definition 3.19. An element x in a ternary seminear ring T is called regular if there exists an element y in T such that xyx=x. A ternary seminear ring T is called regular if all of its elements are regular.

Definition 3.20. An element x of a ternary seminear ring $(T, +, .)$ is completely regular if there exists $y \in T$ satisfying the following criteria
i) x=x+y+x
ii) x+y=y+x
iii) xx(x+y)=x+y

Theorem 3.21. *Every lateral ideal of a regular ternary seminear ring T is a regular ternary seminear ring.*

Proof. Let I be a lateral ideal of a regular ternary seminear ring T. Then each $x \in I$, there exits y in T such that xyx=x. Now

$$\begin{aligned} x &= xyx \\ &= xyxyx \\ &= x(yxy)x \\ x &= xzx, \ where\ z = yxy \end{aligned}$$

which belongs to I, which implies that I is a regular ternary seminear ring. □

Theorem 3.22. *Let T be a regular ternary seminear ring. For every x in T, xy and yx are idempotents.*

Proof. Let T be a regular ternary seminear ring. For every $x \in T$, there exists $y \in T$, such that $x = xyx$. Now

$$\begin{aligned}(xy)^3 &= (xy)(xy)(xy)\\ &= xy.xy.xy\\ &= (xyx)yxy\\ &= xyxy\\ &= (xyx)y\\ (xy)^3 &= xy\end{aligned}$$

Therefore xy is idempotent.

$$\begin{aligned}(yx)^3 &= (yx)(yx)(yx)\\ &= yxyxyx\\ &= yxy(xyx)\\ &= yxyx\\ &= y(xyx)\\ (yx)^3 &= yx\end{aligned}$$

Therefore yx is also an idempotent. Hence xy and yx are idempotents. □

Theorem 3.23. *Homomorphic image of a regular ternary seminear ring is a regular ternary seminear ring.*

Proof. Let ϕ be a homomorphism from a regular ternary seminear ring T onto a ternary seminear ring U as $\phi : T \to U$, then we have to show that T is a regular ternary seminear ring suppose $z \in U$. Since ϕ is onto, there exixts $x \in T$ such that $\phi(x) = z$. Again since T is regular ternary seminear ring, there exists $y \in T$ such that x=xyx. Therefore

$$\begin{aligned}z &= \phi(x)\\ &= \phi(xyx)\\ &= \phi(x)\phi(y)\phi(x)\\ z &= zuz\end{aligned}$$

where $u = \phi(y) \in T$. Thus T is a regular ternary seminear ring. □

4. Commutative Ternary Seminear Ring

This section describes about commutative ternary seminear rings and derived some of their properties.

Definition 4.1. Let T be a ternary seminear ring. T is called commutative ternary seminear ring if xyz=yzx=zxy=yxz=zyx=xzy for all $x, y, z \in T$.

Example 4.2. The set of all negative integers with zero is a commutative ternary seminear ring.

Example 4.3. Let T={x,y,z} defined as [xyz]=x.y.z where binary operation '.' is defined as in the following table

+	x	y	z
x	x	y	z
y	y	x	y
z	z	y	x

.	x	y	z
x	x	x	x
y	y	x	x
z	x	x	x

It can be easily seen that T is a commutative ternary seminear ring.

Definition 4.4. Let T be a ternary seminear ring.T is called quasi commutative if for each $x, y, z \in T$ there exists an odd positive integer number n such that $xyz = y^n xz = yzx = z^n yx = zxy = x^n zy$.

Remark 4.5. $z^{n-2}zzxy = z^{n-2}z(zxy) = z^{n-2}z(zyx) = (z^{n-2}zz)yx = z^n yx$.

Theorem 4.6. *If T is a commutative ternary seminear ring then T is a quasi commutative ternary seminear ring.*

Proof. Suppose T be a commutative ternary seminear ring. Let $x, y, z \in T$. Now

$$xyz = yzx = zxy = yxz = zyx = xzy$$
$$xyz = y^1 xz = yzx = z^1 yx = zxy = x^1 zy$$
$$xyz = y^n xz = yzx = z^n yx = zxy = x^n zy$$

Therefore T is a quasi commutative ternary seminear ring. □

Theorem 4.7. *If T is a quasi commutative ternary seminear ring then T is a normal ternary seminear ring.*

Proof. Let T be a quasi commutative ternary seminear ring. If x,y,z $\in T$ then there exists an odd positive integer n such that $xyz = y^n xz = yzx = z^n yx = zxy = x^n zy$. We have to prove T be a normal ternary seminear ring. If $a \in xyT$, $x, y \in T$, then $a = xyz$ where $z \in T$. As T is a quasi commutative ternary seminear ring, $xyz = z^n xy \in Txy$. Therefore $a \in Txy$. Thus $xyT \subseteq Txy$. Likewise $Txy \subseteq xyT$. Thus we have $xyT = Txy$ for all $x, y \in T$. Hence T is a normal ternary seminear ring. □

Theorem 4.8. *Each commutative ternary seminear ring is a normal ternary seminear ring.*

Proof. Obviously it is true. □

Definition 4.9. A ternary seminear ring T is called pseudo commutative if T is a left pseudo commutative, a right pseudo commutative and a lateral pseudo commutative as $xyzuv = yzxuv = zxyuv = yxzuv = zyxuv = xzyuv$, $xyzuv = xyuvz = xyvzu = xyuzv = xyvuz = xyzvu$ and $xyzuv = xzuyv = xuyzv = xzyuv = xuzyv = xyuzv$ respectively.

Example 4.10. Let T={x,y,z,u,v} be a ternary seminear ring defined as [xyz]=x.y.z where binary operation '.' is defined as in the following table

+	x	y	z	u	v
x	x	y	z	u	v
y	y	x	y	z	y
z	z	y	x	y	z
u	u	z	y	x	v
v	v	y	z	v	x

.	x	y	z	u	v
x	x	x	x	x	x
y	y	x	x	x	x
z	x	x	x	x	x
u	x	x	x	x	x
v	x	y	z	u	v

It can be easily verify that T is a pseudo commutative ternary seminear ring.

Theorem 4.11. *If T be a commutative ternary seminear ring, then it is a pseudo commutative ternary seminear ring.*

Proof. Let T be a commutative ternary seminear ring. Then

$$xyz = yzx = zxy = yxz = zyx = xzy$$
$$xyzuv = (yzx)uv = (zxy)uv = (yxz)uv = (zyx)uv = (xzy)uv$$
$$xyzuv = yzxuv = zxyuv = yxzuv = zyxuv = xzyuv$$

Therefore T is a left pseudo commutative ternary seminear ring.
Then,

$$xyzuv = xy(zuv) = xy(uvz) = xy(vzu) = xy(uzv) = xy(vuz) = xy(zvu)$$
$$xyzuv = xyuvz = xyvzu = xyuzv = xyvuz = xyzvu$$

Again, T be a right pseudo commutative ternary seminear ring.
And,

$$xyzuv = x(yzu)v = x(zuy)v = x(uyz)v = x(zyu)v = x(uzy)v = x(yuz)v$$
$$xyzuv = xzuyv = xuyzv = xzyuv = xuzyv = xyuzv$$

Therefore T is a lateral pseudo commutative ternary seminear ring. Hence T is a pseudo commutative ternary seminear ring. □

References

1. Daddi. V. R. and Pawar. Y. S.: On completely regular ternary semiring, *Novi Sad J. Math.*, **12(2)**, (2012), 1–7.
2. Dutta. T.K. and Kar.S.: On regular ternary semirings, *Advances in Algebra, Proceedings of the ICM Satellite Conference in Algebra and Related Topics, World Scientific, New Jersey*, (2003), 343–355.
3. Madhusudana Rao. D. and Srinivasa Rao. G.: A Study on ternary semirings, *International Journal of Mathematical Archive*, **5(12)**, (2014), 24–30.
4. Perumal. R. Arulprakasam. R. and Radhakrishnan. M.: A Note on normal seminear rings, *Asia Mathematika*, **1(2)**, (2017), 15–21.
5. Radha.D. and Raja Lakshmi. C:, A Study on Pseudo commutative seminear rings, *Journal of Emerging Technologies and Innovative Research*, **6(2)**, (2019), 1082–1088.
6. Rajeswari. R. and Radha. D.: On Quasi weak commutative seminear ring, *International Journal of Science, Engineering and Management*, **4(1)**, (2019), 21–23.
7. Sardar. S.K., Mukherjee. R.: On Additively regular seminearrings, *Semigroup Forum*, **88**, (2014), 541–554.
8. Sardar. S.K., Mukherjee. R.: Erratum to: On Additively regular seminearrings, *Semigroup Forum*, DOI: 10.1007/s00233-015-9749-6.

9. Sen. M.K., Maity. S.K., Shum. K.P.: On Completely regular semirings, *Bulletin of Calcutta Mathematical Society*, **98(4)**, (2006), 319–328.
10. Srinivasa Rao. G., Madhusudhana Rao. D. and Sivaprasad. P.: Some conditions on ternary semirings, *International Journal of Engineering Science Invention*, **7(10)**, (2018), 03–07.
11. Vijayakumar. R. and Dhivya Bharathi. A.: On ternary seminear rings, *International Journal of Mathematics Trends and Technology*, *66(10)*,(2020), 170–177.
12. Vijayakumar. R. and Dhivya Bharathi. A.: Quotient ternary seminear rings, *Malaya Journal of Mathematik*, *9(1)*,(2021), 715–719.

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