2-DISTANCE STRONG B-COLORING OF HELM AND CLOSED HELM GRAPHS

S. SARASWATHI¹, M. POOBALARANJANI²

ABSTRACT

A vertex \( v \) is said to be an at most 2-neighbor of a vertex \( u \), if \( d(u,v) \leq 2 \). For a graph \( G \), a 2-distance strong b-coloring is a proper coloring of \( G \) in which each pair of vertices that are at distance less than or equal to 2 receive different colors in addition to the condition that each color class has a vertex that must have at most 2-neighbor in every other color class. This vertex is called as a strong color dominating vertex. The corresponding parameter termed as 2-distance strong b-chromatic number \( \chi_{sb}(G) \) is the largest integer \( k \) for which \( G \) has a 2-distance strong b-coloring with \( k \) colors. This article discusses a relationship between a b-coloring and a 2b-coloring and the precise values of the Helm and Closed Helm’s 2-distance strong b-chromatic numbers are determined.

Keywords: 2-distance strong b-coloring, 2-distance strong b-chromatic number, strong color dominating vertex.

1. Introduction

This work exclusively considers finite, simple, and undirected graphs. Readers may refer Harary [1, 1988] for terminologies which are not defined here. The 2-distance strong b-coloring (or simply 2b-coloring) was defined by us in 2021 [8] and prior to this coloring, in 2019 [6], we introduced another coloring called a 2-distance b-coloring (or simply 2b-coloring). Both colorings can be viewed as conditional 2-distance coloring. Kramer F. and Kramer H. [3, 1969] defined 2-distance coloring as “a proper coloring in which every pair of vertices that are at least 2 distance apart get different colors”. The smallest integer \( k \) for which \( G \) exhibits 2-distance coloring with \( k \) colors is the 2-distance chromatic number of \( G \). It is denoted by \( \chi_2(G) \). Many results on the boundaries of 2-distance chromatic number were reported in a comprehensive study by Kramers [5, 2008]. The other interesting coloring is b-coloring, which was introduced by Irving and Manlove [2, 1999]. It was defined as a proper coloring of \( G \) such that in each color class there must exist a vertex termed as a color dominating vertex (or \( cdv \)) that has at least one neighbor in each other color class and, the largest integer \( k \) such that \( G \) has a b-coloring with \( k \) colors is the b-chromatic number of \( G \), represented as \( \chi_b(G) \). Both these colorings motivated us to define two types of distance based coloring incorporated with the concept of having a \( cdv \) in each color class in one coloring and a modified version of a \( cdt \) in the next coloring.

The 2-distance b-coloring directly uses the definition of b-coloring. Shorty saying, a 2-distance b-coloring is both a 2-distance coloring and a b-coloring. The definition of 2-distance coloring states that any two vertices which are at distance less than or equal to 2 must have different colors. The concept of at most two distance was critical in defining the 2-distance strong b-coloring, So instead of having a \( cdv \) in each color class, we define that each color class has a vertex to say \( u \) such that there exists a vertex in every other color class which is at distance less than or equal to 2 from \( u \). More explicitly, if \( S_1, S_2, \ldots, S_k \) is a color class partition of the vertex set, then for each \( i \), there exists \( u_i \in S_i \) for which there exists a \( v_j \in S_j \neq i \) such that \( d(u_i, v_j) \leq 2 \).

When we compare a vertex and its neighbors, and the same vertex with its neighbors and 2-neighbors, it is a simple fact that in the second case, the vertex is associated with more vertices and in this context, the vertex in the new coloring having a neighbor or 2-neighbor is called as a \( scdv \).

In [8, 2021], the bounds of the 2sb-number of some of the well-known graphs such as path, cycle were obtained. In [7, 2021], we introduced a new rooted tree called as a perfect \( \Delta \)ary tree having all internal vertices are of degree \( \Delta \) and all pendant vertices are at the same level. In the same paper, the exact value of the 2sb-number of a perfect \( \Delta \)ary tree was obtained. In this
2-DISTANCE STRONG B-COLORING OF HELM AND CLOSED HELM GRAPHS

paper, a relationship between a \( b \)-coloring and a \( 2sb \)-coloring is obtained. The \( b \)-chromatic number of cycle related and wheel related graphs were studied by Vaidya and Shukla [9, 10, 2014]. Also, they [11, 2014] found the \( b \)-chromatic number of Helm and Closed Helm. These papers made us study the new coloring on Helm and Closed Helm. The main outcome of this paper is obtaining the bounds of the \( 2sb \)-number of Helm and Closed Helm.

2. Definitions and Some Prior Results

Definitions and results for 2-distance strong \( b \)-coloring are provided in this section.

**Proposition 2.1.** For \( n \geq 4 \),
(i) \( \chi_2(H_n) = \begin{cases} 5 & \text{if } n = 4 \\ n & \text{if } n \geq 5 \end{cases} \)
(ii) \( \chi_2(CH_n) = \begin{cases} 7 & \text{if } n = 4 \\ n & \text{if } n \geq 5 \end{cases} \)

**Definition 2.2.** In a graph \( G \), for \( u \in V \), a vertex \( v \) is said to be an at most \( k \)-neighbor, if \( d(u, v) \leq k \).

**Definition 2.3.** A set \( S \) of vertices is said to be an \((k \geq 2)\) at most \( k \)-neighborhood (or at most \( k \)-open neighborhood) of a vertex \( u \), if \( S \) consists of all the vertices which are at distance at most \( k \) from \( u \). It is represented by \( N_k(u) \).

An at most \( k \)-closed neighborhood of \( u \) is the set \( N_k[u] = N_k(u) \cup \{u\} \).

**Definition 2.4.** In a graph \( G(V, E) \), for \( u \in V \), the at most \( k \)-degree \((k \geq 2)\) of \( u \) is the number of vertices that are at distance less than or equal to \( k \) from \( u \), and it is denoted by \( d_k(u) \). The at most \( k \)-degree \( d_k(u) \) of a vertex \( u \) is also called the \( d_k \)-degree of \( u \).

In this paper, only the case \( k = 2 \) is considered.

**Definition 2.5.** A vertex in a color class partition is said to be a strong color dominating vertex (or simply \( s \)-dominating) if it has an at most \( 2 \)-neighbor in every other color class.

**Definition 2.6.** A 2-distance strong \( b \)-coloring (\( 2sb \)-coloring) of \( G \) is a 2-distance coloring of \( G \) in which each color class has a \( s \)-dominating vertex.

**Definition 2.7.** For a graph \( G \), the 2-distance strong \( b \)-chromatic number (or simply \( 2sb \)-number) \( \chi_{2sb}(G) \) is the largest \( k \) for which \( G \) has a 2-distance strong \( b \)-coloring with \( k \) colors.

**Definition 2.8.** Let \( V = \{v_1, v_2, \ldots, v_n\} \) be the vertex set of \( G \) in which the vertices are ordered so that \( d_2(v_1) \geq d_2(v_2) \geq \ldots \geq d_2(v_n) \). Then \( m_2 \)-degree of \( G \), presented by \( m_2(G) \), is defined by \( m_2(G) = \max \{1 \leq i \leq n : d_2(v_i) \geq i - 1\} \).

In other words, \( m_2(G) \) is presented as the maximum \( \{1 \leq i \leq n : d_2(v_i) \geq i - 1\} \). \( G \) has at least \( i \) vertices whose at most 2 degree is at least \( i - 1 \).

**Observation 2.9.**
(i) If a graph \( G \) has a 2-distance strong \( b \)-coloring with \( k \) different colors, it must have at least \( k \) vertices whose at most 2-degree is at least \( k - 1 \);
(ii) Any 2-distance coloring with \( k \)-colors is a 2-distance strong \( b \)-coloring;
(iii) 2-distance strong \( b \)-coloring exists for all graphs;
(iv) A 2-distance \( b \)-coloring is equivalent to a 2-distance strong \( b \)-coloring, and the converse is not true.

**Proposition 2.10.** If \( \text{diam}(G) \leq 2 \), then \( \chi_{2sb}(G) = |V(G)| \).

**Corollary 2.11.** If a graph has a dominating vertex, then \( \chi_{2sb}(G) = |V(G)| \).

**Proofs** Since, \( G \) has a dominating vertex, for any pair of vertices \( u \& v \), \( d(u,v) \leq 2 \). Hence, \( G \) is of diameter at most 2. Hence, \( \chi_{2sb}(G) = |V(G)| \).

3. Relationship between a \( b \)-coloring and a \( 2sb \)-coloring

When a new concept and a related parameter are defined based on an existing concept and parameter, there exists a relationship between the two. If \( \chi_{new} \) and \( \chi_{old} \) are two chromatic parameters respectively denoting the new and old as specified in the suffix, then there may exist a relationship that for a given graph \( G \), \( \chi_{new}(G) = \chi_{old}(G) \pm k \) where \( k \) is a positive integer. In some cases, if \( G' \) is a graph obtained from the graph \( G \), then there may exist a relation that \( \chi_{new}(G') = \chi_{new}(G) \). For the well known such kind of results is that \( \chi'(G) = \chi'(L(G)) \) where \( \chi'(G) \) is the chromatic index or edge chromatic number of \( G \), and \( L(G) \) is its line graph. This section establishes that each \( 2sb \)-coloring of a graph \( G \) is equivalent to a \( b \)-coloring of the square graph \( G^2 \) and that any \( b \)-coloring of \( G^2 \) is equivalent to a \( 2sb \)-coloring of \( G \).
Definition 3.1: For a given graph $G = (V, E)$, a $k$-distance graph denoted by $G^k$ is the graph whose vertex set is $V$ and whose edge set contains all the edges connecting any pair of vertices that are at distance less than or equal to $k$ in $G$.

If $k = 2$, the graph $G^k$ is called the square graph.

Theorem 3.2: For any graph $G$, a 2sbd-coloring of $G$ is a $b$-coloring of $G^2$ and vice versa; and hence $X_{2×}(G) = X_b(G^2)$.

Proof: From the definition of $G^2$, for any pair of vertices $u, v$,

$$d_2(u, v) \leq 2 \iff uv \in E(G^2)$$

(1)

The theorem is proved in two claims.

Claim 1: Any 2sbd-coloring of $G$ is a $b$-coloring of $G^2$.
Let $c$ be a 2sbd-coloring of $G$. Since $V(G) = V(G^2)$, give colors to the vertices of $G^2$ as in $G$ given by $c$. If $uv \in E(G^2)$, then from (1) and from the assumption that $c$ is a 2sbd-coloring of $G$, $c(u) \neq c(v)$. Hence, in $G^2$, $c(u) \neq c(v)$. Then $c$ is a proper coloring of $G^2$. Also, each $sbd$ of $G^2$ becomes a $sbd$ in $G^2$ and hence, $c$ is a $b$-coloring of $G^2$.

Claim 2: Any $b$-coloring of $G$ is a 2sbd-coloring of $G$.
Let $c$ be a $b$-coloring of $G^2$. As before, assign colors to the vertices of $G$ as given in $G^2$ by $c$. Let $u, v \in V(G)$ and $d_2(u, v) \leq 2$. Then $uv \in E(G^2)$ and hence, $c(u) \neq c(v)$. Consequently, $u, v$ receive distinct colors in $G$. Hence, $c$ is a 2sbd-coloring. Since, each neighboring vertex of $u$ in $G^2$ is an at most 2 neighbor of $u$ in $G$, each $sbd$ of $G^2$ becomes a $sbd$ in $G$. Hence, in $G$, each color class contains a $sbd$ and hence, $c$ is a 2sbd-coloring of $G$. Hence, the claim.

From claim 1, $X_{2×}(G) \leq X_b(G^2)$ and from claim 2, $X_b(G^2) \leq X_{2×}(G)$. This yields the result.

4. 2-distance strong $b$-coloring of Helm and Closed Helm

In this section, two graphs, the Helm and the Closed Helm are chosen to find the bound for the 2-distance strong $b$-chromatic number.

Proposition 4.1: For $n \geq 4$,

$$X_{2×}(H_n) = \begin{cases} 5, & \text{if } n = 4 \\ n, & \text{if } n \geq 5 \end{cases}$$

Proof: A Helm graph contains a Wheel graph $W_n$ as an induced subgraph and $u$ is the central vertex and the $(n-1)$-cycle is induced by $v_1, v_2, \ldots, v_{n-1}$ of $W_n$. Let $c$ be a 2sbd-coloring with $X_{2×}(n) = r$ colors.

In $H_n$, for $1 \leq t \leq n - 1$, $w_t$ is the pendant vertex adjacent to $v_t$. Then $N_{d_2}(u) = \{v_1; 1 \leq t \leq n - 1\}$ and $N_{d_2}(v_t) = \{w_t, v_{t-1}, v_{t+1}\}$ for $1 \leq t \leq n - 1$. For $1 \leq t \leq n - 1$, $N_{d_2}(v_t) = \{w_t, v_{t-1}, v_{t+1}\}$ where $w_n = w_1 = w_{n-1}, v_1 = v_n = v_1, v_0 = v_1, v_{n-1} = v_1$. The $d_2$-degree of the vertices are given in Table 1.

<table>
<thead>
<tr>
<th>Vertex $v$</th>
<th>$d_2(v)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$u$</td>
<td>$2n - 2$</td>
</tr>
<tr>
<td>$v_1$</td>
<td>$n + 2$</td>
</tr>
<tr>
<td>$w_1$</td>
<td>$4$</td>
</tr>
</tbody>
</table>

If $n = 4$, the degree sequence $(d_2(u), d_2(v_1), d_2(w_1), d_2(v_2)) = (6, 6, 6, 4, 4, 4)$. Therefore, $m_{d_2}(H_4) = 5$.

If $n = 5$, then the degree sequence is given by $(8, 7, 7, 7, 4, 4, 4, 4)$. This gives $m_{d_2}(H_5) = 5$.

If $n \geq 6$, then there are exactly $n$ vertices of $d_2$-degree at least $n + 2$ and $n - 1$ vertices of $d_2$-degree 4. Hence, $m_{d_2}(H_n) = n$. The $m_{d_2}$-degrees of $H_n$ are tabulated in Table 2.

<table>
<thead>
<tr>
<th>$n$</th>
<th>$m_{d_2}(H_n)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>4</td>
<td>5</td>
</tr>
<tr>
<td>$\geq 5$</td>
<td>$n$</td>
</tr>
</tbody>
</table>

Then from proposition 2.1 (6), $X_{2×}(H_n) = m_{d_2}(H_n) = \begin{cases} 5, & \text{if } n = 4 \\ n, & \text{if } n \geq 5 \end{cases}$

$\Rightarrow X_{2×}(H_n) = \begin{cases} 5, & \text{if } n = 4 \\ n, & \text{if } n \geq 5 \end{cases}$

Proposition 4.2: For $n \geq 4$.  

227
Since, \( \text{diam}(CH_n) = 2 \), from proposition 2.10, \( X_{22}(CH_n) = |V| = 7 \).

Case (ii) \( n = 5 \)

Then from Tables 1 & 3, the degree sequence \( d_{22}(u), d_{22}(v_1), \ldots, d_{22}(v_4), \ldots, d_{22}(v_5) \) is given by \( (8, 7, 7, 7, 7, 7, 7, 7) \). i.e., only one vertex is of \( d_{22} \)-degree 8 and for all the remaining 8 vertices are of \( d_{22} \)-degree 7. Hence, \( m_{22}(CH_5) = 8 \). Hence from proposition 2.1 (ii), \( 5 \leq X_{22}(CH_5) = r \leq 8 \).

Claim \( r = 5 \)

Suppose \( r > 5 \). Assign colors 1 to 5 to \( u \) and \( v_1 \) with \( c(v_1) = 1, 1 \leq i \leq 4 \) and \( c(u) = 5 \). Hence, the remaining \( r - 5 \geq 1 \) colors are given to \( w_i \)'s. Let \( c(w_1) = 6 \). Since, for \( i \neq 1, d(w_1, w_i) \leq 2 \) & \( d(v_1, w_1) \leq 2 \), \( c(w_1) \notin \{3, 6\} \). Hence, \( S_1 = \{v_1\} \) and \( S_2 = \{w_1\} \). Since \( d(v_1, w_1) = 3 \), both \( v_1 \) and \( w_1 \) cannot be strong color dominating vertices, a contradiction to the choice of \( c \). Hence, \( r > 5 \) cannot hold and hence \( r \leq 5 \) and the equality holds. Therefore, \( X_{22}(CH_5) = 5 \).

Case (iii) \( 6 \leq n \leq 8 \)

From Tables 1 & 3, \( d_{22}(u), d_{22}(v_1) \) and \( d_{22}(w_1) \) are tabulated in Table 4.

**Table 4:** At most 2 degrees of vertices of \( CH_n 

<table>
<thead>
<tr>
<th>n</th>
<th>d_{22}(u)</th>
<th>d_{22}(v_1)</th>
<th>d_{22}(w_1)</th>
</tr>
</thead>
<tbody>
<tr>
<td>6</td>
<td>10</td>
<td>8</td>
<td>8</td>
</tr>
<tr>
<td>7</td>
<td>12</td>
<td>9</td>
<td>8</td>
</tr>
<tr>
<td>8</td>
<td>14</td>
<td>10</td>
<td>8</td>
</tr>
</tbody>
</table>

From the tabulated values, \( 10 \leq d_{22}(u) \leq 14 \), \( 8 \leq d_{22}(v_1) \leq 10 \) & \( d_{22}(w_1) = 8 \).

Hence, we define the following sets.

\[ X_1 = \{v : d_{22}(v) \geq 8\}, \quad X_2 = \{v : d_{22}(v) \geq 9\}, \quad X_3 = \{v : d_{22}(v) \geq 10\} \] and \( n_i = |X_i| \). Then the following are observed.

(a) For all \( n_i \), \( |X_i| = |V| = 2n - 1 \); 
(b) For \( n = 6, X_2 = X_1 = \{u\} \); 
(c) For \( n = 7, X_2 = \{u, v\}; \quad X_3 = \{u\} \); 
(d) For \( n = 8, X_2 = X_1 = \{u, v\} \).

In Table 5, the computed values of \( n_i \) and \( m_{22}(CH_n) \) are tabulated.

**Table 5:** \( n_i \) & \( m_{22}(CH_n) 

<table>
<thead>
<tr>
<th>n</th>
<th>n_1</th>
<th>n_2</th>
<th>n_3</th>
<th>m_{22}(CH_n)</th>
</tr>
</thead>
<tbody>
<tr>
<td>6</td>
<td>11</td>
<td>1</td>
<td>1</td>
<td>9</td>
</tr>
<tr>
<td>7</td>
<td>13</td>
<td>7</td>
<td>1</td>
<td>9</td>
</tr>
<tr>
<td>8</td>
<td>15</td>
<td>8</td>
<td>8</td>
<td>9</td>
</tr>
</tbody>
</table>

Since Wheel graph \( W_n \) needs \( n \) colors in a 2-distance coloring, in all cases let
Let $w_1, w_2$ be the vertices of colors 7, 8 & 9 are strong color dominating vertices. Hence, $|S_1| = 1, i = 7, 8, 9$. Among $w_i$'s, the colors 7, 8 & 9 may occur in $P_3$ or in $P_2 \cup P_1$. Suppose they occur in $P_3$.

**Fact 1:** If $r > n$, then colors from $(n + 1)^{st}$ onwards occur among $w_i$'s.

Since neither $d_{cd}(w_i) = 8$, the following facts are observed.

**Fact 2:** If $w_i$ is a $s$-$cdv$ in a $2sb$-coloring with 9 colors, then the vertices of $N_{cd}[w_i]$ receive 9 distinct colors.

**Subcase (b):** $n = 6$.

Then from (2), $6 \leq r \leq 9$.

Now the following claim is proved.

**Claim 1:** $r < 9$

On the contrary, let $r = 9$. Then by fact 1, the colors 7, 8 and 9 among the $w_i$'s.

Since, the $w_i$'s induce a $C_9$ and $diam(C_9) = 2$, the colors 7, 8 & 9 occur only once and hence the vertices of color 7, 8 and 9 are strong color dominating vertices. Hence, $|S_1| = 1, i = 7, 8, 9$. Among $w_i$'s, the colors 7, 8 & 9 may occur in $P_3$ or in $P_2 \cup P_1$. Suppose they occur in $P_3$.

**Claim 2:** $2r < 8$

Suppose $r = 8$. Then as in claim 1, $|S_1| = 1, i = 7, 8$. If $w_i$ and $w_j$ are the vertices of color 7 and 8, then $d(w_i, w_j) \leq 2$. Suppose $d(w_i, w_j) = 1$ and let $c(w_i) = 7$ and $c(w_j) = 8$.

Since, $d(v_k, w_i) \leq 2$ for $i = 3, 4, 5, c(w_i) \neq 4$. Therefore, $S_4 = \{v_k\}$. Now $d(v_k, w_j) = 3$ for $i = 1, 2$, a contradiction to $w_i, w_j$ and $v_k$ are strong color dominating vertices. So let $d(w_i, w_j) = 2$ and $c(w_i) = 7$ and $c(w_j) = 8$. Now $w_i$ requires at most $2$-neighbor of colors 3 and 4. Since, $d(v_k, w_i) = 3, c(w_i) = 4$ for $i = 2$ or 4 or 5 and further, that $w_i$ is the 4-$s$-$cdv$.

As $d(v_k, w_j) \leq 2$ for $i = 4, 5, c(w_j) = 4$. Now as $w_i$ is a 4-$s$-$cdv$ and $d(v_k, w_j) = 3$, either $c(w_j) = 5$ or $c(w_j) = 5$, a contradiction. Hence, the claim.

Thus, $6 \leq r \leq 7$. Figure 2 shows a $2sb$-coloring of $CH_6$ with 7 colors. Hence, $X_{2sb}(CH_6) = 7$.

**Subcase (ii):** $n = 7$

Then from (2), $7 \leq X_{2sb}(CH_6) = r \leq 9$.

**Claim 3:** $r < 9$

Let $r = 9$. From (1), $W_6$ is given the colors from 1 to 7. Let $c(w_i) = 8$ & $c(w_j) = 9$.

Since, $w_i$'s induce a $C_9$ and $diam(C_9) = 2, d(w_i, w_j) \leq 3$. Hence, 3 cases arise. Suppose $d(w_i, w_j) = 1$. Let $c(w_i) = 8, c(w_j) = 9$ and $w_i$ is a 8-$s$-$cdv$. With $N_{cd}[w_i] = \{u, v_i, v_p, v_q, w_i, w_2, w_3, w_4\}$, equation (3) and from fact 2, $C((w_i, w_j)) = (3, 4, 5)$. Since neither $c(w_i)$ nor $c(w_j)$ can be 3, $c(w_3) = 3$. This gives $C((w_2, w_3)) = (4, 5)$. As $w_3$ is a 2-neighbor of $v_k, c(w_3) \neq 4$. Hence, $c(w_j) = 5$ and consequently $c(w_3) = 4$. Since $d(w_i, w_j) = 2, c(w_i) = 9$. Hence, $S_9 = \{w_i\}$ and therefore $w_i$ is a 9-$cdv$ in $N_{cd}[w_j] = \{u, v_i, v_p, v_q, w_i, w_2, w_3, w_4\}$ and the existing color scheme, $c(w_i) = 2$. (Refer figure 4)
Then $S_1 = \{w_1\}$, $S_2 = \{w_2\}$ and $S_2 = \{v_3, w_4\}$. Now $d(v_2, w_4) = 3$ implies that $v_2$ cannot be a scdn and $d(w_4, w_4) = 3$ implies that $w_4$ cannot be a scdn. Thus, there does not exist a 2-sc-def, a contradiction.

So suppose $d(w_2, w_2) = 2$. Let $c(w_2) = 8 \& c(w_3) = 9$ and $w_1$ a 8-sc-def. With the given coloring, $C((u, v_2, v_2, v_2, w_1, w_1)) = \{1, 2, 6, 7, 8, 9\}$. Hence, from $N_G[w_1]$, $C((w_2, w_2, w_3, w_3)) = \{3, 4, 5\}$. Clearly, $c(w_3) \neq 3$. Also, $c(w_3) \neq 4$ implies that $c(w_3) = 5$ and this leads to $c(w_3) = 5$. Since $c(w_3) \neq c(w_4) = 9$, $S_4 = \{w_2\}$. Now $w_2$ has at most two neighbors $w_3$ and $w_3$ of color 5, a contradiction to fact 2.

So let $d(w_1, w_2) = 3$. Let $c(w_2) = 8 \& c(w_3) = 9$. Since, $d(w_2, w_2) \leq 2$ for $i = 1, 4$ & $j = 2, 3, 5, 6$. Hence, $c(w_2) \neq 8, 9$ for $j = 2, 3, 5, 6$. Hence, $S_2 = \{w_1\}$ and $S_3 = \{w_3\}$. Since, $d(w_2, w_2) = 3$, both cannot be strong color dominating vertices, a contradiction. Hence, the claim.

Then $7 \leq r \leq 8$. From figure 5, $X_{2ab}(CH_2) = 8$.

Subcase (iii): $n = 8$

By proposition 2.1 (ii), $\chi_r(CH_3) = 8$ and from Table 5, $m_{\Xi2}(CH_3) = 9$. Then $8 \leq X_{2ab}(CH_3) \leq 9$.

Claim 4: $r < 9$

Suppose $r = 9$. As before color 9 is received only by $w_j$’s. Let $w_2$ be a s-cd and $c(w_2) = 9$. Suppose $|S_4| = 1$. Then $S_4 = \{w_2\}$ and all its at most 2-neighbors are also strong color dominating vertices. Now $N_G[w_2] = \{u, v_2, v_2, v_2, w_2, w_2, w_2, w_2\}$ together with (3) and $c(w_2) = 9$,

$C((w_2, w_2, w_2, w_2)) = \{4, 5, 6, 7\}$

Since, $c(w_3) \neq c(v_3) = 4$, either $c(w_3) = 4$ or $c(w_3) = 4$.

Suppose $c(w_3) = 4$. Now $c(w_3) \neq c(v_3)$, $c(v_3) = c(w_3) \neq 6, 7$. Therefore, $c(w_3) = 5$. Then from (5),

$C((w_2, w_3)) = \{6, 7\}$

Now $N_G[w_3] = \{u, v_2, v_2, v_2, v_2, v_2, v_2, v_2\}$ and $w_4$ is a $s-cd$. Then from fact 2, $C((w_2, w_3)) = \{3, 6\}$. Since, $c(w_2) \neq 6 \& c(w_2) \neq 3, c(w_3) = 3 \& c(w_3) = 6$. Then from (6), $C(w_3) = \{7\}$. Hence, $w_3$ is a $s-cd$. Now $v_3$ and $w_6$ are 2-neighbors of $w_4$ and of color 3, a contradiction by fact 2 to the fact that $w_4$ is a $s-cd$.

So let $c(w_3) = 4$. Then from (5),

$C((w_2, w_3, w_4)) = \{5, 6, 7\}$

Now $N_G[w_4] = \{u, v_2, v_2, v_2, v_2, v_2, v_2, v_2\}$ implies that

2-DISTANCE STRONG B-COLORING OF HELM AND CLOSED HELM GRAPHS
\[ C([w_1, w_5, w_6]) = \{2, 3, 5\} \]

From (7) \& (8), \( c(w_4) = 5 \) (Refer figure 8). Then in \( N_{e_2}[w_1] \),
\[ C([u, v_1, v_2, \ldots, v_i, w_1, w_5, w_6]) = \{1, 2, 4, 5, 7, 8, 9\} \] Then \( C([w_5, w_6]) = \{3, 6\} \) and hence, \( c(w_5) = 3 \) \& \( c(w_6) = 6 \). Then from (7), \( c(w_4) = 7 \) (Refer figure 9), contradiction arises as before.

\begin{center}
Figure 8: \( CH_8 \) with \( c(w_5) = 4 \& c(w_9) = 5 \)
\end{center}

\begin{center}
Figure 9: Colouring of \( N_{e_2}[w_1] \) in \( CH_8 \)
\end{center}

So suppose \( |S_d| = 2 \) since \( c(w_4) = 9 \), either \( c(w_5) = 9 \) or \( c(w_6) = 9 \). Since the graph structure is symmetrical, it is enough to discuss one case. Let \( c(w_5) = 9 \). Since, \( w_2 \) is a s-\( cv \), and \( c(w_2) \neq c(w_5) \), either \( c(w_2) = 7 \) or \( c(w_2) = 8 \). For \( w_2, i = 3, 4, w_2 \) and \( w_4 \) are 2-neighbors of \( w_5 \) and of color 9. Hence, by fact 2, neither \( w_2 \) nor \( w_4 \) can be a s-\( cv \). On the other hand, \( d(v_5, w_5) > 2 \) for \( i = 2, 5 \). Hence, \( v_5 \) does not have an at most 2-neighbor of color 9. Hence, there does not exist a 7-s-\( cv \), a contradiction. Hence, the claim. Thus, \( r \leq 8 \). Then from (2), \( x_{2b}(CH_8) = 8 \).

Case (iv) \( n \geq 9 \)

From proposition 2.1 (ii), \( x_{2b}(CH_8) = n \) and from Table 5, \( m_{e_2}(CH_8) = n \). Hence, \( x_{2b}(CH_8) = n \).

5. Conclusion

In this paper, it is proved that for any graph \( G \), any 2\( k \)-coloring of \( G \) is a \( b \)-coloring of \( G^2 \) and vice versa. Further, the exact bounds of the 2\( k \)-number of Helm and Closed Helm are obtained.

References

S. SARASWATHI¹, M. POOBALARANJANI¹,² PG & Research
Department of Mathematics Seethalakshmi Ramaswami College
(Autonomous), Affiliated to Bharathidasan University, Tiruchirappalli, Tamil Nadu, India.
Email: ¹sarasw75@gmail.com, ²mpranjani@hotmail.com