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ON TOKEN SIGNED GRAPHS

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Abstract. The new concept of a token signed graph of a signed graph was presented in this research, and its characteristics were examined. The structural characterization of this novel concept was also acquired, and some switching equivalent characterizations were offered.

1. Introduction

By a graph $G = (V, E)$ we mean a finite undirected graph with neither loops nor multiple edges. The order $|V|$ and the size $|E|$ are denoted by n and m respectively. For standard terminology and notion in graph theory, we refer the reader to the text-book of Harary [5]. The non-standard will be given in this paper as and when required.

Numerous issues in mathematics and computer science are represented by the movement of objects along the vertices of a graph in accordance with specific rules. In "graph pebbling," a pebbling step is removing two pebbles off a vertex and inserting one pebble on an adjacent vertex.

For any set V, we denote by $P_k(V)$ the set of all k-element subsets of V. Monray et al. [8] introduced the notion of k-token graph of a graph G.

Let $G = (V, E)$ be a graph and let $k \geq 1$ be an integer. The k-token graph $\mathcal{F}_k(G)$ of G is the graph with vertex set $P_k(V)$ and two vertices A and B are adjacent if $A\Delta B = \{a, b\}$ where $ab \in E(G)$. Here $A\Delta B$ is a symmetric difference and is a pair $\{a, b\}$ such that $a \in A$, $b \in B$ and $ab \in E(G)$ (See [8]).

To model individuals' preferences towards each other in a group, Harary [6] introduced the concept of signed graphs in 1953. A signed graph $S = (G, \sigma)$ is a graph $G = (V, E)$ whose edges are labeled with positive and negative signs (i.e., $\sigma : E(G) \to \{+, -\}$). The vertexes of a graph represent people and an edge connecting two nodes signifies a relationship between individuals. The signed graph captures the attitudes between people, where a positive (negative edge) represents liking (disliking). An unsigned graph is a signed graph with the signs removed. Similar to an unsigned graph, there are many active areas of research for signed

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graphs.

The sign of a cycle (this is the edge set of a simple cycle) is defined to be the product of the signs of its edges; in other words, a cycle is positive if it contains an even number of negative edges and negative if it contains an odd number of negative edges. A signed graph S is said to be balanced if every cycle in it is positive. A signed graph S is called totally unbalanced if every cycle in S is negative. A chord is an edge joining two non adjacent vertices in a cycle.

A *marking* of S is a function $\zeta : V(G) \to \{+, -\}$. Given a signed graph S one can easily define a marking ζ of S as follows: For any vertex $v \in V(S)$,

$$\zeta(v) = \prod_{uv \in E(S)} \sigma(uv),$$

the marking ζ of S is called *canonical marking* of S.

The following are the fundamental results about balance, the second being a more advanced form of the first. Note that in a bipartition of a set, $V = V_1 \cup V_2$, the disjoint subsets may be empty.

Theorem 1.1. *A signed graph S is balanced if and only if either of the following equivalent conditions is satisfied:*

- **(i):** *Its vertex set has a bipartition $V = V_1 \cup V_2$ such that every positive edge joins vertices in V_1 or in V_2, and every negative edge joins a vertex in V_1 and a vertex in V_2 (Harary [6]).*
- **(ii):** *There exists a marking μ of its vertices such that each edge uv in Γ satisfies $\sigma(uv) = \zeta(u)\zeta(v)$. (Sampathkumar [9]).*

Switching S with respect to a marking ζ is the operation of changing the sign of every edge of S to its opposite whenever its end vertices are of opposite signs. The resulting signed graph $S_\zeta(S)$ is said switched signed graph. A signed graph S is called to switch to another signed graph S'written $S \sim S'$, whenever their exists a marking ζ such that $S_\zeta(S) \cong S'$, where $\cong$ denotes the usual equivalence relation of isomorphism in the class of signed graphs. Hence, if $S \sim S'$, we shall say that S and S' are switching equivalent. Two signed graphs S_1 and S_2 are signed isomorphic (written $S_1 \cong S_2$) if there is a one-to-one correspondence between their vertex sets which preserve adjacency as well as sign.

Two signed graphs $S_1 = (G_1, \sigma_1)$ and $S_2 = (G_2, \sigma_2)$ are said to be *weakly isomorphic* (see [12]) or *cycle isomorphic* (see [13]) if there exists an isomorphism $\phi : G_1 \to G_2$ such that the sign of every cycle Z in S_1 equals to the sign of $\phi(Z)$ in S_2. The following result is well known (see [13]):

Theorem 1.2. (**T. Zaslavsky** [13]) *Given a graph G, any two signed graphs in $\psi(G)$, where $\psi(G)$ denotes the set of all the signed graphs possible for a graph G, are switching equivalent if and only if they are cycle isomorphic.*

2. Token Signed Graphs

Motivated by the existing definition of complement of a signed graph, we now extend the notion of toke graphs to signed graphs as follows: The *token signed graph* $\mathcal{F}_k(S)$ of a signed graph $S = (G, \sigma)$ is a signed graph whose underlying graph is $\mathcal{F}_k(G)$ and sign of any edge uv is $\mathcal{F}_k(S)$ is $\zeta(u)\zeta(v)$, where ζ is the canonical marking of S. Further, a signed graph $S = (G, \sigma)$ is called distance divisor signed graph, if $S \cong \mathcal{F}_k(S')$ for some signed graph S'. The following result restricts the class of token graphs.

Theorem 2.1. *For any signed graph $S = (G, \sigma)$, its token signed graph $\mathcal{F}_k(S)$ is balanced.*

Proof. Since sign of any edge $e = uv$ in $\mathcal{F}_k(S)$ is $\zeta(u)\zeta(v)$, where ζ is the canonical marking of S, by Theorem 1.1, $\mathcal{F}_k(S)$ is balanced. □

For any positive integer m, the m^{th} iterated token signed graph, $\mathcal{F}_k^m(S)$ of S is defined as follows:

$$\mathcal{F}_k^0(S) = S,\ \mathcal{F}_k^m(S) = \mathcal{F}_k(\mathcal{F}_k^{m-1}(S)).$$

Corollary 2.2. *For any signed graph $S = (G, \sigma)$ and for any positive integer m, $\mathcal{F}_k^m(S)$ is balanced.*

Theorem 2.3. *For any two signed graphs S_1 and S_2 with the same underlying graph, their token signed graphs are switching equivalent.*

Proof. Suppose $S_1 = (G, \sigma)$ and $S_2 = (G', \sigma')$ be two signed graphs with $G \cong G'$. By Theorem 2.1, $\mathcal{T}_k(S_1)$ and $\mathcal{T}_k(S_2)$ are balanced and hence, the result follows from Theorem 1.2. □

The following result characterize signed graphs which are token signed graphs.

Theorem 2.4. *A signed graph $S = (G, \sigma)$ is a token signed graph if, and only if, S is balanced signed graph and its underlying graph G is a token graph.*

Proof. Suppose that S is balanced and G is a token graph. Then there exists a graph G' such that $\mathcal{F}_k(G') \cong G$. Since S is balanced, by Theorem 1.1, there exists a marking ζ of G such that each edge uv in S satisfies $\sigma(uv) = \zeta(u)\zeta(v)$. Now consider the signed graph $S' = (G', \sigma')$, where for any edge e in G', $\sigma'(e)$ is the marking of the corresponding vertex in G. Then clearly, $\mathcal{F}_k(S') \cong S$. Hence S is a token signed graph.

Conversely, suppose that $S = (G, \sigma)$ is a token signed graph. Then there exists a signed graph $S' = (G', \sigma')$ such that $\mathcal{F}_k(S') \cong S$. Hence, G is the token graph of G' and by Theorem 2.1, S is balanced. □

The notion of *negation* $\eta(S)$ of a given signed graph S defined in [7] as follows: $\eta(S)$ has the same underlying graph as that of S with the sign of each edge opposite to that given to it in S. However, this definition does not say anything about what to do with nonadjacent pairs of vertices in S while applying the unary operator $\eta(.)$ of taking the negation of S.

For a signed graph $S = (G, \sigma)$, the $\mathcal{FB}(S)$ is balanced (Theorem 2.1). We now examine, the conditions under which negation $\eta(S)$ of $\mathcal{FB}(S)$ is balanced.

Proposition 2.5. *Let $S = (G, \sigma)$ be a signed graph. If $\mathcal{F}_k(G)$ is bipartite then $\eta(\mathcal{F}_k(S))$ is balanced.*

Proof. Since, by Theorem 2.1, $\mathcal{F}_k(S)$ is balanced, it follows that each cycle C in $\mathcal{F}_k(S)$ contains even number of negative edges. Also, since $\mathcal{F}_k(G)$ is bipartite, all cycles have even length; thus, the number of positive edges on any cycle C in $\mathcal{F}_k(S)$ is also even. Hence $\eta(\mathcal{F}_k(S))$ is balanced. □

3. Switching Equivalence of 2-Token Signed Graphs and Line Signed Graphs

Behzad and Chartrand [2] introduced the notion of line signed graph $L(S)$ of a given signed graph S as follows: Given a signed graph $S = (G, \sigma)$ its *line signed graph* $L(S) = (L(G), \sigma')$ is the signed graph whose underlying graph is $L(G)$, the line graph of G, where for any edge e_ie_j in $L(S)$, $\sigma'(e_ie_j)$ is negative if, and only if, both e_i and e_j are adjacent negative edges in S. Another notion of line signed graph introduced in [4], is as follows: The *line signed graph* of a signed graph $S = (G, \sigma)$ is a signed graph $L(S) = (L(G), \sigma')$, where for any edge ee' in $L(S)$, $\sigma'(ee') = \sigma(e)\sigma(e')$. In this paper, we follow the notion of line signed graph defined by M. K. Gill [4] (See also E. Sampathkumar et al. [10, 11]).

Theorem 3.1. (**M. Acharya** [1]) *For any signed graph $S = (G, \sigma)$, its line signed graph $L(S) = (L(G), \sigma')$ is balanced.*

In [3], the authors remarked that $\mathcal{F}_2(G)$ and $L(G)$ are isomorphic if and only if G is a K_n. We now give a characterization of signed graphs whose 2-token signed graphs are switching equivalent to their line signed graphs.

Theorem 3.2. *For any nontrivial connected signed graph $S = (G, \sigma)$, $\mathcal{F}_2(S) \sim L(S)$ if and only if G is a K_n.*

Proof. Suppose $\mathcal{F}_2(S) \sim L(S)$. This implies, $\mathcal{F}_2(G) \cong L(G)$ and hence G is a K_n.

Conversely, suppose that G is a K_n. Then $\mathcal{F}_2(G) \cong L(G)$. Now, if S any signed graph with G is a K_n, By Theorem 2.1 and 3.1, $\mathcal{F}_2(S)$ and $L(S)$ are balanced and hence, the result follows from Theorem 1.2. This completes the proof.

□

Theorem 2.3 & 3.2 provides easy solutions to other signed graph switching equivalence relations, which are given in the following results:

Corollary 3.3. *For any two signed graphs S_1 and S_2 with the same underlying graph, $\mathcal{T}_k(S_1)$ and $\mathcal{T}_k(\eta(S_2))$ are switching equivalent.*

Corollary 3.4. *For any two signed graphs S_1 and S_2 with the same underlying graph, $\mathcal{T}_k(\eta(S_1))$ and $\mathcal{T}_k(S_2)$ are switching equivalent.*

Corollary 3.5. *For any two signed graphs S_1 and S_2 with the same underlying graph, $\mathcal{T})k(\eta(S_1))$ and $\mathcal{T}_k(\eta(S_2))$ are switching equivalent.*

Corollary 3.6. *For any two signed graphs $S_1 = (G_1, \sigma_1)$ and $S_2 = (G_2, \sigma_2)$ with the $G_1 \cong G_2$ and G_1, G_2 are bipartite, $\eta(\mathcal{T}_k(S_1))$ and $\mathcal{T}_k(S_2)$ are switching equivalent.*

Corollary 3.7. *For any two signed graphs $S_1 = (G_1, \sigma_1)$ and $S_2 = (G_2, \sigma_2)$ with the $G_1 \cong G_2$ and G_1, G_2 are bipartite, $\mathcal{T}_k(S_1)$ and $\eta(\mathcal{T}_k(S_2))$ are switching equivalent.*

Corollary 3.8. *For any two signed graphs $S_1 = (G_1, \sigma_1)$ and $S_2 = (G_2, \sigma_2)$ with the $G_1 \cong G_2$ and G_1, G_2 are bipartite, $\eta(\mathcal{T}_k(S_1))$ and $\eta(\mathcal{T}_k(S_2))$ are switching equivalent.*

Corollary 3.9. *For any nontrivial connected signed graph $S = (G, \sigma)$, $\mathcal{F}_2(\eta(S)) \sim L(S)$ if and only if G is a K_n.*

Corollary 3.10. *For any nontrivial connected signed graph $S = (G, \sigma)$, $\mathcal{F}_2(\eta(S)) \sim L(\eta(S))$ if and only if G is a K_n.*

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