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2-DISTANCE STRONG B-COLORING OF HELM AND CLOSED HELM GRAPHS

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ABSTRACT

A vertex v is said to be an at most 2-neighbor of a vertex u, if $\mathrm{d(u,v)} \leq 2$. For a graph G, a 2-distance strong b-coloring is a proper coloring of G in which each pair of vertices that are at distance less than or equal to 2 receive different colors in addition to the condition that each color class has a vertex that must have at most 2-neighbor in every other color class. This vertex is called as a strong color dominating vertex. The corresponding parameter termed as 2-distance strong b-chromatic number $\chi_{2\mathrm{sb}}(\mathrm{G})$ is the largest integer k for which G has a 2-distance strong b-coloring with k colors. This article discusses a relationship between a b-coloring and a $2\mathrm{sb}$-coloring and the precise values of the Helm and Closed Helm's 2-distance strong b-chromatic numbers are determined.

Keywords: 2-distance strong b-coloring, 2-distance strong b-chromatic number, strong color dominating vertex.

1. Introduction

This work exclusively considers finite, simple, and undirected graphs. Readers may refer Harary [1, 1988] for terminologies which are not defined here. The 2-distance strong b-coloring (or simply $2sb$-coloring) was defined by us in 2021 [8] and prior to this coloring, in 2019 [6], we introduced another coloring called a 2-distance b-coloring (or simply $2b$-coloring). Both colorings can be viewed as conditional 2-distance coloring. Kramer F. and Kramer H. [3,4, 1969] defined 2-distance coloring as "a proper coloring in which every pair of vertices that are at least 2 distance apart get different colors". The smallest integer k for which G exhibits 2-distance coloring with k colors is the 2-distance chromatic number of G. It is denoted by $\chi_2(G)$. Many results on the boundaries of 2-distance chromatic number were reported in a comprehensive study by Kramers [5, 2008]. The other interesting coloring is b-coloring, which was introduced by Irving and Manlove [2, 1999]. It was defined as a proper coloring of G such that in each color class there must exist a vertex termed as a color dominating vertex (or cdv) that has at least one neighbor in each other color class and, the largest integer k such that G has a b-coloring with k colors is the b-chromatic number of G, represented as $\chi_b(G)$. Both these colorings motivated us to define two types of distance based coloring incorporated with the concept of having a cdv in each color class in one coloring and a modified version of a cdv in the next coloring.

The 2-distance b-coloring directly uses the definition of b-coloring. Shortly saying, a 2-distance b-coloring is both a 2-distance coloring and a b-coloring. The definition of the 2-distance coloring states that any two vertices which are at distance less than or equal to 2 must have different colors. The concept of at most two distance was critical in defining the 2-distance strong b-coloring. So instead of having a cdv in each color class, we define that each color class has a vertex say u such that there exists a vertex in every other color class which is at distance less than or equal to 2 from u. More explicitly, if $S_1, S_2, \dots, S_k$, is a color class partition of the vertex set, then for each i, there exists $u_i \in S_i$ for which there exists a $v_j \in S_j, j \neq i$ such that $d(u_i, v_j) \leq 2$. When we compare a vertex and its neighbors, and the same vertex with its neighbors and 2-neighbors, it is a simple fact that in the second case, the vertex is associated with more vertices and in this context, the vertex in the new coloring having a neighbor or 2-neighbor is called as a s-cdv.

In [8, 2021], the bounds of the $2sb$-number of some of the well-known graphs such as path, cycle were obtained. In [7, 2021], we introduced a new rooted tree called as a perfect Δ-ary tree having all internal vertices are of degree Δ and all pendant vertices are at the same level. In the same paper, the exact value of the $2sb$-number of a perfect Δ-ary tree was obtained. In this paper, a relationship between a b-coloring and a $2sb$-coloring is obtained. The b-chromatic number of cycle related and wheel related graphs were studied by Vaidya and Shukla [9, 10, 2014]. Also, they [11, 2014] found the b-chromatic number of Helm and Closed Helm. These papers made us study the new coloring on Helm and Closed Helm. The main outcome of this paper is obtaining the bounds of the $2sb$-number of Helm and Closed Helm.

2. Definitions and Some Prior Results

Definitions and results for 2-distance strong b-coloring are provided in this section.

Proposition 2.1: For $n \geq 4$,

(i) $\chi_2(H_n) = \begin{cases} 5, & if\ n = 4 \\ n, & if\ n \geq 5 \end{cases}$;

(ii) $\chi_2(CH_n) = \begin{cases} 7, & if\ n = 4 \\ n, & if\ n \geq 5 \end{cases}$.

Definition 2.2: In a graph G, for $u \in V$, a vertex v is said to be an at most k-neighbor, if $d(u, v) \leq k$.

Definition 2.3: A set S of vertices is said to be an $(k \geq 2)$at most k-neighborhood (or at most k-open neighborhood) of a vertex u, if S consists of all the vertices which are at distance at most k from u. It is represented by $N_{\leq k}(u)$.

i.e., $N_{\leq k}(u) = \{v \in V : d(u, v) \leq k\}$.

An at most k-closed neighborhood of u is the set$N_{\leq k}[u] = N_{\leq k}(u) \cup \{u\}$.

Definition 2.4: In a graph $G = (V, E)$, for$u \in V$, the at most k-degree $(k \geq 2)$of u is the number of vertices that are at distance less than or equal to k from u and it is denoted by $d_{\leq k}(u)$.

The at most k-degree $d_{\leq k}(u)$ of a vertex u is also called the $d_{\leq k}$-degree of u.

In this paper, only the case $k = 2$ is considered.

Definition 2.5: A vertex in a color class partition is said to be a strong color dominating vertex (or simply s-cdv) if it has an at most 2-neighbor in every other color class.

Definition 2.6: A 2-distance strong b-coloring ($2sb$-coloring) of G is a 2-distance coloring of G in which each color class has a s-cdv.

Definition 2.7: For a graph G, the 2-distance strong b-chromatic number (or simply $2sb$-number) $\chi_{2sb}(G)$ is the largest k for which G has a 2-distance strong b-coloring with k colors.

Definition 2.8: Let $V = \{v_1, v_2, \dots, v_n\}$ be the vertex set of G in which the vertices are ordered so that $d_{\leq 2}(v_1) \geq d_{\leq 2}(v_2) \geq \dots \geq d_{\leq 2}(v_n)$. Then $m_{\leq 2}$-degree of G, presented by $m_{\leq 2}(G)$, is defined by $m_{\leq 2}(G) = max\{i : 1 \leq i \leq n,\ d_{\leq 2}(v_i) \geq i - 1\}$.

In other words, $m_{\leq 2}(G)$ is presented as the $maximum\ \{i : 1 \leq i \leq |V(G)|,\ G$ has at least i vertices whose at most 2 degree is at least $i - 1\}$.

Observation 2.9:

(i) If a graph G has a 2-distance strong b-coloring with k different colors, it must have at least k vertices whose at most 2-degree is at least $k - 1$;

(ii) Any 2-distance coloring with $\chi_2$colors is a 2-distance strong b-coloring;

(iii) 2-distance strong b-coloring exists for all graphs;

(iv) A 2-distance b-coloring is equivalent to a 2-distance strong b-coloring, and the converse is not true.

Proposition 2.10: If $diam(G) \leq 2$, then $\chi_{2sb}(G) = |V(G)|$.

Corollary 2.11: If a graph has a dominating vertex, then $\chi_{2sb}(G) = |V(G)|$.

Proof: Since, G has a dominating vertex, for any pair of vertices u & $v, d(u, v) \leq 2$. Hence, G is of diameter at most 2. Hence, $\chi_{2sb}(G) = |V(G)|$.

3. Relationship between a b-coloring and a $2sb$-coloring

When a new concept and a related parameter are defined based on an existing concept and parameter, there exists a relationship between the two. If χ_{new} and χ_{old} are two chromatic parameters respectively denoting the new and old as specified in the suffix, then there may exist a relation that for a given graph $G, \chi_{new}(G) = \chi_{old}(G) \pm k$ where k is a positive integer. In some cases, if G' is a graph obtained from the graph G, then there may exist a relation that $\chi_{new}(G') = \chi_{old}(G)$or $\chi_{new}(G) = \chi_{old}(G')$. One of the well known such kind of results is that $\chi'(G) = \chi(L(G))$ where $\chi'(G)$is the chromatic index or edge chromatic number of G, and $L(G)$ is its line graph. This section establishes that each $2sb$-coloring of a graph G is equivalent to a b-coloring of the square graph $G^2$and that any b-coloring of G^2 is equivalent to a $2sb$-coloring of G.

Definition 3.1: For a given graph $G = (V, E)$, a k-distance graph denoted by G^k is the graph whose vertex set is V and whose edge set contains all the edges connecting any pair of vertices that are at distance less than or equal to k in G.

If $k = 2$, the graph G^k is called the square graph.

Theorem 3.2: For any graph G, a $2sb$-coloring of G is a b-coloring of G^2 and vice versa; and hence $\chi_{2sb}(G) = \chi_b(G^2)$.

Proof: From the definition of G^2, for any pair of vertices u, v,

$d_G(u, v) \leq 2 \iff uv \in E(G^2)$ ---------- (1)

The theorem is proved in two claims.

Claim 1:Any $2sb$-coloring of G is a b-coloring of G^2

Let c be a $2sb$-coloring of G. Since $V(G) = V(G^2)$, give colors to the vertices of G^2 as in G given by c. If $uv \in E(G^2)$, then from (1) and from the assumption that c is a $2sb$-coloring of G, $c(x) \neq c(y)$. Hence, in G^2, $c(x) \neq c(y)$. Then c is a proper coloring of G^2. Also, each *s-cdv* of G becomes a cdv in G^2 and hence,c is a b-coloring of G^2.

Claim 2:Any b-coloring of G^2 is a $2sb$-coloring of G

Let c be a b-coloring of G^2. As before, assign colors to the vertices of G as given in G^2 by c. Let $u, v \in V(G)$ and $d_G(u, v) \leq 2$. Then $uv \in E(G^2)$ and hence, $c(u) \neq c(v)$ in G^2.Consequently, u, v receive distinct colors in G. Hence, c is a 2-distance coloring. Since, each neighboring vertex of u in G^2 is an at most 2-neighbor of u in G, each cdv of G^2 becomes a *s-cdv* in G. Hence, in G, each color class contains a *s-cdv* and hence, c is a $2sb$-coloring of G. Hence, the claim.

From claim 1, $\chi_{2sb}(G) \leq \chi_b(G^2)$ and from claim 2, $\chi_b(G^2) \leq \chi_{2sb}(G)$. This yields the result.

4. 2-distance strong *b*-coloring of Helm and Closed Helm

In this section, two graphs, the Helm and the Closed Helm are chosen to find the bound for the 2-distance strong b-chromatic number.

Proposition 4.1: For $n \geq 4$,

$$\chi_{2sb}(H_n) = \begin{cases} 5, & if\ n = 4 \\ n, & if\ n \geq 5 \end{cases}$$

Proof: A Helm graph contains a Wheel graph W_n as an induced subgraph and u is the central vertex and the $(n-1)$-cycle is induced by $v_1, v_2, \dots v_{n-1}$ of W_n. Let c be a $2sb$-coloring with $\chi_{2sb} = r$ colors.

In H_n, for $1 \leq i \leq n-1$, w_i is the pendant vertex adjacent to v_i. Then $N_{\leq 2}(u) = \{v_i: 1 \leq i \leq n-1\} \cup \{w_i: 1 \leq i \leq n-1\} = \{v_i, w_i: 1 \leq i \leq n-1\}$. For $1 \leq i \leq n-1$,$N_{\leq 2}(v_i) = \{u, w_i, w_{i-1}, w_{i+1}\} \cup \{v_j: 1 \leq j \leq n-1, j \neq i\}$ and $N_{\leq 2}(w_i) = \{u, v_i, v_{i-1}, v_{i+1}\}$ where $w_n = w_1, w_0 = w_{n-1}, v_n = v_1 \& v_0 = v_{n-1}$. The $d_{\leq 2}$-degree of the vertices are given in Table 1.

Table 1: At most 2-degrees in H_n

Vertex v	$d_{\leq 2}(v)$
u	$2n-2$
v_i	$n+2$
w_i	4

If $n = 4$, the degree sequence $(d_{\leq 2}(u), d_{\leq 2}(v_1), d_{\leq 2}(v_2), d_{\leq 2}(v_3), d_{\leq 2}(w_1), d_{\leq 2}(w_1), d_{\leq 2}(w_2), d_{\leq 2}(w_3)) = (6,6,6,6,4,4,4)$. Therefore, $m_{\leq 2}(H_n) = 5$.

If $n = 5$, then the degree sequence is given by $(8,7,7,7,7,4,4,4,4)$. This gives $m_{\leq 2}(H_n) = 5$.

If $n \geq 6$, then there are exactly n vertices of $d_{\leq 2}$-degree at least $n+2$ and $n-1$ vertices of $d_{\leq 2}$-degree 4. Hence, $m_{\leq 2}(H_n) = n$. The $m_{\leq 2}$-degrees of H_n are tabulated in Table 2.

Table 2: Values of $m_{\leq 2}(H_n)$

n	$m_{\leq 2}(H_n)$
4	5
≥ 5	n

Then from proposition 2.1 (i), $\chi_2(H_n) = m_{\leq 2}(H_n) = \begin{cases} 5, & if\ n = 4 \\ n, & if\ n \geq 5 \end{cases}$

$$\Rightarrow \chi_{2sb}(H_n) = \begin{cases} 5, & if\ n = 4 \\ n, & if\ n \geq 5 \end{cases}$$

Proposition 4.2: For $n \geq 4$,

$$\chi_{2sb}(CH_n) = \begin{cases} 7, & if\ n = 4 \\ 5, & if\ n = 5 \\ 7, & if\ n = 6 \\ 8, & if\ n = 7,8 \\ n, & if\ n \geq 9 \end{cases}$$

Proof: In CH_n, for $1 \leq i \leq n-1$, $\{w_i: 1 \leq i \leq n-1\}$ induces the $(n-1)$-cycle given by $w_1, w_2, \dots w_{n-1}, w_1$. The sets $N_{\leq 2}(u)$ and for $1 \leq i \leq n-1$, $N_{\leq 2}(v_i)$ are same as in (i). Hence,$d_{\leq 2}(u)$ & $d_{\leq 2}(v_i)$ are same as given in Table 1. Since, each w_i has the same property, $d_{\leq 2}(w_i)$ is the same for all $w_i's$.For computational purpose, Table 3 lists $N_{\leq 2}(w_1)$ and $d_{\leq 2}(w_1)$ for various values of n.

Table 3:$N_{\leq 2}(w_1)$&$d_{\leq 2}(w_1)$

n	$N_{\leq 2}(w_1)$	$d_{\leq 2}(w_1)$
4	$\{u, v_1, v_2, v_3, w_2, w_3\}$	6
5	$\{u, v_1, v_2, v_4, w_2, w_3, w_4\}$	7
6	$\{u, v_1, v_2, v_5, w_2, w_3, w_4, w_5\}$	8
$n \geq 7$	$\{u, v_1, v_2, v_{n-1}, w_2, w_3, w_{n-2}, w_{n-1}\}$	8

Case (i):$n = 4$

Since, $diam(CH_n) = 2$, from proposition 2.10, $\chi_{2sb}(CH_n) = |V| = 7$.

Case (ii):$n = 5$

Then from Tables 1 & 3, the degree sequence$(d_{\leq 2}(u), d_{\leq 2}(v_1), \dots, d_{\leq 2}(v_4), d_{\leq 2}(w_1), \dots d_{\leq 2}(w_4))$ is given by $(8,7,7,7,7,7,7,7,7)$. $i.e.,$ only one vertex is of $d_{\leq 2}$-degree 8 and for all the remaining 8 vertices are of $d_{\leq 2}$-degree 7. Hence, $m_{\leq 2}(CH_n) = 8$. Hence from proposition 2.1 (ii), $5 \leq \chi_{2sb}(CH_n) = r \leq 8$.

Claim :$r = 5$

Suppose $r > 5$. Assign colors 1 to 5 to u and v_i with $c(v_i) = i$, $1 \leq i \leq 4$ and $c(u) = 5$. Hence, the remaining $r - 5 \geq 1$ colors are given to $w_i's$. Let $c(w_1) = 6$. Since, for $i \neq 1$, $d(w_1, w_i) \leq 2$ &$d(v_3, w_i) \leq 2$, $c(w_i) \notin \{3,6\}$. Hence, $S_3 = \{v_3\}$ and $S_6 = \{w_1\}$. Since $d(v_3, w_1) = 3$, both v_3 and w_1 cannot be strong color dominating vertices, a contradiction to the choice of c. Hence, $r > 5$ cannot hold and hence $r \leq 5$ and the equality holds. Therefore, $\chi_{2sb}(CH_n) = 5$.

Case (iii): $6 \leq n \leq 8$

From Tables 1 & 3, $d_{\leq 2}(u), d_{\leq 2}(v_i)$ and $d_{\leq 2}(w_i)$ are tabulated in Table 4.

Table 4: At most 2-degrees of vertices of CH_n

n	$d_{\leq 2}(u)$	$d_{\leq 2}(v_i)$	$d_{\leq 2}(w_i)$
6	10	8	8
7	12	9	8
8	14	10	8

Table 5: n_i& $m_{\leq 2}(CH_n)$

n	n_1	n_2	n_3	$m_{\leq 2}(CH_n)$
6	11	1	1	9
7	13	7	1	9
8	15	8	8	9

From the tabulated values, $10 \leq d_{\leq 2}(u) \leq 14$,
$8 \leq d_{\leq 2}(v_i) \leq 10$&$d_{\leq 2}(w_i) = 8$.

Hence, we define the following sets.

$X_1 = \{v: d_{\leq 2}(v) \geq 8\}$, $X_2 = \{v: d_{\leq 2}(v) \geq 9\}$, $X_3 = \{v: d_{\leq 2}(v) \geq 10\}$ and $n_i = |X_i|$. Then the following are observed.

(a) For all n, $|X_1| = |V| = 2n - 1$;

(b) For $n = 6$, $X_2 = X_3 = \{u\}$;

(c) For $n = 7$, $X_2 = \{u, v_i\}$& $X_3 = \{u\}$;

(d) For $n = 8$, $X_2 = X_3 = \{u, v_i\}$.

In Table 5, the computed values of n_i and $m_{\leq 2}(CH_n)$ are tabulated.

From Table 5, $m_{\leq 2}(CH_n) = 9$ and from proposition 2.1 (ii), $\chi_2(CH_n) = n$.

$\therefore n \leq \chi_{2sb}(CH_n) = r \leq 9$ ------------- (2)

Since Wheel graph W_n needs n colors in a 2-distance coloring, in all cases let

$c(v_i) = i,\ 1 \le i \le n-1$ and $c(u) = n$ ---------- (3)

From (3) and from the fact that $d_{\le 2}(w_i) = 8$, the following facts are observed.

Fact 1: If $r > n$, then colors from $(n+1)^{th}$(if exist) onwards occur among $w_i s$.

Fact 2: If w_i is a *s-cdv* in a $2sb$-coloring with 9 colors, then the vertices of $N_{\le 2}[w_i]$ receive 9 distinct colors.

Subcase (i): $n = 6$.

Then from(2),$6 \le r \le 9$.

Now the following claim is proved.

Claim 1: $r < 9$

On the contrary, let $r = 9$. Then by fact 1, the colors 7, 8 and 9 among the $w_i's$. Since, the $w_i's$ induce a C_5 and $diam(C_5) = 2$, the colors 7, 8 & 9 occur only once and hence the vertices of colors 7, 8 and 9 are strong color dominating vertices. Hence, $|S_i| = 1, i = 7, 8, 9$. Among $w_i's$, the colors 7, 8 & 9 may occur in P_3 or in $P_2 \cup P_1$.Suppose they occur in P_3.

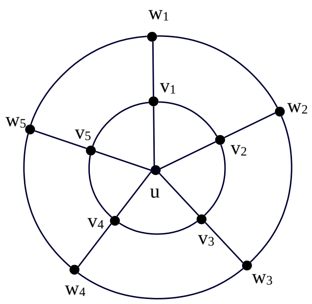

Figure 1: Labelling of CH_6

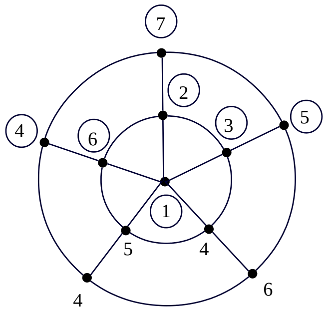

Figure 2: 7-$2sb$-coloring of CH_6

Without loss of generality, let

$c(w_1) = 7, c(w_2) = 8$ & $c(w_3) = 9$ ---------- (4)

From Table 3, $N_{\le 2}[w_1] = \{u, v_1, v_2, v_5, w_1, w_2, w_3, w_4, w_5\}$. Then from fact 2, and from (3) & (4), $C(\{w_4, w_5\}) = \{3, 4\}$. Since, for $i = 4, 5,\ d(v_4, w_i) \le 2,\ c(w_i) \ne 4$, a contradiction. So let the colors 7, 8 & 9 appear in $P_2 \cup P_1$. Without loss of generality, let $c(w_1) = 7, c(w_2) = 8$ and $c(w_4) = 9$.Then as w_1 is a *s-cdv*, by fact 2, $C(\{w_3, w_5\}) = \{3, 4\}$. Since, for $i = 3, 5,\ d(v_4, w_i) \le 2, c(w_i) \ne 4$, again a contradiction. Hence, the claim holds.

Claim 2: $r < 8$

Suppose $r = 8$. Then as in claim 1, $|S_i| = 1, i = 7, 8$. If $w_i$and w_j are the vertices of color 7 and 8, then $d(w_i, w_j) \le 2$. Suppose $d(w_i, w_j) = 1$ and let $c(w_1) = 7$ and $c(w_2) = 8$. Since, $d(v_4, w_i) \le 2$ for $i = 3, 4, 5, c(w_i) \ne 4$. Therefore, $S_4 = \{v_4\}$. Now $d(v_4, w_i) = 3$ for $i = 1, 2$, a contradiction to w_1, w_2 and v_4 are strong color dominating vertices. So let $d(w_i, w_j) = 2$ and $c(w_1) = 7$ and $c(w_3) = 8$. Now w_1 requires at most 2-neighbors of colors 3 and 4. Since, $d(v_4, w_1) = 3,\ c(w_i) = 4$ for $i = 2$ or 4 or 5 and further, that w_i is the 4-*s-cdv*. As $d(v_4, w_i) \le 2$ for $i = 4,\ 5,\ c(w_2) = 4$. Now as w_2 is a 4-*s-cdv* and $d(v_5, w_2) = 3$, either $c(w_4) = 5$ or $c(w_5) = 5$, a contradiction. Hence, the claim.

Thus, $6 \le r \le 7$. Figure 2 shows a $2sb$-coloring of CH_6 with 7 colors. Hence, $\chi_{2sb}(CH_6) = 7$.

Subcase (ii): $n = 7$

Then from (2), $7 \le \chi_{2sb}(CH_n) = r \le 9$.

Claim 3: $r < 9$

Let $r = 9$. From (3), W_n is given the colors from 1 to 7. Let $c(w_i) = 8$&$c(w_j) = 9$. Since, $w_i's$ induce a C_6 and $diam(C_6) = 3, d(w_i, w_j) \le 3$. Hence, 3 cases arise. Suppose $d(w_i, w_j) = 1$. Let $c(w_1) = 8, c(w_6) = 9$ and w_1 a 8-*s-cdv*. With $N_{\le 2}[w_1] = \{u, v_1, v_2, v_6, w_1, w_2, w_3, w_5, w_6\}$, equation (3) and from fact 2, $C(\{w_2, w_3, w_5\}) = \{3, 4, 5\}$. Since neither $c(w_2)$ nor $c(w_3)$ can be 3, $c(w_5) = 3$. This gives $C(\{w_2, w_3\}) = \{4, 5\}$.As w_3 is a 2-neighbors of $v_4, c(w_3) \ne 4$. Hence, $c(w_3) = 5$ and consequently $c(w_2) = 4$. Since $d(w_4, w_6) = 2, c(w_4) \ne 9$. Hence, $S_9 = \{w_6\}$ and therefore w_6 is a *s-cdv*. With $N_{\le 2}[w_6] = \{u, v_1, v_5, v_6, w_1, w_2, w_4, w_5, w_6\}$ and the existing color scheme, $c(w_4) = 2$. (Refer figure 4).

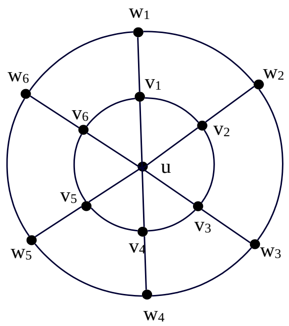

Figure 3: Labelling of CH_7

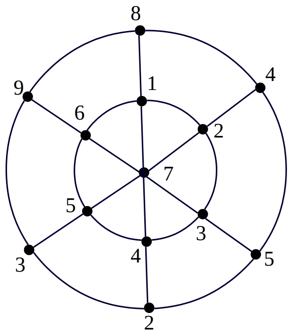

Figure 4: Colouring under the condition $d(w_i, w_j) = 1$

Then $S_8 = \{w_1\}$, $S_9 = \{w_6\}$ and $S_2 = \{v_2, w_4\}$. Now $d(v_2, w_6) = 3$ implies that v_2 cannot be a *s-cdv*and $d(w_4, w_1) = 3$ implies that w_4 cannot be a *s-cdv*. Thus, there does not exist a 2-*s-cdv*, a contradiction.

So suppose $d(w_i, w_j) = 2$. Let $c(w_1) = 8$ &$c(w_5) = 9$ and w_1 a 8-*s-cdv*. With the given coloring, $C(\{u, v_1, v_2, v_6, w_1, w_5\} = \{1,2,6,7,8,9\}$. Hence, from $N_{\leq 2}[w_1]$, $C(\{w_2, w_3, w_6\} = \{3, 4, 5\}$. Clearly, $c(w_2), c(w_3) \neq 3$.Hence, $c(w_6) = 3$. Also $c(w_3) \neq 4$ implies that $c(w_3) = 5$and this leads to $c(w_2) = 4$. Since, $c(w_4) \neq c(w_5) = 9$, $S_9 = \{w_5\}$. Now w_5 has two at most 2-neighbors w_3 and v_5 of color 5, a contradiction to fact 2.

So let $d(w_i, w_j) = 3$. Let $c(w_1) = 8$ &$c(w_4) = 9$. Since, $d(w_i, w_j) \leq 2$ for $i = 1, 4$ & $j = 2, 3, 5, 6$. Hence, $c(w_j) \neq 8, 9$ for $j = 2, 3, 5, 6$. Hence, $S_8 = \{w_1\}$ and $S_9 = \{w_4\}$. Since, $d(w_1, w_4) = 3$, both cannot be strong color dominating vertices, a contradiction. Hence, the claim.

Then $7 \leq r \leq 8$. From figure 5, $\chi_{2sb}(CH_7) = 8$.

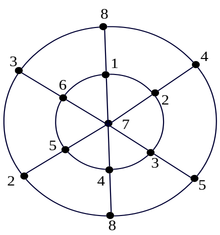

Figure 5: 8-2*sb*-coloring of CH_7

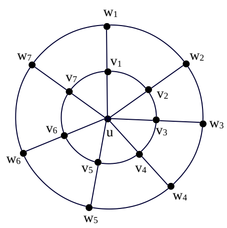

Figure 6: Labelling of CH_8

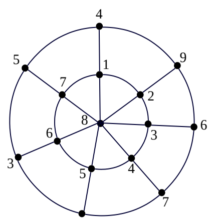

Figure 7: Coloring of CH_8 with $c(w_1) = 4$

Subcase (iii): $n = 8$

By proposition 2.1 (ii), $\chi_2(CH_8) = 8$ and from Table 5, $m_{\leq 2}(CH_8) = 9$. Then $8 \leq \chi_{2sb}(CH_n) \leq 9$.

Claim 4: $r < 9$

Suppose $r = 9$. As before color 9 is received only by w_i's. Let $w_2$be a *s-cdv* and$c(w_2) = 9$.Suppose $|S_9| = 1$. Then $S_9 = \{w_2\}$ and all its at most 2-neighbors are also strong color dominating vertices. Now $N_{\leq 2}[w_2] = \{u, v_1, v_2, v_3, w_1, w_2, w_3, w_4, w_7\}$
together with (3) and $c(w_2) = 9$,

$C(\{w_1, w_3, w_4, w_7\}) = \{4, 5, 6, 7\}$ ------------ (5)

Since, $c(w_3), c(w_4) \neq c(v_4) = 4$, either $c(w_1) = 4$ or $c(w_7) = 4$.

Suppose $c(w_1) = 4$. Now $c(w_7) \neq c(v_6), c(v_7) \Longrightarrow c(w_7) \neq 6, 7$. Therefore, $c(w_7) = 5$. Then from (5),

$C(\{w_3, w_4\}) = \{6, 7\}$ ------------ (6)

Now $N_{\leq 2}[w_1] = \{u, v_1, v_2, v_7, w_1, w_2, w_3, w_6, w_7\}$ and w_1 is a 4-*s-cdv*. Then from fact 2, $C(\{w_3, w_6\}) = \{3, 6\}$. Since, $c(w_6) \neq 6$ &$c(w_3) \neq 3, c(w_6) = 3$ &$c(w_3) = 6$.Then from (6), $c(w_4) = 7$ (Refer figure 7). Now w_4 is an at most 2-neighbor of w_2 and hence a *s-cdv*. Now v_3 and w_6 are 2-neighbors of w_4 and of color 3, a contradiction by fact 2 to the fact that w_4 is a *s-cdv*.

So let $c(w_7) = 4$. Then from (5),

$C(\{w_1, w_3, w_4\}) = \{5, 6, 7\}$ ------------ (7)

Now $N_{\leq 2}[w_7] = \{u, v_1, v_6, v_7, w_1, w_2, w_5, w_6, w_7\}$ implies that

$C(\{w_1, w_5, w_6\}) = \{2,3,5\}$ ------------- (8)

From (7) & (8), $c(w_1) = 5$ (Refer figure 8). Then in $N_{\leq 2}[w_1]$, $C(\{u, v_1, v_2, v_7, w_1, w_2, w_7\}) = \{1,2,4,5,7,8,9\}$. Then $C(\{w_3, w_6\}) = \{3,6\}$ and hence, $c(w_6) = 3$ &$c(w_3) = 6$. Then from (7), $c(w_4) = 7$ (Refer figure 9), contradiction arises as before.

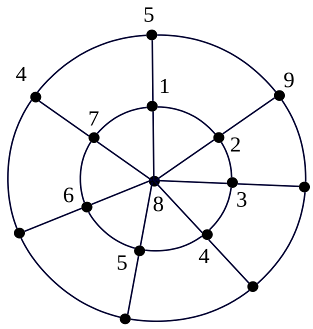

Figure 8: CH_8 with $c(w_7) = 4$&$c(w_1) = 5$

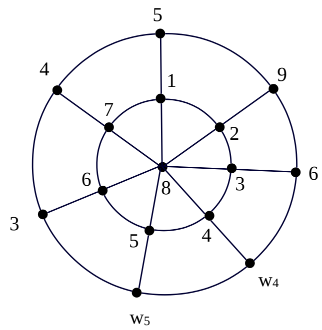

Figure 9: Colouring of $N_{\leq 2}[w_1]$ in CH_8

So suppose $|S_9| = 2$.Since $c(w_2) = 9$, either $c(w_5) = 9$ or $c(w_6) = 9$. Since the graph structure is symmetrical, it is enough to discuss one case. Let $c(w_5) = 9$. Since, w_2 is a *s-*cdv, and $c(w_1), c(w_7) \neq 7$, either $c(w_3) = 7$ or $c(w_4) = 7$. For $w_i, i = 3,4,$ w_2 and w_5 are 2-neighbors of $w_i$and of color 9. Hence, by fact 2, neither w_3 nor w_4 can be a *s-*cdv. On the other hand, $d(v_7,w_i) > 2$ for $i = 2,5$. Hence, v_7 does not have an at most 2-neighbor of color 9. Hence, there does not exist a 7-*s-*cdv, a contradiction. Hence, the claim.

Thus, $r \leq 8$. Then from (2), $\chi_{2sb}(CH_n) = 8$.

Case (iv): $n \geq 9$

From proposition 2.1 (ii), $\chi_2(CH_n) = n$ and from Table 5, $m_{\leq 2}(CH_n) = n$. Hence, $\chi_{2sb}(CH_n) = n$.

5. Conclusion

In this paper, it is proved that for any graph G, any $2sb$-coloring of G is a b-coloring of G^2 and vice versa. Further, the exact bounds of the $2sb$-number of Helm and Closed Helm are obtained.

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