Stochastic Modeling and Applications
Vol. 22 No.2 (December, 2018)

NOTE ON MIDDLE NEIGHBORHOOD SIGNED GRAPHS

S. R. RAMACHANDRA* AND M. J. JYOTHI

Abstract. In this paper we introduced the new notion called middle neighborhood signed graph of a signed graph and its properties are studied. Also, we obtained the structural characterization of this new notion and presented some switching equivalent characterizations.

1. Introduction

By a graph $G = (V, E)$ we mean a finite undirected graph with neither loops nor multiple edges. The order $|V|$ and the size $|E|$ are denoted by n and m respectively. For standard terminology and notion in graph theory, we refer the reader to the text-book of Harary [1]. The non-standard will be given in this paper as and when required.

To model individuals' preferences towards each other in a group, Harary [2] introduced the concept of signed graphs in 1953. A signed graph $S = (G, \sigma)$ is a graph $G = (V, E)$ whose edges are labeled with positive and negative signs (i.e., $\sigma : E(G) \to \{+, -\}$). The vertexes of a graph represent people and an edge connecting two nodes signifies a relationship between individuals. The signed graph captures the attitudes between people, where a positive (negative edge) represents liking (disliking). An unsigned graph is a signed graph with the signs removed. Similar to an unsigned graph, there are many active areas of research for signed graphs.

The sign of a cycle (this is the edge set of a simple cycle) is defined to be the product of the signs of its edges; in other words, a cycle is positive if it contains an even number of negative edges and negative if it contains an odd number of negative edges. A signed graph S is said to be balanced if every cycle in it is positive. A signed graph S is called totally unbalanced if every cycle in S is negative. A chord is an edge joining two non adjacent vertices in a cycle.

2000 *Mathematics Subject Classification.* 05C22.

Key words and phrases. Signed graphs, Balance, Switching, Middle Neighborhood Signed Graph, Negation.

*Corresponding author.

A *marking* of S is a function $\zeta : V(G) \to \{+,-\}$. Given a signed graph S one can easily define a marking ζ of S as follows: For any vertex $v \in V(S)$,

$$\zeta(v) = \prod_{uv \in E(S)} \sigma(uv),$$

the marking ζ of S is called *canonical marking* of S.

The following are the fundamental results about balance, the second being a more advanced form of the first. Note that in a bipartition of a set, $V = V_1 \cup V_2$, the disjoint subsets may be empty.

Theorem 1.1. *A signed graph S is balanced if and only if either of the following equivalent conditions is satisfied:*

- **(i):** *Its vertex set has a bipartition $V = V_1 \cup V_2$ such that every positive edge joins vertices in V_1 or in V_2, and every negative edge joins a vertex in V_1 and a vertex in V_2 (Harary* [2]*).*
- **(ii):** *There exists a marking μ of its vertices such that each edge uv in Γ satisfies $\sigma(uv) = \zeta(u)\zeta(v)$. (Sampathkumar* [5]*).*

Switching S with respect to a marking ζ is the operation of changing the sign of every edge of S to its opposite whenever its end vertices are of opposite signs. The resulting signed graph $S_\zeta(S)$ is said switched signed graph. A signed graph S is called to switch to another signed graph S'written $S \sim S'$, whenever their exists a marking ζ such that $S_\zeta(S) \cong S'$, where $\cong$ denotes the usual equivalence relation of isomorphism in the class of signed graphs. Hence, if $S \sim S'$, we shall say that S and S' are switching equivalent. Two signed graphs S_1 and S_2 are signed isomorphic (written $S_1 \cong S_2$) if there is a one-to-one correspondence between their vertex sets which preserve adjacency as well as sign.

Two signed graphs $S_1 = (G_1, \sigma_1)$ and $S_2 = (G_2, \sigma_2)$ are said to be *weakly isomorphic* (see [6]) or *cycle isomorphic* (see [7]) if there exists an isomorphism $\phi : G_1 \to G_2$ such that the sign of every cycle Z in S_1 equals to the sign of $\phi(Z)$ in S_2. The following result is well known (see [7]):

Theorem 1.2. (**T. Zaslavsky** [7]) *Given a graph G, any two signed graphs in $\psi(G)$, where $\psi(G)$ denotes the set of all the signed graphs possible for a graph G, are switching equivalent if and only if they are cycle isomorphic.*

2. Middle Neighborhood Signed Graph of a Signed Graph

In [4], the author introduced the new notion called middle neighborhood graph of a graph by the motivation of dominating graph. Let $G = (V, E)$ be any graph, the graph $MDND(G)$ is called middle neighborhood graph with vertex set $V \cup S$ and any two vertices p and q in middle neighborhood graph are adjacent if and only if q is an open neighborhood set containing p or $p, q \in S$ and $p \cap q \neq \phi$. Here, the set S stands for the set of all open neighborhood sets of G.

Let $G = (V, E)$ be any 1-component graph with $|V| \geq 2$, then the corresponding middle neighborhood graph contains more than one component. In [4], the author mentioned that: G is a complete bipartite graph with $1, p-1$, where $p \geq 2$ vertices, then the corresponding middle neighborhood graph is isomorphic to $G \cup K_p$ and vice versa. For any graph G, the vertex neighborhood graph $VND(G)$ is a spanning subgraph of middle neighborhood graph $MDND(G)$. Suppose G is 1-component graph with $|V| \geq 2$, then the corresponding middle neighborhood graph is also 1-component if and only if the graph G contains a cycle of odd length. Suppose G is a bipartite graph $|V| \geq 4$, then the corresponding middle neighborhood graph $MDND(G)$ contains more than one component. The middle neighborhood graph $MDND(G)$ and the union of G and the complete graph with p vertices are isomorphic if and only if the graph G is complete bipartite graph with 1, p vertices, where $p \geq 2$. Suppose v is an end vertex in the middle neighborhood graph $MDND(G)$ and the corresponding vertex in the graph G is also an end vertex.

By the motivation of middle neighborhood graph introduced by Kulli [4], in this section we defined the new notion called middle neighborhood signed graph of signed graph as:
The middle neighborhood signed graph $MDND(S) = (MDND(G), \sigma')$ of a signed graph $S = (G, \sigma)$ is a signed graph, the sign of any edge $pq \in E(MDND(S))$ is the product of canonical marking of the vertices p and q. If any signed graph S is isomorphic to middle neighborhood signed graph of some signed S' (i.e., $MDND(S) \cong S'$), then S is called a middle neighborhood signed graph.

In general signed graphs can be partitioned into groups as: positive signed graphs (i.e., balanced signed graphs) and negative signed graphs (i.e., unbalanced signed graphs). Given signed graph $S = (G, \sigma)$ is either positive or negative, the middle neighborhood signed graph is always positive.

Theorem 2.1. *The middle neighborhood signed graph $MDND(S)$ is balanced, for any signed graph $S = (G, \sigma)$.*

Proof. Let $S = (G, \sigma)$ be any signed graph and S_ζ is a signed marked graph subsequently employ the canonical marking. Through the elucidation of middle neighborhood signed graph $MDND(S)$, we examined in order that the sign of any edge uv in $MDND(S)$ is $\sigma(uv) = \zeta(u)\zeta(v)$. From Theorem 1.1, it follows that middle neighborhood signed graph $MDND(S)$ is balanced. □

Consider the $\mathbb{Z}^+$ and $k \in \mathbb{Z}^+$, the k^{th} iterated middle neighborhood signed graph $MDND(S)$ of S is defined as follows:

$$MDND^0(S) = S,\ MDND^k(S) = MDND(MDND^{k-1}(S)).$$

Corollary 2.2. *the k^{th} iterated middle neighborhood signed graph $MDND^k(S)$ is always positive, for any signed graph $S = (G, \sigma)$.*

Theorem 2.3. *The middle neighborhood signed graphs of $S_1 = (G_1, \sigma)$ and $S_2 = (G_2, \sigma)$ are switching equivalent (i.e., $MDND(S_1) \sim MDND(S_2)$), if G_1 and G_2 are isomorphic.*

Proof. Consider two signed graphs S_1 and S_2 with their underlying graphs are isomorphic. Thereupon, the corresponding middle neighborhood signed graphs

$MDND(S_1)$ and $MDND(S_2)$ are positive. From Theorem 1.2, it follows that $MDND(S_1)$ and $MDND(S_2)$ are switching equivalent. □

In [4], the author proved that: the middle neighborhood graph $MDND(G)$ and $2pK_2$ are isomorphic if and only if G is isomorphic to pK_2, where $p \geq 1$. In view of this, we have the following:

Theorem 2.4. *Let $S = (G, \sigma)$ be any signed graph. Then $MDND(S) \sim 2pK_2(S)$ if and only if G is isomorphic to pK_2, where $p \geq 1$.*

Proof. Suppose $MDND(S) \sim 2pK_2(S)$. Then $MDND(G) \cong 2pK_2$, form the above observation, we have G is isomorphic to pK_2, where $p \geq 1$.

Conversely, suppose that G is isomorphic to pK_2, where $p \geq 1$. Consider a signed graph with underlying graph is isomorphic to pK_2, where $p \geq 1$. Then the corresponding vertex neighborhood signed graph is positive. Since G is pK_2, where $p \geq 1$, then $MDND(G) \cong 2pK_2$. From the structural characterization of Harary (Theorem 1.1), every signed graph with underlying graph as $2pK_2(S)$ is always positive. Hence, from Theorem 1.2, it follows that $MDND(S)$ and $2pK_2(S)$ are cycle isomorphic. □

In [4], Kulli characterize the graphs for which the graphs and its corresponding middle neighborhood graphs are isomorphic.

Theorem 2.5. *For any graph $G = (V, E)$, the middle neighborhood graph $MDND(G)$ and the graph G are isomorphic if and only if every pair of open neighborhood sets of vertices of G are disjoint.*

In this context, we now characterize the signed graphs for which the signed graphs and middle neighborhood signed graphs are cycle isomorphic.

Theorem 2.6. *For any signed graph $S = (G, \sigma)$, the signed graphs and middle neighborhood signed graphs are cycle isomorphic if and only if S is balanced and each pair of open neighborhood sets of vertices of G are disjoint.*

Proof. Suppose S is positive and each pair of open neighborhood sets of vertices of the underlying graph of the signed graph S are disjoint. Then, G and $MDND(G)$ are isomorphic. Now the middle neighborhood signed graph $MDND(S)$ of a signed graph S with underlying graph having each pair of open neighborhood sets of vertices of G are disjoint. From the hypothesis, S is positive and just now we have seen that $MDND(S)$ is also positive and hence S and $VND(S)$ are cycle isomorphic, from the Theorem 1.2.

Conversely, suppose that signed graph and its middle neighborhood signed graph are cycle isomorphic. Then $G \cong MDND(G)$. Therefore G satisfies property that each pair of open neighborhood sets of vertices of G are disjoint. Since $MDND(S)$ and S are cycle isomorphic. This satisfies only when S is positive. □

The notion of *negation* $\eta(S)$ of a given signed graph S defined in [3] as follows: $\eta(S)$ has the same underlying graph as that of S with the sign of each edge opposite to that given to it in S. However, this definition does not say anything about what to do with nonadjacent pairs of vertices in S while applying the unary operator $\eta(.)$ of taking the negation of S.

For a signed graph $S = (G, \sigma)$, the $MDND(S)$ is balanced (Theorem 2.1). We now examine, the conditions under which negation $\eta(S)$ of $MDND(S)$ is balanced.

Theorem 2.7. *Suppose the middle neighborhood graph $MDND(G)$ is bipartite. Then the negation of middle neighborhood signed graph $\eta(MDND(S))$ is positive, where S is any signed graph.*

Proof. Since, by Theorem 2.1, middle neighborhood signed graph $MDND(S)$ is positive. Then all the cycles in middle neighborhood signed graph $MDND(S)$ are positive. By the hypothesis, the middle neighborhood graph $MDND(G)$ is bipartite. Then each cycle C_n (where n is even) in $MDND(S)$ is positive. Therefore, the negation of middle neighborhood signed graph $MDND(S)$ is positive. □

We now give the structural characterization of vertex neighborhood signed graphs.

Theorem 2.8. *Suppose $S = (G, \sigma)$ be any signed graph. Then S is positive and its underlying graph is middle neighborhood graph $MDND(G)$ if and only if S is a middle neighborhood signed graph $MDND(S)$.*

Proof. Let us consider that S is a middle neighborhood signed graph $MDND(S)$. Then the signed graph S and the middle neighborhood signed graph of some signed graph S_1 (i.e., $MDND(S_1)$) are isomorphic. Since, the middle neighborhood signed graph of any signed graph is positive and we have $S \cong MDND(S_1)$. Consequently, S is positive and its underlying graph is a middle neighborhood graph.

Conversely, suppose that S is positive and its underlying graph is middle neighborhood graph $MDND(G)$. Since, the signed graph S is positive, then establish the S_ζ. With the evidence of Sampathkumar's result (Theorem 1.1), every edge pq in S_ζ amuse $\sigma(pq) = \zeta(p)\zeta(q)$. Deliberate, the signed graph $S_1 = (G_1, \sigma_1)$ in which each edge $e = (pq)$ in G_1, $\sigma_1(e) = \zeta(p)\zeta(q)$. Therefore, the signed graph S and the middle neighborhood signed graph of S_1 are isomorphic. Hence, S is a middle neighborhood signed graph $MDND(S)$. □

In view of the negation operator introduced by Harary [3], we have the following cycle isomorphic characterizations:

Corollary 2.9. *The negation of middle neighborhood signed graphs of $S_1 = (G_1, \sigma)$ and $S_2 = (G_2, \sigma)$ are cycle isomorphic, if G_1 and G_2 are isomorphic.*

Corollary 2.10. *For any two signed graphs $S_1 = (G_1, \sigma)$ and $S_2 = (G_2, \sigma)$, $MDND(\eta(S_1))$ and $MDND(\eta(S_2))$ are are cycle isomorphic, if G_1 and G_2 are isomorphic.*

Corollary 2.11. *For any signed graph $S = (G, \sigma)$, the signed graph S and middle neighborhood signed graph of $\eta(S)$ are cycle isomorphic if and only if S is balanced and G satisfies the property that each pair of open neighborhood sets of vertices of G are disjoint.*

Acknowledgment. The authors would like to thank the referees for their invaluable comments and suggestions which led to the improvement of the manuscript.

References

1. Harary, F.: *Graph Theory*, Addison Wesley, Reading, Mass, (1972).
2. Harary, F.: On the notion of balance of a sigraph, *Michigan Math. J.*, **2** (1953), 143-146.
3. Harary, F.: Structural duality, *Behav. Sci.*, **2**(4) (1957), 255-265.
4. Kulli, V. R.: On Middle Neighborhood Graphs, *International Journal of Mathematics And its Applications*, **3**(4-D) (2015), 79-83.
5. Sampathkumar, E.: Point signed and line signed graphs, *Nat. Acad. Sci. Letters*, **7**(3) (1984), 91-93.
6. Sozánsky, T.: Enueration of weak isomorphism classes of signed graphs, *J. Graph Theory*, **4**(2)(1980), 127-144.
7. Zaslavsky, T.: Signed graphs, *Discrete Appl. Math.*, **4** (1982), 47-74.

S. R. Ramachandra: Department of Mathematics, Government College, M.C. Road, Mandya-571 401, India.

E-mail address: srrc16@gmail.com

M. J. Jyothi: Department of Mathematics, Maharani's Science College for Women, Mysuru-570 005, India.

E-mail address: jyothimj49@gmail.com