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SOME TRANSFORMS OF DOUBLE INTEGRALS INVOLVING GENERAL POLYNOMIALS, MULTIVARIABLE MITTAGE–LEFFLER FUNCTION AND MODIFIED I–FUNCTION OF TWO VARIABLES

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Abstract: In literature, a lot of remarkable definite and indefinite integral formulas whose integrand include various special functions have been given. In this paper first our aim is to establish two double infinite integral formulas which involve Srivastava's polynomial, multivariable Mittage–Leffler function and modified I-function. After that we give the Beta transform, Laplace transform, Verma transform of these two double infinite integrals.

Keywords and phrases: Gamma function, beta function, Srivastava's polynomial, multivariable Mittage–Leffler function, modified I-function of two variables, double finite integral formulas.

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1. Introduction and Preliminaries

In 1979, Prasad and Prasad [7] introduced modified H-function of two variables and in 2012, Shantha Kumari et al.[1] defined I-function of two variables. In this paper we are defining modified I-function of two variables, which is the generalization of both the modified H-function of two variables and I-function of two variables, in the following manner:

$$I[z_1,.z_r] = I^{m,n:m_1,n_1:m_2,n_2:m_3,n_3}_{p,q:p_1,q_1:p_2,q_2:p_3,q_3}$$

$$\left[\begin{matrix} z_1 \\ z_2 \end{matrix}\middle| \begin{matrix} (a_j;\alpha_j,A_j;\xi_j)_{1,p} : (c_j;\gamma_j,C_j;\xi'_j)_{1,p_1} : (e_j,E_j;U_j)_{1,p_2};(g_j,G_j;P_j)_{1,p_3} \\ (b_j;\beta_j,B_j;\eta_j)_{1,q} : (d_j;\delta_j,D_j;\eta'_j)_{1,q_1} : (f_j,F_j;V_j)_{1,q_2},(h_j,H_j;Q_j)_{1,q_3} \end{matrix}\right]$$

$$= \frac{1}{(2\pi i)^2}\int_{L_1}\int_{L_2}\psi(s_1,s_2)\theta_1(s_1)\theta_2(s_2)z_1^{s_1}z_2^{s_2}ds_1ds_2 \tag{1}$$

where

$$\psi(s_1,s_2) = \frac{\prod_{j=1}^{m}\Gamma^{\eta_j}(b_j-\beta_js_1-B_js_2)\prod_{j=1}^{n}\Gamma^{\xi_j}(1-a_j+\alpha_js_1+A_js_2)}{\prod_{j=1}^{q}\Gamma^{\eta_j}(1-b_j+\beta_js_1+B_js_2)\prod_{j=n+1}^{p}\Gamma^{\xi_j}(a_j-\alpha_js_1-A_js_2)}$$

$$\times \frac{\prod_{j=1}^{m_1} \Gamma^{\eta_j^{'}}(d_j - \delta_j s_1 + D_j s_2) \prod_{j=1}^{n_1} \Gamma^{\xi_j^{'}}(1 - c_j + \gamma_j s_1 - C_j s_2)}{\prod_{j=1}^{q_1} \Gamma^{\eta_j^{'}}(1 - d_j + \delta_j s_1 - D_j s_2) \prod_{j=n+1}^{p_1} \Gamma^{\xi_j^{'}}(c_j - \gamma_j s_1 + C_j s_2)} \tag{2}$$

$$\theta_1(s_1) = \frac{\prod_{j=1}^{m_2} \Gamma^{U_j}(f_j - F_j s_1) \prod_{j=1}^{n_2} \Gamma^{V_j}(1 - e_j + E_j s_1)}{\prod_{j=m_2+1}^{q_2} \Gamma^{U_j}(1 - f_j + F_j s_1) \prod_{j=n_2+1}^{p_2} \Gamma^{V_j}(e_j - E_j s_1)} \tag{3}$$

$$\theta_2(s_2) = \frac{\prod_{j=1}^{m_3} \Gamma^{P_j}(h_j - H_j s_2) \prod_{j=1}^{n_3} \Gamma^{Q_j}(1 - g_j + G_j s_2)}{\prod_{j=m_3+1}^{q_3} \Gamma^{P_j}(1 - h_j + H_j s_2) \prod_{j=n_3+1}^{p_3} \Gamma^{Q_j}(g_j - G_j s_2)} \tag{4}$$

Here, the variables z_1 and z_2 are non-zero real or complex numbers and an empty product is interpreted as unity. $m, n, m_1, n_1, m_2, n_2, m_3, n_3, p, q, p_1, q_1, p_2, q_2, p_3, q_3$ are all non-negative integers such that $0 \le n \le p, 0 \le m \le q, 0 \le n_1 \le p_1, 0 \le m_1 \le q_1, 0 \le n_2 \le p_2$ $0 \le m_2 \le q_2, 0 \le n_3 \le p_3, 0 \le m_3 \le q_3$ and $\alpha_j, \beta_j, \gamma_j, \delta_j, A_j, B_j, C_j, D_j, E_j, F_j, G_j$ and H_j are all positive. $a_j, b_j, c_j, d_j, e_j, f_j, g_j, h_j$ are all complex numbers. The integration path L_1 in the complex s_1 plane runs from $\sigma_1 - i\infty$ to $\sigma_1 + i\infty$ so that all the poles of $\Gamma^{U_j}(f_j - F_j s_1)$ for $(j = 1, 2, \cdots, m_2)$ lie to the right of L_1 while all the poles of $\Gamma^{V_j}(1 - e_j - E_j s_1)$ for $(j = 1, 2, \cdots, n_2)$, $\Gamma^{\xi_j}(1 - a_j + \alpha_j s_1 - A_j s_2)$ for $(j = 1, 2, \cdots, n)$ and $\Gamma^{\xi_j^{'}}(1 - c_j + \gamma_j s_1 - C_j s_2)$ for $(j = 1, 2, \cdots, n_1)$ lie to the left of L_1. The integration path L_2 in the complex s_2 plane runs from $\sigma_2 - i\infty$ to $\sigma_2 + i\infty$ so that all the poles of $\Gamma^{P_j}(h_j - H_j s_2)$ for $(j = 1, 2, \cdots, m_3)$ lie to the right of L_2 while all the poles of $\Gamma^{Q_j}(1 - g_j + G_j s_2)$ for $(j = 1, 2, \cdots, n_3)$, $\Gamma^{\xi_j}(1 - a_j + \alpha_j s_1 - A_j s_2)$ for $(j = 1, 2, \cdots, n)$ and $\Gamma^{\xi_j^{'}}(1 - c_j + \gamma_j s_1 - C_j s_2)$ for $(j = 1, 2, \cdots, n_1)$ lie to the left of L_2.

The function $I[z_1, .z_r]$ defined by the equation (1) is an analytic function of z_1 and z_2 if

$$V_1 = \sum_{j=1}^{p} \xi_j \alpha_j + \sum_{j=1}^{p_1} \xi_j^{'} \gamma_j + \sum_{j=1}^{p_2} U_j E_j - \sum_{j=1}^{q} \eta_j \beta_j - \sum_{j=1}^{q_1} \eta_j^{'} \delta_j - \sum_{j=1}^{q_2} V_j F_j < 0 \tag{5}$$

$$V_2 = \sum_{j=1}^{p} \xi_j A_j + \sum_{j=1}^{q_1} \eta_j^{'} D_j + \sum_{j=1}^{p_3} P_j G_j - \sum_{j=1}^{q} \eta_j B_j - \sum_{j=1}^{p_1} \xi_j^{'} C_j - \sum_{j=1}^{q_3} Q_j H_j < 0 \tag{6}$$

exist. The double integral defined in the equation (1) converges absolutely if

$|\arg z_1| < \frac{1}{2}\pi\Omega_1$ and $|\arg z_2| < \frac{1}{2}\pi\Omega_2$ where

$$\Omega_1 = \sum_{j=1}^{n} \xi_j \alpha_j - \sum_{j=n+1}^{p} \xi_j \alpha_j + \sum_{j=1}^{m} \eta_j \beta_j - \sum_{j=m+1}^{q} \eta_j \beta_j + \sum_{j=1}^{n_1} \xi_j^{'} \gamma_j - \sum_{j=n_1+1}^{p_1} \xi_j^{'} \gamma_j + \sum_{j=1}^{m_1} \eta_j^{'} \delta_j$$

$$- \sum_{m_1+1}^{q_1} \eta_j^{'} \delta_j + \sum_{j=1}^{n_2} U_j E_j - \sum_{j=n_2+1}^{p_2} U_j E_j + \sum_{j=1}^{m_2} V_j F_j - \sum_{j=m_2+1}^{q_2} V_j F_j > 0 \tag{7}$$

and

$$\Omega_2 = \sum_{j=1}^{n} \xi_j A_j - \sum_{j=n+1}^{p} \xi_j A_j + \sum_{j=1}^{m} \eta_j B_j - \sum_{j=m+1}^{q} \eta_j B_j + \sum_{j=1}^{n_1} \xi_j^{'} C_j - \sum_{j=n_1+1}^{p_1} \xi_j^{'} C_j + \sum_{j=1}^{m_2} \eta_j^{'} D_j$$

$$- \sum_{m_1+1}^{q_2} \eta_j^{'} D_j + \sum_{j=1}^{n_3} P_j G_j - \sum_{j=n_3+1}^{p_3} P_j G_j + \sum_{j=1}^{m_3} Q_j H_j - \sum_{j=m_3+1}^{q_3} Q_j H_j > 0 \tag{8}$$

For simplicity, we shall use the following notations;

$$U = m_1, n_1 : m_2, n_2 : m_3, n_3 \tag{9}$$
$$V = p_1, q_1 : p_2, q_2 : p_3, q_3 \tag{10}$$
$$A_1 = (a_j; \alpha_j, A_j; \xi_j)_{1,p} \tag{11}$$
$$A_2 = (c_j; \gamma_j, C_j; \xi_j^{'})_{1,p_1} \tag{12}$$
$$A_3 = (e_j, E_j; U_j)_{1,p_2} \tag{13}$$
$$A_4 = (g_j; G_j; P_j)_{1,p_3} \tag{14}$$
$$B_1 = (b_j; \beta_j, B_j; \eta_j)_{1,q} \tag{15}$$
$$B_2 = (d_j; \delta_j, D_j; \eta_j^{'})_{1,q_1} \tag{16}$$
$$B_3 = (f_j, F_j; V_j)_{1,q_3} \tag{17}$$
$$B_4 = (h_j, H_j; Q_j)_{1,q_3} \tag{18}$$

In 1903, Mittage-Leffler [4] introduced the function $E_\alpha(y)$ in the following manner:

$$E_{\alpha}(y)=\sum_{k=0}^{\infty}\frac{y^{k}}{\Gamma(\alpha k+1)} \quad \text{where } \alpha,\ y\in C,\ \Re(\alpha)>0 \tag{19}$$

In 1905, Wiman [12] generalized the function $E_{\alpha}(y)$ and gave the function $E_{\alpha,\beta}(y)$ in the following manner:

$$E_{\alpha,\beta}(y)=\sum_{k=0}^{\infty}\frac{y^{k}}{\Gamma(\alpha k+\beta)} \quad \text{where } \alpha,\ \beta,\ y\in C,\ \Re(\alpha)>0,\ \Re(\beta)>0 \tag{20}$$

In 1971, Prabhakar [6] generalized the function $E_{\alpha,\beta}(y)$ and gave the function $E_{\alpha,\beta}^{\lambda}(y)$ in the following manner:

$$E_{\alpha,\beta}^{\lambda}(y)=\sum_{k=0}^{\infty}\frac{(\lambda)_{k}}{\Gamma(\alpha k+\beta)}\frac{y^{k}}{k!} \tag{21}$$

where $\alpha,\ \beta,\ \lambda,\ y\in C, \Re(\alpha)>0, \Re(\beta)>0, \Re(\lambda)>0$

Saxena et al. [8] gave the multivariable analogue of multivariable Mittage-leffler function in the following manner:

$$E_{\mu_i,\nu_1}^{\lambda_i}(y_1,\ldots,y_m)=E_{\mu_1,\ldots,\mu_m,\nu_1}^{\lambda_1,\ldots,\lambda_m}(y_1,\ldots,y_m)=\sum_{k_1,\cdots,k_m=0}^{\infty}\frac{(\lambda_1)_{k_1},\cdots,(\lambda_m)_{k_m}}{\Gamma(\nu_1+\sum_{i=1}^{m}\mu_i k_i)}\frac{y_1^{k_1}}{k_1!}\cdots\frac{y_m^{k_m}}{k_m!} \tag{22}$$

where $\nu_1,\ \lambda_i,\ \mu_i\in C,\ \Re(\mu_j)>0,\ \forall i=1,2,\cdots,m$

For the sake of convenience, let $F_{\mu_i,\nu_1}^{\lambda_i}=\dfrac{(\lambda_1)_{k_1},\cdots,(\lambda_m)_{k_m}}{\Gamma(\nu_1+\sum_{i=1}^{m}\mu_i k_i)}$

Srivastava [11] introduced the general class of polynomials in the following manner:

$$S_N^M(y)=\sum_{k=0}^{[N/M]}\frac{(-N)_{Mk}}{k!}A_{N,k}y^{k} \quad N=0,1,2,\cdots \tag{23}$$

where M is an arbitrary positive integer and the coefficients $A_{N,k}(N,k\geq 0)$ are arbitrary constants, real or complex and $(\lambda)_N$ is the pochhammer symbol.

2. Required Formulas

The following some known integral formulas are required for our present study.

Manoj Singh [2] gave the following formula for the Bessel-Maitland function,

$$\int_0^1 u^{\alpha-1}(1-u)^{\beta-1}J_{v,q}^{\mu,\gamma}(xu^{\delta})du = \sum_{n=0}^{\infty}\frac{(\gamma)_{qn}(-x)^n}{n!\Gamma(\mu n+v+1)}\frac{\Gamma(\alpha+\delta n)\Gamma(\beta)}{\Gamma(\alpha+\beta+\delta n)} \tag{24}$$

where $\mu,\ v,\ \gamma\in C,\ \Re(\mu)\geq 0,\ \Re(v)\geq -1,\ \Re(\gamma)\geq 0,\ \Re(\alpha)\geq 0,\ \Re(\beta)\geq 0$ and $q\in(0,1)\bigcup N$

Nielsen [5].

$$\int_0^{\pi}(\sin\theta)^{\rho}\cos a\theta d\theta = \frac{\pi\cos\left(\frac{\pi a}{2}\right)\Gamma(1+\rho)}{2^{\rho}\Gamma\left(1+\frac{\rho+a}{2}\right)\Gamma\left(1+\frac{\rho-a}{2}\right)} \tag{25}$$

where $\rho>-1$

Nielsen [5].

$$\int_0^{\pi}(\sin\theta)^{\rho}\sin a\theta d\theta = \frac{\pi\sin\left(\frac{\pi a}{2}\right)\Gamma(1+\rho)}{2^{\rho}\Gamma\left(1+\frac{\rho+a}{2}\right)\Gamma\left(1+\frac{\rho-a}{2}\right)} \tag{26}$$

where $\rho>-1$

3. Main Integrals

Here we are giving two double integrals involving the Srivastava's polynomial (23), multivariable Mittage-Leffler function (22) and the modified I-function of two variables (1).

Theorem 1: Let $d,f,d_i,f_i,h_j,\sigma_j\in R^{+}$ for $i=1,\ldots,m$ and $j=1,2$, $\Re(\alpha)>0$, $\Re(\beta)>0$ and $\Re(\rho)>-1$. Also let

$$\Re(\alpha)+\left(h_1+h_2\right)\min_{1\leq j\leq m_2}\Re\left(V_j\frac{f_j}{F_j}\right)>0,\ \Re(\rho)+\left(\sigma_1+\sigma_2\right)\min_{1\leq j\leq m_3}\Re\left(Q_j\frac{h_j}{H_j}\right)>-1.$$

Further, let $\left|\arg(z_i u^{h_i}\sin^{\sigma_i}\theta)\right|<\frac{1}{2}\Omega_i\pi\ (i=1,2)$ with Ω_i same as in the equations (7) and (8). Then

$$\int_0^1\int_0^{\pi}u^{\alpha-1}(1-u)^{\beta-1}J_{v,q}^{\mu,\gamma}(xu^{\delta})(\sin\theta)^{\rho}\cos a\theta\, S_N^M(yu^d(\sin\theta)^f)$$

$$\times E_{\mu_i,v_1}^{\lambda_i}(y_1u^{d_1}\sin^{f_1}\theta,\ldots,y_m u^{d_m}\sin^{f_m}\theta)I\begin{pmatrix}z_1u^{h_1}(\sin\theta)^{\sigma_1}\\ z_2u^{h_2}(\sin\theta)^{\sigma_2}\end{pmatrix}dud\theta$$

$$=\sum_{n=0}^{\infty}\frac{(\gamma)_{qn}\Gamma(\beta)(-x)^n}{n!\Gamma(\mu n+v+1)}\pi\cos\left(\frac{\pi a}{2}\right)\sum_{k=0}^{[N/M]}\sum_{k_1,\ldots,k_r=0}^{\infty}\frac{(-N)_{Mk}}{k!}A_{N,k}F_{\mu_i,v_1}^{\lambda_i}\left(\frac{y}{2^f}\right)^k\prod_{i=1}^{m}\frac{1}{k_i!}\left(\frac{y_i}{2^{f_i}}\right)^{k_i}$$

$$\times I^{m,n+2;U}_{p+2,q+3;V}\left(\begin{array}{c|c} 2^{-\sigma_1} z_1 & A_1, X_1, X_2 : A_2 : A_3; A_4 \\ 2^{-\sigma_2} z_2 & B_1, Y_1, Y_2, Y_3 : B_2 : B_3; B_4 \end{array}\right) \tag{27}$$

where

$$X_1 = (1-\alpha-\delta n - dk - \textstyle\sum_{i=1}^{m} k_i d_i;\, h_1,\, h_2;\, 1) \tag{28}$$

$$X_2 = (-\rho - fk - \textstyle\sum_{i=1}^{m} k_i f_i;\, \sigma_1,\, \sigma_2;\, 1) \tag{29}$$

$$Y_1 = (1-\alpha-\beta-\delta n - dk - \textstyle\sum_{i=1}^{m} k_i d_i;\, h_1,\, h_2;\, 1) \tag{30}$$

$$Y_2 = \left(-\frac{\rho}{2} - \frac{a}{2} - \frac{fk}{2} + \sum_{i=1}^{m} \frac{k_i f_i}{2};\, \frac{\sigma_1}{2}, \frac{\sigma_2}{2}; 1\right) \tag{31}$$

$$Y_3 = \left(-\frac{\rho}{2} + \frac{a}{2} - \frac{fk}{2} + \sum_{i=1}^{m} \frac{k_i f_i}{2};\, \frac{\sigma_1}{2}, \frac{\sigma_2}{2}; 1\right) \tag{32}$$

Proof: We denote the left hand side of the equation (27) by Δ. Then we express the general class of polynomial and multivariable Mittage-Leffler function in the series form with the help of equations (23) and (22) and expressing the modified I-function of two variables in terms of Mellin-Barnes contour integral with the help of equation (1) and changing the order of integration, which is permissible under the given conditions, we obtain

$$\Delta = \sum_{k=0}^{[N/M]} \sum_{k_1,\ldots,k_l=0}^{\infty} \frac{(-N)_{Mk}}{k!} A_{N,k} F^{\lambda_i}_{\mu_i,v} y^k \frac{y_1^{k_1}}{k_1!} \cdots \frac{y_m^{k_m}}{k_m!} \frac{1}{(2\pi i)^2} \int_{L_1}\int_{L_2} \Phi(s_1,s_2)\theta_1(s_1)\theta_2(s_2) z_1^{s_1} z_2^{s_2}$$

$$\times\left[\int_0^1 u^{\alpha+dk+\sum_{i=1}^{m} k_i d_i + h_1 s_1 + h_2 s_2 - 1}(1-u)^{\beta-1} J^{\mu,\gamma}_{v,q}(xu^{\delta})du\right]\left[\int_0^{\pi} (\sin\theta)^{\rho+fk+\sum_{i=1}^{m} k_i f_i + \sigma_1 s_1 + \sigma_2 s_2} \cos a\theta d\theta\right] ds_1 ds_2 \tag{33}$$

now, evaluating the inner u and θ-integrals in the equation (33) with the help of the integral formulas given in the equations (24) and (25). We obtain

$$\int_0^1 u^{\alpha+dk+\sum_{i=1}^{m} k_i d_i + h_1 s_1 + h_2 s_2 - 1}(1-u)^{\beta-1} J^{\mu,\gamma}_{v,q}(xu^{\delta})du =$$

$$\sum_{n=0}^{\infty} \frac{(\gamma)_{qn}(-x)^n}{n!\Gamma(\mu n + v + 1)} \frac{\Gamma(\alpha+\delta n + dk + \sum_{i=1}^{m} k_i d_i + h_1 s_1 + h_2 s_2)\Gamma(\beta)}{\Gamma(\alpha+\beta+\delta n + dk + \sum_{i=1}^{m} k_i d_i + h_1 s_1 + h_2 s_2)} \tag{34}$$

and

$$\int_0^{\pi} (\sin\theta)^{\rho+fk+\sum_{i=1}^{m} k_i f_i + \sigma_1 s_1 + \sigma_2 s_2} \cos a\theta d\theta =$$

$$\frac{2^{-(\rho+fk+\sum_{i=1}^{m}k_i f_i+\sigma_1 s_1+\sigma_2 s_2)}\pi\cos\left(\frac{\pi a}{2}\right)\Gamma(1+\rho+fk+\sum_{i=1}^{m}k_i f_i+\sigma_1 s_1+\sigma_2 s_2)}{\Gamma\left(1+\frac{\rho+a+fk+\sum_{i=1}^{m}k_i f_i+\sigma_1 s_1+\sigma_2 s_2}{2}\right)\Gamma\left(1+\frac{\rho-a+fk+\sum_{i=1}^{m}k_i f_i+\sigma_1 s_1+\sigma_2 s_2}{2}\right)} \tag{35}$$

now, putting the equations (34) and (35) in the equation (33) and interpreting the result with the help of equation (1) in terms of double contour integral, we get the required result (27).

Theorem 2: Let $d,\ f,\ d_i,\ f_i,\ h_j,\ \sigma_j \in R^+$ for $i=1,\dots,m$ and $j=1,2$, $\Re(\alpha)>0$, $\Re(\beta)>0$ and $\Re(\rho)>-1$. Also let

$$\Re(\alpha)+\left(h_1+h_2\right)\min_{1\le j\le m_2}\Re\left(V_j\frac{f_j}{F_j}\right)>0,\ \Re(\rho)+\left(\sigma_1+\sigma_2\right)\min_{1\le j\le m_3}\Re\left(Q_j\frac{h_j}{H_j}\right)>-1.$$

Further, let $\left|\arg(z_i u^{h_i}\sin^{\sigma_i}\theta)\right|<\frac{1}{2}\Omega_i\pi\ (i=1,2)$ with Ω_i same as in the equations (7) and (8). Then

$$\int_0^1\int_0^\pi u^{\alpha-1}(1-u)^{\beta-1}J_{v,q}^{\mu,\gamma}(xu^\delta)(\sin\theta)^\rho\sin a\theta S_N^M(yu^d(\sin\theta)^f)$$

$$\times E_{\mu_i,v_1}^{\lambda_i}(y_1u^{d_1}\sin^{f_1}\theta,\dots,y_mu^{d_m}\sin^{f_m}\theta)I\left(\begin{matrix}z_1u^{h_1}(\sin\theta)^{\sigma_1}\\ z_2u^{h_2}(\sin\theta)^{\sigma_2}\end{matrix}\right)dud\theta$$

$$=\sum_{n=0}^{\infty}\frac{(\gamma)_{qn}\Gamma(\beta)(-x)^n}{n!\Gamma(\mu n+v+1)}\pi\sin\left(\frac{\pi a}{2}\right)\sum_{k=0}^{[N/M]}\sum_{k_1,\dots,k_r=0}^{\infty}\frac{(-N)_{Mk}}{k!}A_{N,k}F_{\mu_i,v_1}^{\lambda_i}\left(\frac{y}{2^f}\right)^k\prod_{i=1}^{m}\frac{1}{k_i!}\left(\frac{y_i}{2^{f_i}}\right)^{k_i}$$

$$\times I_{p+2,q+3;V}^{m,n+2;U}\left(\begin{matrix}2^{-\sigma_1}z_1\\ 2^{-\sigma_2}z_2\end{matrix}\middle|\begin{matrix}A_1,X_1,X_2:A_2:A_3;A_4\\ B_1,Y_1,Y_2,Y_3:B_2:B_3;B_4\end{matrix}\right) \tag{36}$$

where $X_1,\ X_2,\ Y_1,\ Y_2,\ Y_3$ are given in the equations from (28) to (32) respectively.

4. Integral Transform Formulas

In this section we shall obtain integral transform like Beta transform, Laplace transform, Verma transform of the double integrals obtained in the above two theorems.

4.1 Beta Transform

Definition: The beta transform of a function $f(z)$ is defined as (see [10])

$$B\left\{f(z);s,p\right\}=\int_0^1 z^{s-1}(1-z)^{p-1}f(z)dz \tag{37}$$

Theorem 3: Let $d,\, f,\, d_i,\, f_i,\, h_j,\, \sigma_j \in R^{+}$ for $i=1,\ldots,m$ and $j=1,2$, $\Re(\alpha)>0$, $\Re(\beta)>0$ and $\Re(\rho)>-1$. Also let

$$\Re(\alpha)+\left(h_1+h_2\right)\min_{1\le j\le m_2}\Re\left(V_j\frac{f_j}{F_j}\right)>0,\ \Re(\rho)+\left(\sigma_1+\sigma_2\right)\min_{1\le j\le m_3}\Re\left(Q_j\frac{h_j}{H_j}\right)>-1.$$

Further, let $\left|\arg(z_i u^{h_i}\sin^{\sigma_i}\theta)\right|<\frac{1}{2}\Omega_i\pi\ (i=1,2)$ with Ω_i same as in the equations (7) and (8). Then

$$B\left\{\int_0^1\int_0^\pi u^{\alpha-1}(1-u)^{\beta-1}J_{v,q}^{\mu,\gamma}(xu^{\delta})(\sin\theta)^{\rho}\cos a\theta S_N^M(yz^b u^d(\sin\theta)^f)\right.$$

$$\left.\times E_{\mu_i,v_1}^{\lambda_i}(y_1 z^{\eta_1}u^{d_1}\sin^{f_1}\theta,\ldots,y_m z^{\eta_m}u^{d_m}\sin^{f_m}\theta)I\begin{pmatrix} z_1 z^{l_1}u^{h_1}(\sin\theta)^{\sigma_1}\\ z_2 z^{l_2}u^{h_2}(\sin\theta)^{\sigma_2}\end{pmatrix}dud\theta : s,p\right\}$$

$$=\sum_{n=0}^{\infty}\frac{(\gamma)_{qn}\Gamma(\beta)(-x)^n\Gamma(p)}{n!\Gamma(\mu n+v+1)}\pi\cos\left(\frac{\pi a}{2}\right)\sum_{k=0}^{[N/M]}\sum_{k_1,\ldots,k_r=0}^{\infty}\frac{(-N)_{Mk}}{k!}A_{N,k}F_{\mu_i,v_1}^{\lambda_i}\left(\frac{y}{2^f}\right)^k\prod_{i=1}^{m}\frac{1}{k_i!}\left(\frac{y_i}{2^{f_i}}\right)^{k_i}$$

$$\times I_{p+3,q+4;V}^{m,n+3;U}\left(\begin{matrix}2^{-\sigma_1}z_1\\ 2^{-\sigma_2}z_2\end{matrix}\left|\begin{matrix}A_1,X_1,X_2,X_3:A_2:A_3;A_4\\ B_1,Y_1,Y_2,Y_3,Y_4:B_2:B_3;B_4\end{matrix}\right.\right) \tag{38}$$

where $X_1,\, X_2,\, Y_1,\, Y_2,\, Y_3$ are given in the equations from (28) to (32) respectively and

$$X_3=(1-s-bk-\sum_{i=1}^{m}\eta_i k_i;\, l_1,\, l_2;\, 1) \tag{39}$$

$$Y_4=(1-s-p-bk-\sum_{i=1}^{m}\eta_i k_i;\, l_1,\, l_2;\, 1) \tag{40}$$

Proof: In order to prove the equation (38), using the definition (37) of beta transform, we get

$$B\left\{\int_0^1\int_0^\pi u^{\alpha-1}(1-u)^{\beta-1}J_{v,q}^{\mu,\gamma}(xu^{\delta})(\sin\theta)^{\rho}\cos a\theta S_N^M(yz^b u^d(\sin\theta)^f)\right.$$

$$\left.\times E_{\mu_i,v_1}^{\lambda_i}(y_1 z^{\eta_1}u^{d_1}\sin^{f_1}\theta,\ldots,y_m z^{\eta_m}u^{d_m}\sin^{f_m}\theta)I\begin{pmatrix} z_1 z^{l_1}u^{h_1}(\sin\theta)^{\sigma_1}\\ z_2 z^{l_2}u^{h_2}(\sin\theta)^{\sigma_2}\end{pmatrix}dud\theta : s,p\right\}$$

$$=\int_0^1 z^{s-1}(1-z)^{p-1}\left\{\int_0^1\int_0^\pi u^{\alpha-1}(1-u)^{\beta-1}J_{v,q}^{\mu,\gamma}(xu^{\delta})(\sin\theta)^{\rho}\cos a\theta S_N^M(yz^b u^d(\sin\theta)^f)\right.$$

$$\times E^{\lambda_i}_{\mu_i,\nu_1}(y_1 z^{\eta_1} u^{d_1} \sin^{f_1}\theta,\ldots,y_m z^{\eta_m} u^{d_m} \sin^{f_m}\theta) I\begin{pmatrix} z_1 z^{l_1} u^{h_1}(\sin\theta)^{\sigma_1} \\ z_2 z^{l_2} u^{h_2}(\sin\theta)^{\sigma_2}\end{pmatrix} du d\theta\} dz \tag{41}$$

now, expressing the general class of polynomials and multivariable Mittage-Leffler function in series form with the help of equations (23) and (22) and expressing the modified I-function of two variables in terms of Mellin-Barnes contour integral with the help of equation (1) and changing the order of integration, which is permissible under the given conditions, we get

$$= \sum_{k=0}^{[N/M]} \sum_{k_1,\ldots,k_l=0}^{\infty} \frac{(-N)_{Mk}}{k!} A_{N,k} F^{\lambda_i}_{\mu_i,\nu} y^k \frac{y_1^{k_1}}{k_1!}\cdots\frac{y_m^{k_m}}{k_m!}\frac{1}{(2\pi i)^2}\int_{L_1}\int_{L_2}\Phi(s_1,s_2)\theta_1(s_1)\theta_2(s_2) z_1^{s_1} z_2^{s_2}$$

$$\times\left[\int_0^1 u^{\alpha+dk+\sum_{i=1}^m k_i d_i + h_1 s_1 + h_2 s_2 - 1}(1-u)^{\beta-1} J^{\mu,\gamma}_{v,q}(xu^\delta) du\right]\left[\int_0^\pi (\sin\theta)^{\rho+fk+\sum_{i=1}^m k_i f_i + \sigma_1 s_1 + \sigma_2 s_2}\cos a\theta d\theta\right]$$

$$\times\left[\int_0^1 z^{s+bk+\sum_{i=1}^m \eta_i k_i + l_1 s_1 + l_2 s_2}(1-z)^{p-1} dz\right] ds_1 ds_2 \tag{42}$$

now evaluating the inner z-integral, we obtain

$$\int_0^1 z^{s+bk+\sum_{i=1}^m \eta_i k_i + l_1 s_1 + l_2 s_2 - 1}(1-z)^{p-1} dz = \frac{\Gamma(s+bk+\sum_{i=1}^m \eta_i k_i + l_1 s_1 + l_2 s_2)\Gamma(p)}{\Gamma(s+p+bk+\sum_{i=1}^m \eta_i k_i + l_1 s_1 + l_2 s_2)} \tag{43}$$

now putting the equations (34), (35) and (43) in the above equation (42) and interpreting the result with the help of equation (1) in terms of double contour integral, we get the required result (38).

Theorem 4: Let $d, f, d_i, f_i, h_j, \sigma_j \in R^+$ for $i = 1,\ldots,m$ and $j = 1,2$, $\Re(\alpha) > 0$, $\Re(\beta) > 0$ and $\Re(\rho) > -1$. Also let

$$\Re(\alpha) + (h_1 + h_2)\min_{1\le j\le m_2}\Re\left(V_j \frac{f_j}{F_j}\right) > 0,\ \Re(\rho) + (\sigma_1+\sigma_2)\min_{1\le j\le m_3}\Re\left(Q_j\frac{h_j}{H_j}\right) > -1.$$

Further, let $\left|\arg(z_i u^{h_i}\sin^{\sigma_i}\theta)\right| < \frac{1}{2}\Omega_i\pi\ (i=1,2)$ with Ω_i same as in the equations (7) and (8). Then

$$B\left\{\int_0^1\int_0^\pi u^{\alpha-1}(1-u)^{\beta-1} J^{\mu,\gamma}_{v,q}(xu^\delta)(\sin\theta)^\rho \sin a\theta S_N^M(yz^b u^d(\sin\theta)^f)\right.$$

$$\left.\times E^{\lambda_i}_{\mu_i,\nu_1}(y_1 z^{\eta_1} u^{d_1}\sin^{f_1}\theta,\ldots,y_m z^{\eta_m} u^{d_m}\sin^{f_m}\theta) I\begin{pmatrix} z_1 z^{l_1} u^{h_1}(\sin\theta)^{\sigma_1} \\ z_2 z^{l_2} u^{h_2}(\sin\theta)^{\sigma_2}\end{pmatrix} du d\theta : s, p\right\}$$

$$=\sum_{n=0}^{\infty}\frac{(\gamma)_{qn}\Gamma(\beta)(-x)^n\Gamma(p)}{n!\Gamma(\mu n+v+1)}\pi\sin\left(\frac{\pi a}{2}\right)\sum_{k=0}^{[N/M]}\sum_{k_1,\ldots,k_r=0}^{\infty}\frac{(-N)_{Mk}}{k!}A_{N,k}F_{\mu_i,v_1}^{\lambda_i}\left(\frac{y}{2^f}\right)^k\prod_{i=1}^{m}\frac{1}{k_i!}\left(\frac{y_i}{2^{f_i}}\right)^{k_i}$$

$$\times I_{p+3,q+4;V}^{m,n+3;U}\left(\begin{matrix}2^{-\sigma_1}z_1\\2^{-\sigma_2}z_2\end{matrix}\middle|\begin{matrix}A_1,X_1,X_2,X_3:A_2:A_3;A_4\\B_1,Y_1,Y_2,Y_3,Y_4:B_2:B_3;B_4\end{matrix}\right) \tag{44}$$

where $X_1,\ X_2,\ X_3,\ Y_1,\ Y_2,\ Y_3,\ Y_4$ are given in the equations (28), (29), (39), (30), (31), (32) and (40) respectively.

4.2 Laplace Transform

Definition: The Laplace transform of a function $f(z)$ is defined as (see [9])

$$L\{f(z)\}=\int_0^{\infty}e^{-qz}f(z)dz \tag{45}$$

Theorem 5: Let $d,\ f,\ d_i,\ f_i,\ h_j,\ \sigma_j\in R^+$ for $i=1,\ldots,m$ and $j=1,2$, $\Re(\alpha)>0$, $\Re(\beta)>0$ and $\Re(\rho)>-1$. Also let

$$\Re(\alpha)+(h_1+h_2)\min_{1\le j\le m_2}\Re\left(V_j\frac{f_j}{F_j}\right)>0,\ \Re(\rho)+(\sigma_1+\sigma_2)\min_{1\le j\le m_3}\Re\left(Q_j\frac{h_j}{H_j}\right)>-1.$$

Further, let $\left|\arg(z_iu^{h_i}\sin^{\sigma_i}\theta)\right|<\frac{1}{2}\Omega_i\pi\ (i=1,2)$ with Ω_i same as in the equations (7) and (8). Then

$$L\left\{\int_0^1\int_0^{\pi}u^{\alpha-1}(1-u)^{\beta-1}J_{v,q}^{\mu,\gamma}(xu^{\delta})(\sin\theta)^{\rho}\cos a\theta S_N^M(yz^bu^d(\sin\theta)^f)\right.$$

$$\left.\times E_{\mu_i,v_1}^{\lambda_i}(y_1z^{\eta_1}u^{d_1}\sin^{f_1}\theta,\ldots,y_mz^{\eta_m}u^{d_m}\sin^{f_m}\theta)I\left(\begin{matrix}z_1z^{l_1}u^{h_1}(\sin\theta)^{\sigma_1}\\z_2z^{l_2}u^{h_2}(\sin\theta)^{\sigma_2}\end{matrix}\right)dud\theta\right\}$$

$$=\sum_{n=0}^{\infty}\frac{(\gamma)_{qn}\Gamma(\beta)(-x)^n}{qn!\Gamma(\mu n+v+1)}\pi\cos\left(\frac{\pi a}{2}\right)\sum_{k=0}^{[N/M]}\sum_{k_1,\ldots,k_r=0}^{\infty}\frac{(-N)_{Mk}}{k!}A_{N,k}F_{\mu_i,v_1}^{\lambda_i}\left(\frac{y}{2^fq^b}\right)^k$$

$$\times\prod_{i=1}^{m}\frac{1}{k_i!}\left(\frac{y_i}{q^{\eta_i}2^{f_i}}\right)^{k_i}I_{p+3,q+3;V}^{m,n+3;U}\left(\begin{matrix}q^{-l_1}2^{-\sigma_1}z_1\\q^{-l_2}2^{-\sigma_2}z_2\end{matrix}\middle|\begin{matrix}A_1,X_1,X_2,X_4:A_2:A_3;A_4\\B_1,Y_1,Y_2,Y_3:B_2:B_3;B_4\end{matrix}\right) \tag{46}$$

where X_1, X_2, Y_1, Y_2, Y_3 are given in the equations from (28) to (32) respectively and

$$X_4 = (-s - bk - \sum_{i=1}^{m} \eta_i k_i; \, l_1, \, l_2; \, 1) \tag{47}$$

Proof: In order to prove the equation (46), using the definition (45) of Laplace transform, we get

$$L\left\{\int_0^1 \int_0^\pi u^{\alpha-1}(1-u)^{\beta-1} J_{\nu,q}^{\mu,\gamma}(xu^\delta)(\sin\theta)^\rho \cos a\theta S_N^M(yz^b u^d (\sin\theta)^f)\right.$$

$$\left.\times E_{\mu_i,\nu_1}^{\lambda_i}(y_1 z^{\eta_1} u^{d_1} \sin^{f_1}\theta, \ldots, y_m z^{\eta_m} u^{d_m} \sin^{f_m}\theta) I\left(\begin{matrix} z_1 z^{l_1} u^{h_1} (\sin\theta)^{\sigma_1} \\ z_2 z^{l_2} u^{h_2} (\sin\theta)^{\sigma_2} \end{matrix}\right) du d\theta\right\}$$

$$= \int_0^\infty e^{-qz} \left\{\int_0^1 \int_0^\pi u^{\alpha-1}(1-u)^{\beta-1} J_{\nu,q}^{\mu,\gamma}(xu^\delta)(\sin\theta)^\rho \cos a\theta S_N^M(yz^b u^d (\sin\theta)^f)\right.$$

$$\left.\times E_{\mu_i,\nu_1}^{\lambda_i}(y_1 z^{\eta_1} u^{d_1} \sin^{f_1}\theta, \ldots, y_m z^{\eta_m} u^{d_m} \sin^{f_m}\theta) I\left(\begin{matrix} z_1 z^{l_1} u^{h_1} (\sin\theta)^{\sigma_1} \\ z_2 z^{l_2} u^{h_2} (\sin\theta)^{\sigma_2} \end{matrix}\right) du d\theta\right\} dz \tag{48}$$

Now expressing the general class of polynomial and multivariable Mittage-Leffler function in series with the help of equations (23) and (22) and expressing the modified I-function of two variables in terms of Mellin-Barnes contour integral with the help of the equation (1) and changing the order of integration, which is permissible under the given conditions, we get

$$= \sum_{k=0}^{[N/M]} \sum_{k_1,\ldots,k_l=0}^{\infty} \frac{(-N)_{Mk}}{k!} A_{N,k} F_{\mu_i,\nu}^{\lambda_i} y^k \frac{y_1^{k_1}}{k_1!} \cdots \frac{y_m^{k_m}}{k_m!} \frac{1}{(2\pi i)^2} \int_{L_1} \int_{L_2} \Phi(s_1,s_2)\theta_1(s_1)\theta_2(s_2) z_1^{s_1} z_2^{s_2}$$

$$\times\left[\int_0^1 u^{\alpha+dk+\sum_{i=1}^m k_i d_i + h_1 s_1 + h_2 s_2 - 1}(1-u)^{\beta-1} J_{\nu,q}^{\mu,\gamma}(xu^\delta) du\right]\left[\int_0^\pi (\sin\theta)^{\rho+fk+\sum_{i=1}^m k_i f_i + \sigma_1 s_1 + \sigma_2 s_2} \cos a\theta d\theta\right]$$

$$\times\left[\int_0^\infty e^{-qz} z^{s+bk+\sum_{i=1}^m \eta_i k_i + l_1 s_1 + l_2 s_2} dz\right] ds_1 ds_2 \tag{49}$$

now evaluating the inner z-integral, we obtain

$$\int_0^\infty e^{-qz} z^{1+bk+\sum_{i=1}^m \eta_i k_i + l_1 s_1 + l_2 s_2 - 1} dz = \frac{\Gamma(1+bk+\sum_{i=1}^m \eta_i k_i + l_1 s_1 + l_2 s_2)}{q^{1+bk+\sum_{i=1}^m \eta_i k_i + l_1 s_1 + l_2 s_2}} \tag{50}$$

now putting the equations (34), (35) and (50) in the above equation (49) and interpreting the result with the help of equation (1) in terms of double contour integral, we get the required result (46).

Theorem 6: Let $d, f, d_i, f_i, h_j, \sigma_j \in R^+$ for $i = 1, \ldots, m$ and $j = 1, 2$, $\Re(\alpha) > 0$, $\Re(\beta) > 0$ and $\Re(\rho) > -1$. Also let

$$\Re(\alpha) + (h_1 + h_2) \min_{1 \le j \le m_2} \Re\left(V_j \frac{f_j}{F_j}\right) > 0, \ \Re(\rho) + (\sigma_1 + \sigma_2) \min_{1 \le j \le m_3} \Re\left(Q_j \frac{h_j}{H_j}\right) > -1.$$

Further, let $\left|\arg(z_i u^{h_i} \sin^{\sigma_i} \theta)\right| < \frac{1}{2} \Omega_i \pi \ (i = 1, 2)$ with Ω_i same as in the equations (7) and (8). Then

$$L\left\{\int_0^1 \int_0^\pi u^{\alpha-1}(1-u)^{\beta-1} J_{v,q}^{\mu,\gamma}(xu^\delta)(\sin\theta)^\rho \sin a\theta S_N^M(yz^b u^d (\sin\theta)^f)\right.$$

$$\left.\times E_{\mu_i,v_1}^{\lambda_i}(y_1 z^{\eta_1} u^{d_1} \sin^{f_1}\theta, \ldots, y_m z^{\eta_m} u^{d_m} \sin^{f_m}\theta) I\left(\begin{matrix} z_1 z^{l_1} u^{h_1} (\sin\theta)^{\sigma_1} \\ z_2 z^{l_2} u^{h_2} (\sin\theta)^{\sigma_2} \end{matrix}\right) du d\theta\right\}$$

$$= \sum_{n=0}^\infty \frac{(\gamma)_{qn} \Gamma(\beta)(-x)^n}{qn!\Gamma(\mu n + v + 1)} \pi \sin\left(\frac{\pi a}{2}\right) \sum_{k=0}^{[N/M]} \sum_{k_1, \ldots, k_r = 0}^\infty \frac{(-N)_{Mk}}{k!} A_{N,k} F_{\mu_i, v_1}^{\lambda_i}\left(\frac{y}{2^f q^b}\right)^k$$

$$\times \prod_{i=1}^m \frac{1}{k_i!}\left(\frac{y_i}{q^{\eta_i} 2^{f_i}}\right)^{k_i} I_{p+3,q+3;V}^{m,n+3;U}\left(\begin{matrix} q^{-l_1} 2^{-\sigma_1} z_1 \\ q^{-l_2} 2^{-\sigma_2} z_2 \end{matrix}\middle| \begin{matrix} A_1, X_1, X_2, X_4 : A_2 : A_3; A_4 \\ B_1, Y_1, Y_2, Y_3 : B_2 : B_3; B_4 \end{matrix}\right) \quad (51)$$

where X_1, X_2, X_4, Y_1, Y_2, Y_3 are given in the equations (28), (29), (47), (30), (31), and (32) respectively.

4.3 Verma Transform

Definition: Verma transform of a function $f(z)$ is defined as (see [3])

$$V\{f(z)\} = \int_0^\infty (sz)^{r-1} e^{-\frac{1}{2}sz} W_{\kappa,v}(sz) f(z) dz \quad (52)$$

where $W_{\kappa,v}(z)$ represents Whittaker function.

Theorem 7: Let $d, f, d_i, f_i, h_j, \sigma_j \in R^+$ for $i = 1, \ldots, m$ and $j = 1, 2$, $\Re(\alpha) > 0$, $\Re(\beta) > 0$ and $\Re(\rho) > -1$. Also let

$$\Re(\alpha)+\left(h_1+h_2\right)\min_{1\le j\le m_2}\Re\left(V_j\frac{f_j}{F_j}\right)>0,\ \Re(\rho)+\left(\sigma_1+\sigma_2\right)\min_{1\le j\le m_3}\Re\left(Q_j\frac{h_j}{H_j}\right)>-1.$$

Further, let $\left|\arg(z_i u^{h_i}\sin^{\sigma_i}\theta)\right|<\frac{1}{2}\Omega_i\pi\ (i=1,2)$ with Ω_i same as in the equations (7) and (8). Then

$$V\left\{\int_0^1\int_0^\pi u^{\alpha-1}(1-u)^{\beta-1}J_{v,q}^{\mu,\gamma}(xu^\delta)(\sin\theta)^\rho\cos a\theta S_N^M(yz^bu^d(\sin\theta)^f)\right.$$

$$\left.\times E_{\mu_i,v_1}^{\lambda_i}(y_1z^{\eta_1}u^{d_1}\sin^{f_1}\theta,\ldots,y_mz^{\eta_m}u^{d_m}\sin^{f_m}\theta)I\begin{pmatrix}z_1z^{l_1}u^{h_1}(\sin\theta)^{\sigma_1}\\ z_2z^{l_2}u^{h_2}(\sin\theta)^{\sigma_2}\end{pmatrix}dud\theta\right\}$$

$$=\sum_{n=0}^{\infty}\frac{(\gamma)_{qn}\Gamma(\beta)(-x)^n}{n!\Gamma(\mu n+v+1)s^r}\pi\cos\left(\frac{\pi a}{2}\right)\sum_{k=0}^{[N/M]}\sum_{k_1,\ldots,k_r=0}^{\infty}\frac{(-N)_{Mk}}{k!}A_{N,k}F_{\mu_i,v_1}^{\lambda_i}\left(\frac{y}{s^b2^f}\right)^k\prod_{i=1}^{m}\frac{1}{k_i!}\left(\frac{y_i}{s^{\eta_i}2^{f_i}}\right)^{k_i}$$

$$\times I_{p+4,q+4;V}^{m,n+4;U}\left(\begin{matrix}s^{-l_1}2^{-\sigma_1}z_1\\ s^{-l_2}2^{-\sigma_2}z_2\end{matrix}\middle|\begin{matrix}A_1,X_1,X_2,X_5,X_6:A_2:A_3;A_4\\ B_1,Y_1,Y_2,Y_3,Y_5:B_2:B_3;B_4\end{matrix}\right) \tag{53}$$

where $X_1,\ X_2,\ Y_1,\ Y_2,\ Y_3$ are given in the equations from (28) to (32) respectively and

$$X_5=(\tfrac{1}{2}-v-r-bk-\textstyle\sum_{i=1}^{m}\eta_ik_i;\ l_1,\ l_2;\ 1) \tag{54}$$

$$X_6=(\tfrac{1}{2}+v-r-bk-\textstyle\sum_{i=1}^{m}\eta_ik_i;\ l_1,\ l_2;\ 1) \tag{55}$$

$$Y_5=(\kappa-r-bk-\textstyle\sum_{i=1}^{m}\eta_ik_i;\ l_1,\ l_2;\ 1) \tag{56}$$

Proof: In order to prove the equation (53), using the definition (52) of Verma transform, we get

$$V\left\{\int_0^1\int_0^\pi u^{\alpha-1}(1-u)^{\beta-1}J_{v,q}^{\mu,\gamma}(xu^\delta)(\sin\theta)^\rho\cos a\theta S_N^M(yz^bu^d(\sin\theta)^f)\right.$$

$$\left.\times E_{\mu_i,v_1}^{\lambda_i}(y_1z^{\eta_1}u^{d_1}\sin^{f_1}\theta,\ldots,y_mz^{\eta_m}u^{d_m}\sin^{f_m}\theta)I\begin{pmatrix}z_1z^{l_1}u^{h_1}(\sin\theta)^{\sigma_1}\\ z_2z^{l_2}u^{h_2}(\sin\theta)^{\sigma_2}\end{pmatrix}dud\theta\right\}$$

$$=\int_0^\infty z^{r-1}e^{-\frac{1}{2}sz}W_{\kappa,v}(sz)\left\{\int_0^1\int_0^\pi u^{\alpha-1}(1-u)^{\beta-1}J_{v,q}^{\mu,\gamma}(xu^\delta)(\sin\theta)^\rho\cos a\theta S_N^M(yz^bu^d(\sin\theta)^f)\right.$$

$$\left.\times E_{\mu_i,v_1}^{\lambda_i}(y_1z^{\eta_1}u^{d_1}\sin^{f_1}\theta,\ldots,y_mz^{\eta_m}u^{d_m}\sin^{f_m}\theta)I\begin{pmatrix}z_1z^{l_1}u^{h_1}(\sin\theta)^{\sigma_1}\\ z_2z^{l_2}u^{h_2}(\sin\theta)^{\sigma_2}\end{pmatrix}dud\theta\right\}dz \tag{57}$$

now expressing the general class of polynomial and multivariable Mittage-Leffler function in series with the help of equations (23) and (22) and expressing the modified I-function of two variables in terms of Mellin-Barnes contour integral with the help of the equation (1) and changing the order of integration, which is permissible under the given conditions, we get

$$=\sum_{k=0}^{[N/M]}\sum_{k_1,\ldots,k_l=0}^{\infty}\frac{(-N)_{Mk}}{k!}A_{N,k}F_{\mu_i,v}^{\lambda_i}y^k\frac{y_1^{k_1}}{k_1!}\cdots\frac{y_m^{k_m}}{k_m!}\frac{1}{(2\pi i)^2}\int_{L_1}\int_{L_2}\Phi(s_1,s_2)\theta_1(s_1)\theta_2(s_2)z_1^{s_1}z_2^{s_2}$$

$$\times\left[\int_0^1 u^{\alpha+dk+\sum_{i=1}^m k_id_i+h_1s_1+h_2s_2-1}(1-u)^{\beta-1}J_{v,q}^{\mu,\gamma}(xu^\delta)du\right]\left[\int_0^\pi(\sin\theta)^{\rho+fk+\sum_{i=1}^m k_if_i+\sigma_1s_1+\sigma_2s_2}\cos a\theta d\theta\right]$$

$$\times\left[\int_0^\infty z^{r+bk+\sum_{i=1}^m\eta_ik_i+l_1s_1+l_2s_2-1}e^{-\frac{1}{2}sz}W_{\kappa,v}(sz)dz\right]ds_1ds_2 \tag{58}$$

now evaluating the inner z-integral, we obtain

$$\int_0^\infty z^{r+bk+\sum_{i=1}^m\eta_ik_i+l_1s_1+l_2s_2-1}e^{-\frac{1}{2}sz}W_{\kappa,v}(sz)dz=s^{-r-bk-\sum_{i=1}^m\eta_ik_i-l_1s_1-l_2s_2}$$

$$\times\frac{\Gamma(\frac{1}{2}+v+r+bk+\sum_{i=1}^m\eta_ik_i+l_1s_1+l_2s_2)\Gamma(\frac{1}{2}-v+r+bk+\sum_{i=1}^m\eta_ik_i+l_1s_1+l_2s_2)}{\Gamma(1-\kappa+r+bk+\sum_{i=1}^m\eta_ik_i+l_1s_1+l_2s_2)} \tag{59}$$

now putting the equations (34), (35) and (59) in the above equation (58) and interpreting the result with the help of equation (1) in terms of double contour integral, we get the required result (53).

Theorem 8: Let $d, f, d_i, f_i, h_j, \sigma_j \in R^+$ for $i=1,\ldots,m$ and $j=1,2$, $\Re(\alpha)>0$, $\Re(\beta)>0$ and $\Re(\rho)>-1$. Also let

$$\Re(\alpha)+(h_1+h_2)\min_{1\le j\le m_2}\Re\left(V_j\frac{f_j}{F_j}\right)>0,\ \Re(\rho)+(\sigma_1+\sigma_2)\min_{1\le j\le m_3}\Re\left(Q_j\frac{h_j}{H_j}\right)>-1.$$

Further, let $\left|\arg(z_iu^{h_i}\sin^{\sigma_i}\theta)\right|<\frac{1}{2}\Omega_i\pi\ (i=1,2)$ with Ω_i same as in the equations (7) and (8). Then

$$V\left\{\int_0^1\int_0^\pi u^{\alpha-1}(1-u)^{\beta-1}J_{v,q}^{\mu,\gamma}(xu^\delta)(\sin\theta)^\rho\cos a\theta S_N^M(yz^bu^d(\sin\theta)^f)\right.$$

$$\left.\times E_{\mu_i,v_1}^{\lambda_i}(y_1z^{\eta_1}u^{d_1}\sin^{f_1}\theta,\ldots,y_mz^{\eta_m}u^{d_m}\sin^{f_m}\theta)I\begin{pmatrix}z_1z^{l_1}u^{h_1}(\sin\theta)^{\sigma_1}\\ z_2z^{l_2}u^{h_2}(\sin\theta)^{\sigma_2}\end{pmatrix}dud\theta\right\}$$

$$=\sum_{n=0}^{\infty}\frac{(\gamma)_{qn}\Gamma(\beta)(-x)^n}{n!\Gamma(\mu n+v+1)s^r}\pi\cos\left(\frac{\pi a}{2}\right)\sum_{k=0}^{[N/M]}\sum_{k_1,\ldots,k_r=0}^{\infty}\frac{(-N)_{Mk}}{k!}A_{N,k}F_{\mu_i,v_1}^{\lambda_i}\left(\frac{y}{s^b 2^f}\right)^k\prod_{i=1}^{m}\frac{1}{k_i!}\left(\frac{y_i}{s^{\eta_i}2^{f_i}}\right)^{k_i}$$

$$\times I_{p+4,q+4;V}^{m,n+4;U}\left(\begin{matrix} s^{-l_1}2^{-\sigma_1}z_1 \\ s^{-l_2}2^{-\sigma_2}z_2\end{matrix}\middle| \begin{matrix} A_1, X_1, X_2, X_5, X_6 : A_2 : A_3; A_4 \\ B_1, Y_1, Y_2, Y_3, Y_5 : B_2 : B_3; B_4\end{matrix}\right) \tag{60}$$

where X_1, X_2, X_5, X_6, Y_1, Y_2, Y_3, Y_4 are given in the equations (28), (29), (54), (55), (30), (31), (32) and (56) respectively.

References

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