VERSAL DEFORMATIONS OF AFFINE VECTOR FIELDS ON TORUS

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1. Universal deformations of vector fields and differential forms

1.1. Deformations. We consider a smooth vector field \( A \in \Gamma(T(\mathbb{T}^n)) \) on the \( n \)-dimensional torus \( \mathbb{T}^n \). A deformation of the vector field \( A \in \Gamma(T(\mathbb{T}^n)) \) we will call a vector field \( A(\tau) \in \Gamma(T(\mathbb{T}^n)) \), which depends analytically on the parameter \( \tau \in \mathbb{C}^k, k \in \mathbb{Z}_+ \), in some vicinity of the point \( \tau = 0 \in \mathbb{C}^k \), and such that \( A(0) = A \). The space of parameters \( \mathcal{Y} \{ \tau \in \mathbb{C}^k \} \) is often called a base of the deformation. Similarly will consider a differential 1-form \( l(\tau) \in \Lambda^1(\mathbb{T}^n) \), which depends analytically on the parameter \( \tau \in \mathbb{C}^k, k \in \mathbb{Z}_+ \), in some vicinity of the point \( \tau = 0 \in \mathbb{C}^k \) and such that \( l(0) = l \).

**Definition 1.1.** Two vector fields deformations \( A(\tau) \) and \( B(\tau) \in \Gamma(T(\mathbb{T}^n)) \) are called equivalent, if there exists such a deformation \( g(\tau) \in Diff(\mathbb{T}^n) \) of the identity \( Id \in Diff(\mathbb{T}^n) \), that \( Ad_{g(\tau)} A(\tau) = B(\tau) \), where \( ad : Diff(\mathbb{T}^n) \times \Gamma(T(\mathbb{T}^n)) \rightarrow \Gamma(T(\mathbb{T}^n)) \) is the usual \([2, 3, 5, 4]\) adjoint mapping of the space \( Diff(\mathbb{T}^n) \) on \( \Gamma(T(\mathbb{T}^n)) \). Similarly, two 1-form deformations \( l(\tau) \) and \( p(\tau) \in \Lambda^1(\mathbb{T}^n) \) are called equivalent, if there exists such a deformation \( g(\tau) \in Diff(\mathbb{T}^n) \) of the identity \( Id \in Diff(\mathbb{T}^n) \), that \( Ad^*_{g(\tau)} l(\tau) = p(\tau) \), where \( Ad^* : Diff(\mathbb{T}^n) \times \Lambda^1(\mathbb{T}^n) \rightarrow \Lambda^1(\mathbb{T}^n) \) is the usual adjoint mapping of the space \( Diff(\mathbb{T}^n) \) on \( \Lambda^1(\mathbb{T}^n) \).

Let \( \varphi \) - a germ of a holomorphic at zero mapping \( \mathbb{C}^m \rightarrow \mathbb{C}^k \), that is a set of converging at \( 0 \in \mathbb{C}^m \) degree series of complex variables, and assume that \( \varphi(0) = 0 \). The mapping \( \varphi : \mathcal{Y} \{ \sigma \in \mathbb{C}^m \} \rightarrow \mathcal{Y} \{ \tau \in \mathbb{C}^m \} \) defines evidently a new deformation \( \check{\varphi}(\sigma) \in \Lambda^1(\mathbb{T}^n) \) of the 1-form \( l(\tau) \in \Lambda^1(\mathbb{T}^n) \) and a new deformation \( \check{\varphi}(\sigma) \) of the vector field \( A(\tau) \in \Gamma(T(\mathbb{T}^n)) \) via the expressions

\[
(1.1) \quad (\check{\varphi})l(\sigma) = l(\varphi(\sigma)), \quad (\check{\varphi}A)(\sigma) = A(\varphi(\sigma))
\]

on the deformation base \( \mathcal{Y} \{ \sigma \in \mathbb{C}^k \} \).

**Definition 1.2.** The deformation \( (\check{\varphi})l(\sigma) \in \Lambda^1(\mathbb{T}^n) \) is called induced from the deformation \( l(\tau) \in \Lambda^1(\mathbb{T}^n) \) under the mapping \( \varphi : \mathcal{Y} \{ \sigma \in \mathbb{C}^m \} \rightarrow \mathcal{Y} \{ \tau \in \mathbb{C}^m \} \). Similarly, the deformation \( (\check{\varphi}A)(\sigma) \in \Gamma(T(\mathbb{T}^n)) \) is called induced from the deformation \( A(\tau) \in \Gamma(T(\mathbb{T}^n)) \) under the mapping \( \varphi : \mathcal{Y} \{ \sigma \in \mathbb{C}^k \} \rightarrow \mathcal{Y} \{ \tau \in \mathbb{C}^m \} \).

1.2. Versal deformations.

**Definition 1.3.** A vector field deformation \( A(\tau) \in \Gamma(T(\mathbb{T}^n)) \) is called \([2, 3, 5]\) versal, if it generates every other deformation \( B(\sigma) \in \Gamma(T(\mathbb{T}^n)) \) of the vector field \( A \in \Gamma(T(\mathbb{T}^n)) \), that is there exists such a mapping \( \varphi : \mathcal{Y} \{ \sigma \in \mathbb{C}^k \} \rightarrow \mathcal{Y} \{ \tau \in \mathbb{C}^m \} \) and a deformation \( g(\tau) \in Diff(\mathbb{T}^n) \) of the identity \( Id \in Diff(\mathbb{T}^n) \) that \( A(\varphi(\tau)) \in \Gamma(T(\mathbb{T}^n)) \) :

\[
(1.2) \quad B(\sigma) = Ad_{g(\sigma)}(\check{\varphi}A)(\sigma)
\]
on the deformation base $\mathcal{Y}\{\sigma \in \mathbb{C}^k\}$. Similarly, a 1-form deformation $l(\tau) \in \Lambda^1(T^n)$ is called versal, if it generates every other 1-form deformation $p(\tau) \in \Lambda^1(T^n)$ of the 1-form $l \in \Lambda^1(T^n)$, that is there exists such a mapping $\varphi: \mathcal{Y}\{\sigma \in \mathbb{C}^k\} \to \mathcal{Y}\{\tau \in \mathbb{C}^m\}$ and a deformation $g(\sigma) \in Diff(T^n)$ of the identity $I \in Diff(T^n)$ that it is equivalent to the deformation obtained from the induced deformation $p(\varphi(\sigma)) \in \Lambda^1(T^n)$:

$$p(\sigma) = Ad^*_g(\varphi)(\tau)$$

on the deformation base $\mathcal{Y}\{\sigma \in \mathbb{C}^k\}$.

2. **Versality and Transversality**

2.1. **Transversality.** Let $N \subset M$ - a smooth submanifold of a manifold $M$. Consider a smooth mapping $A: \mathcal{Y} \to M$, and let a point $\tau \in \mathcal{Y}$ for which $A(\tau) \in N$.

**Definition 2.1.** A mapping $A: \mathcal{Y} \to M$ is called transversal [2, 3] to the submanifold $N \subset M$, if

$$(2.1) \quad T_{A(\tau)}(M) = T_{A(\tau)}(N) + A_*T_{\tau}(\mathcal{Y}).$$

As the diffeomorphism group $Diff(T^n)$ naturally acts on a fixed vector field $A \in \Gamma(T(T^n))$, its orbit $Or(A; Diff(T^n)) = Ad_{Diff(T^n)}A \subset \Gamma(T(T^n))$. Thus, a deformation $A(\tau) \in \Gamma(T(T^n))$ can be considered as a mapping $A: \mathcal{Y} \to \Gamma(T(T^n))$ of the deformation base $\mathcal{Y}\{\sigma \in \mathbb{C}^m\}$ into the space of vector fields $\Gamma(T(T^n))$ on the torus $T^n$. The following lemma [3, 5] holds.

**Lemma 2.2.** A deformation $A(\tau) \in \Gamma(T(T^n))$ is versal iff the mapping $A: \mathcal{Y} \to \Gamma(T(T^n))$ is transversal to the orbit of the corresponding element $A \in \Gamma(T(T^n))$, that is any deformation $B(\tau) = Ad^*_g(\tau)(A)(\sigma)$ on the deformation base $\mathcal{Y}\{\sigma \in \mathbb{C}^k\}$ for some mapping $\varphi: \mathcal{Y}\{\sigma \in \mathbb{C}^k\} \to \mathcal{Y}\{\tau \in \mathbb{C}^m\}$.

**Proof.** Really, owing to the versality condition (1.2), for any deformation $B(\tau) \in \Gamma(T(T^n))$ of the vector field $A \in \Gamma(T(T^n))$ one has

$$(2.2) \quad B(\tau) = Ad^*_g(\tau)(\varphi)(\tau)$$

on the deformation base $\mathcal{Y}\{\tau \in \mathbb{C}^m\}$. Then, upon differentiating (2.2) with respect to $\tau \in \mathcal{Y}$ one obtains that

$$(2.3) \quad B_\xi(0) = A_\xi(0)\varphi(0) + [C_\xi(0)\xi, A]$$

for any $\xi \in T(\mathcal{Y})$, where $[\cdot, \cdot]$ is the usual commutator of vector fields on $T^n$ and $\nabla_{\tau}\varphi(\tau)|_{\tau=0} := \xi \in T_0(\mathcal{Y}), \nabla_\tau g(\tau)|_{\tau=0} := C_\xi(0) \in \Gamma(T(T^n))$. Now it is easy to see that (2.3) is equivalent to the transversality condition (2.1), if to put $M := \Gamma(T(T^n)), N := Or(A; Diff(T^n)) \subset \Gamma(T(T^n))$.

Consider now a smooth mapping $\alpha: Diff(T^n) \to \Gamma(T(T^n))$, where

$$(2.4) \quad \alpha(g) := Ad^*_gA,$$

which induces the tangent mapping $\alpha_*: diff(T^n) \to T_A(\Gamma(T(T^n)))$, where $diff(T^n) := T_{Id}(Diff(T^n))$ is the Lie algebra of vector fields on the torus $T^n$ and acts as

$$(2.5) \quad \alpha_*C = [C, A].$$

The kernel $\text{Ker}_\alpha$ is a Lie subalgebra of vector fields commuting with the vector field $A \in \Gamma(T(T^n))$ and is called its centralizer. It is also interesting to observe that the codimension $\text{codim} Or(A; Diff(T^n)) = \text{dim} \text{Ker}_\alpha$. As a result from reasoning in [2, 3, 5] for small enough $\tau \in \mathcal{Y}$ there exists an invertible mapping $\beta: V \times \mathcal{Y}\{\tau \in \mathbb{C}^m\} \to \Gamma(T(T^n))$ for $V$ to be a submanifold of $Diff(T^n)$, transversal to the centralizer $\text{Ker}_\alpha$ and of maximal dimension $dim V = dim Or(A; Diff(T^n))$, allowing the representation

$$(2.6) \quad \beta(g, \tau) = Ad^*_gA(\tau)$$

on the deformation base $\mathcal{Y}\{\tau \in \mathbb{C}^m\}$ for some $g \in V$. Let now $B(\sigma) \in \Gamma(T(T^n))$ be an arbitrary transversal deformation. Then it can be represented as $B(\sigma) = \beta(v, \tau)$, giving rise to the following expression:

$$(2.7) \quad B(\sigma) = Ad^*_gA(\varphi(\sigma)),$$

where $\varphi(\sigma) := \pi_2\beta^{-1}(B(\sigma)), g(\sigma) := \pi_1\beta^{-1}(B(\sigma))$ and $\pi_1$ and $\pi_2$ are projections of $V \times \mathcal{Y}\{\tau \in \mathbb{C}^k\}$ on the first and the second factor, respectively. The obtained expression (2.7) exactly means that this arbitrary deformation $B(\sigma) \in \Gamma(T(T^n))$ is versal, thus proving the lemma. \qed
Consider now a 1-form deformation $l(\tau) \in \Lambda^1(T^n)$ on the deformation base $\mathcal{Y}\{\tau \in \mathbb{C}^m\}$. The same way as above one can prove the following dual to Lemma (2.2) proposition.

**Proposition 2.3.** A 1-form deformation $l(\tau) \in \Lambda^1(T^n)$ is versal iff the mapping $l : \mathcal{Y}\{\tau \in \mathbb{C}^m\} \to \Lambda^1(T^n)$ is transversal to the orbit of the corresponding element $l \in \Lambda^1(T^n)$, that is any deformation $p(\sigma) = Ad^*_p(\varphi)(\sigma)$ on the deformation base $\mathcal{Y}\{\sigma \in \mathbb{C}^k\}$ for some mapping $\varphi : \mathcal{Y}\{\sigma \in \mathbb{C}^k\} \to \mathcal{Y}\{\tau \in \mathbb{C}^m\}$.

Being interested in describing versal deformations of pencils of differential forms, analytically depending on the "spectral" parameter $\lambda \in \mathbb{C}$, we will proceed below first to studying their orbits from the Marsden-Weinstein reduction theory point of view.

### 3. Torus Diffeomorphism Group and its Orbits

Let us now consider the action of the diffeomorphism group $Diff(T^n)$ on the space $\mathcal{G} := \text{diff}(T^n) \ltimes \text{diff}(T^n)^*$, being the semidirect product $\Gamma(T(T^n)) \ltimes \Lambda^1(T^n) \simeq \text{diff}(T^n) \ltimes \text{diff}(T^n)^*$.

It is well known [7, 8] that the semidirect sum $\mathcal{G} = \text{diff}(T^n) \ltimes \text{diff}(T^n)^*$ is a metrized Lie algebra with the Lie structure

$$[a_1 \ltimes l_1, a_2 \ltimes l_2] := [a_1, a_2] \ltimes (ad^*_{a_1} l_2 - ad^*_{a_2} l_1),$$

allowing to identify it with its adjoint space $\mathcal{G}^* \simeq \mathcal{G}$ via the nondegenerate and symmetric scalar product

$$\langle a_1 \ltimes l_1, a_2 \ltimes l_2 \rangle = (l_1, a_2) + (l_2, a_1)$$

for arbitrary $a_1 \ltimes l_1, a_2 \ltimes l_2 \in \mathcal{G}^* \simeq \mathcal{G}$, where $\langle \cdot, \cdot \rangle : \Lambda^1(T^n) \times \Gamma(T(T^n)) \to \mathbb{C}$ is the standard pairing.

Consider now the point product $\mathcal{G} := \prod_{z \in \mathbb{S}^1} \tilde{\mathcal{G}}$ of Lie algebra $\mathcal{G}$ and endow it with the central extension generated by a two-cocycle $\omega_2 : \mathcal{G} \times \mathcal{G} \to \mathbb{C}$, where

$$\omega_2(a_1 \ltimes l_1, a_2 \ltimes l_2) := \int_{S^1} [(l_1, \partial a_2 / \partial z) - (l_2, \partial a_1 / \partial z)] dz$$

for arbitrary $a_1 \ltimes l_1, a_2 \ltimes l_2 \in \mathcal{G}$. Thus, the adjoint space $\mathcal{G}^*$ is a Poisson manifold [2, 11, 4, 10] endowed with the canonical Lie-Poisson structure

$$\{f, h\} : = (a \ltimes l, [\nabla f(a \ltimes l), \nabla h(a \ltimes l)]) + \int_{S^1} | < \nabla f_a(a \ltimes l), \partial / \partial z \nabla h_a(a \ltimes l) > - < \nabla h_a(a \ltimes l), \partial / \partial z \nabla f_a(a \ltimes l) > | dz,$$

where $f, h \in D(\mathcal{G}^*)$, $\nabla f(a \ltimes l) := \nabla f_a(a \ltimes l) \ltimes \nabla f_a(a \ltimes l) \in \mathcal{G}$, $\nabla h(a \ltimes l) := \nabla h_a(a \ltimes l) \ltimes \nabla h_a(a \ltimes l) \in \mathcal{G}$, and $\nabla : D(\mathcal{G}^*) \to \mathcal{G}$ is the usual functional gradient mapping. If to take now a constant vector field $d(a \ltimes l) / ds = J(a) := \sum_{j,k=1}^{m} \alpha_{jk} \partial / \partial x_j \times dx_k \in \tilde{\mathcal{G}}$, depending on the constant parameters $\alpha_{jk} \in \mathbb{C}, j, k = \sum_{1,n}$, one can construct [6, 9] by means of the Lie differentiation $L_{J(a)}$ of the bracket (3.4) a new Poisson bracket

$$\{f, h\}_1 := L_{J(a,\beta)}\{f, h\}_0 - \{f, L_{J(a,\beta)}h\}_0 =$$

$$= (J(a), [\nabla f(a \ltimes l), \nabla h(a \ltimes l)])$$

defined for any $f, h \in D(\mathcal{G}^*)$ and satisfying the Jacobi condition.

Consider now the infinitesimal $Diff(T^n)$-actions on the space $\mathcal{G}^* \simeq \mathcal{G}$ subject to the Poisson brackets (3.4) and (3.5):

$$d(a \ltimes l) / d\tau = \{h, a \ltimes l\}_0 = (-[\nabla h_t, a] + \partial / \partial z \nabla h_t) \ltimes (ad^*_{\nabla h_t} l - ad^*_{\nabla h_a} - \partial / \partial z \nabla h_a)$$

subject to any function $h \in D(\mathcal{G}^*)$ and

$$d(a \ltimes l) / d\xi = \{f, a \ltimes l\}_1 = - \sum_{j=1}^{1,m} \alpha_{jk} [\nabla f_t, \partial / \partial x_j] \ltimes ad^*_{\nabla f_t} dx_k$$


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subject to a Casimir function \( f \in \mathcal{D}(\mathcal{G}^*) \), respectively to the evolution parameters \( \tau \) and \( \xi \in \mathbb{C} \). Making use of the vector fields \((3.6)\) and \((3.7)\), one can construct the following integrable on the space \( \mathcal{G}^* \) distributions:

\[
\mathcal{D}_0 = \{ (-[\nabla h_l, a] + \frac{\partial}{\partial z} \nabla h_l) \times (ad_{\nabla h_l} l - ad_{\nabla h_a} l - \frac{\partial}{\partial z} \nabla h_a) : h \in \mathcal{I}_1(\mathcal{G}^*) \},
\]

where \( \mathcal{I}_1(\mathcal{G}^*) \) is the space of Casimir functions for the Poisson bracket \((3.5)\), and

\[
\mathcal{D}_1 = \{ -\sum_{j,k=1}^n \alpha_{jk} \left[ \nabla f_i, \frac{\partial}{\partial x_j} \right] \times ad_{\nabla f_i} dx_k : f \in \mathcal{D}(\mathcal{G}^*) \},
\]

as \([\mathcal{D}_0, \mathcal{D}_0] \subset \mathcal{D}_0\) and \([\mathcal{D}_1, \mathcal{D}_1] \subset \mathcal{D}_1\). The set of maximal integral submanifolds of \((3.9)\) generates the foliation \( \mathcal{G}^*_1 \mathcal{G}^* \), whose leaves are the intersections of fixed integral submanifolds \( \mathcal{G}^*_j \mathcal{G}^* \) of the distribution \( \mathcal{D}_1 \) passing through an element \( a \times l \mathcal{G}^* \) with the leaves of the distribution \( \mathcal{D}_0 \). If the foliation \( \mathcal{G}^*_1 \mathcal{G}^* \) is sufficiently smooth, one can define the quotient manifold \( \mathcal{G}^*_r \mathcal{G}^* := \mathcal{G}^*_1(\mathcal{G}^* \mathcal{D}_0) \) with its associated projection mapping \( \mathcal{G}^*_r \mathcal{G}^* \rightarrow \mathcal{G}^*_r \mathcal{G}^* \). The structure of the reduced manifold \( \mathcal{G}^*_r \mathcal{G}^* \) is characterized by the following theorem.

**Theorem 3.1.** On the manifold \( \mathcal{G}^*_r \mathcal{G}^* \) the pair of Poisson structures \( \{\cdot, \cdot\}_0 \) and \( \{\cdot, \cdot\}_1 \) are compatible, that is for any parameter \( \lambda \in \mathbb{R} \) the algebraic sum \( \{\cdot, \cdot\}_0 + \lambda \{\cdot, \cdot\}_1 \) is Poisson too.

A proof of Theorem 3.1 is strongly based on the classical differential-geometric Marsden-Weinstein reduced space construction.

As a consequence of Theorem 3.1 and reasonings, based on the structure of the distribution \((3.8)\), one can describe its invariants on and a leave \( \mathcal{G}^*_r \mathcal{G}^* \) and generate the related coordinates on the reduced manifold \( \mathcal{G}^*_r \mathcal{G}^* = \mathcal{G}^*_1(\mathcal{G}^* \mathcal{D}_0) \). Thus, the related with \((3.6)\) reduced flow on the manifold \( \mathcal{G}^*_r \mathcal{G}^* \) will present the canonical representation of the studied versal deformation subject to a metric Lie algebra generated by the semidirect sum \( \Gamma(T(T^n)) \ltimes \Lambda^1(T^n) \ltimes \text{diff}(T^n) \ltimes \text{diff}(T^n)^* \) of the smooth affine vector fields \( \Gamma(T(T^n)) \) on the torus \( T^n \) and its adjoint space \( \Lambda^1(T^n) \). Their detailed analytical structure is under preparation and will be presented in other place.

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References


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