Stochastic Modeling and Applications
Vol. 22 No.2 (December, 2018)

VERSAL DEFORMATIONS OF AFFINE VECTOR FIELDS ON TORUS

DENIS BLACKMORE[1)], HANNA BARANSKA[2)] AND ANATOLIJ K. PRYKARPATSKI[2)]

Received: 23rd July 2017 Revised: 14th December 2017 Accepted: 15th January 2018

ABSTRACT. We study a classical problem of describing the versal deformations of a centrally extended metrized Lie algebra generated by the direct sum of affine vector fields and differential forms on torus

1. UNIVERSAL DEFORMATIONS OF VECTOR FIELDS AND DIFFERENTIAL FORMS

1.1. **Deformations.** We consider a smooth vector field $A \in \Gamma(T(\mathbb{T}^n))$ on the n-dimensional torus $\mathbb{T}^n$. A deformation of the vector field $A \in \Gamma(T(\mathbb{T}^n))$ we will call a vector field $A(\tau) \in \Gamma(T(\mathbb{T}^n))$, which depends analytically on the parameter $\tau \in \mathbb{C}^k, k \in \mathbb{Z}_+$, in some vicinity of the point $\tau = 0 \in \mathbb{C}^k$, and such that $A(0) = A$. The space of parameters $\Upsilon\{\tau \in \mathbb{C}^k\}$ is often called a base of the deformation. Similarly will consider a differential 1-form $l \in \Lambda^1(\mathbb{T}^n)$ on the n-dimensional torus $\mathbb{T}^n$ its related deformation $l(\tau) \in \Lambda^1(\mathbb{T}^n)$, which depends analytically on the parameter $\tau \in \mathbb{C}^k, k \in \mathbb{Z}_+$, in some vicinity of the point $\tau = 0 \in \mathbb{C}^k$ and such that $l(0) = l$.

Definition 1.1. Two vector fields deformations $A(\tau)$ and $B(\tau) \in \Gamma(T(\mathbb{T}^n))$ are called equivalent, if there exists such a deformation $g(\tau) \in Diff(\mathbb{T}^n)$ of the identity $Id \in Diff(\mathbb{T}^n)$, that $Ad_{g(\tau)}A(\tau) = B(\tau)$, where $ad : Diff(\mathbb{T}^n) \times \Gamma(T(\mathbb{T}^n)) \to \Gamma(T(\mathbb{T}^n))$ is the usual [2, 3, 5, 4] adjoint mapping of the space $Diff(\mathbb{T}^n)$ on $\Gamma(T(\mathbb{T}^n))$. Similarly, two 1-form deformations $l(\tau)$ and $p(\tau) \in \Lambda^1(\mathbb{T}^n)$ are called equivalent, if there exists such a deformation $g(\tau) \in Diff(\mathbb{T}^n)$ of the identity $I \in Diff(\mathbb{T}^n)$, that $Ad^*_{g(\tau)}l(\tau) = p(\tau)$, where $Ad^* : Diff(\mathbb{T}^n) \times \Lambda^1(\mathbb{T}^n) \to \Lambda^1(\mathbb{T}^n)$ is the usual adjoint mapping of the space $Diff(\mathbb{T}^n)$ on $\Lambda^1(\mathbb{T}^n)$.

Let φ - a germ of a holomorphic at zero mapping $\mathbb{C}^m \to \mathbb{C}^k$, that is a set of converging at $0 \in \mathbb{C}^m$ degree series of complex variables, and assume that $\varphi(0) = 0$. The mapping $\varphi : \Upsilon\{\sigma \in \mathbb{C}^m\} \to \Upsilon\{\tau \in \mathbb{C}^m\}$ defines evidently a new deformation $\check{\varphi}l(\sigma) \in \Lambda^1(\mathbb{T}^n)$ of the 1-form $l \in \Lambda^1(\mathbb{T}^n)$ and a new deformation $\hat{\varphi}A(\sigma)$ of the vector field $A \in \Gamma(T(\mathbb{T}^n))$ via the expressions

$$(\check{\varphi}l)(\sigma) = l(\varphi(\sigma)), \quad (\hat{\varphi}A)(\sigma) = A(\varphi(\sigma)) \tag{1.1}$$

on the deformation base $\Upsilon\{\sigma \in \mathbb{C}^k\}$.

Definition 1.2. The deformation $(\check{\varphi}l)(\sigma) \in \Lambda^1(\mathbb{T}^n)$ is called induced from the deformation $l(\tau) \in \Lambda^1(\mathbb{T}^n)$ under the mapping $\varphi : \Upsilon\{\sigma \in \mathbb{C}^m\} \to \Upsilon\{\tau \in \mathbb{C}^m\}$. Similarly, the deformation $(\hat{\varphi}A)(\sigma) \in \Gamma(T(\mathbb{T}^n))$ is called induced from the deformation $A(\tau) \in \Gamma(T(\mathbb{T}^n))$ under the mapping $\varphi : \Upsilon\{\sigma \in \mathbb{C}^k\} \to \Upsilon\{\tau \in \mathbb{C}^m\}$.

1.2. **Versal deformations.**

Definition 1.3. A vector field deformation $A(\tau) \in \Gamma(T(\mathbb{T}^n))$ is called [2, 3, 5] *versal,* if it generates every other deformation $B(\sigma) \in \Gamma(T(\mathbb{T}^n))$ of the vector field $A \in \Gamma(T(\mathbb{T}^n))$, that is there exists such a mapping $\varphi : \Upsilon\{\sigma \in \mathbb{C}^k\} \to \Upsilon\{\tau \in \mathbb{C}^m\}$ and a deformation $g(\tau) \in Diff(\mathbb{T}^n)$ of the identity $Id \in Diff(\mathbb{T}^n)$ that it is equivalent to the deformation obtained from the induced deformation $A(\varphi(\tau)) \in \Gamma(T(\mathbb{T}^n))$:

$$B(\sigma) = Ad_{g(\sigma)}(\hat{\varphi}A)(\sigma) \tag{1.2}$$

Date: Date of Submission: January 11, 2017; Date of Acceptance: May 15, 2017 .
1991 *Mathematics Subject Classification.* 17B68,17B80,35Q53, 35G25, 35N10, 37K35, 58J70,58J72, 34A34,37K05,37K10.
Key words and phrases. torus diffeomorphisms, vector fields, differential forms, versal deformations, central extension, metric Lie alagebra, Lie-Poisson structures, central extension, Casimir invariants.

on the deformation base $\Upsilon\{\sigma \in \mathbb{C}^k\}$. Similarly, a 1-form deformation $l(\tau) \in \Lambda^1(\mathbb{T}^n)$ is called *versal,* if it generates every other 1-form deformation $p(\tau) \in \Lambda^1(\mathbb{T}^n)$ of the 1-form $l \in \Lambda^1(\mathbb{T}^n)$, that is there exists such a mapping $\varphi : \Upsilon\{\sigma \in \mathbb{C}^k\} \to \Upsilon\{\tau \in \mathbb{C}^m\}$ and a deformation $g(\sigma) \in Diff(\mathbb{T}^n)$ of the identity $I \in Diff(\mathbb{T}^n)$ that it is equivalent to the deformation obtained from the induced deformation $p(\varphi(\sigma)) \in \Lambda^1(\mathbb{T}^n)$:

$$p(\sigma) = Ad^*_{g(\sigma)}(\check{\varphi} l)(\sigma) \tag{1.3}$$

on the deformation base $\Upsilon\{\sigma \in \mathbb{C}^k\}$.

2. Versality and transversality

2.1. **Transversality.** Let $N \subset M$ - a smooth submanifold of a manifold M. Consider a smooth mapping $A : \Upsilon \to M$, and let a point $\tau \in \Upsilon$ for which $A(\tau) \in N$.

Definition 2.1. A mapping $A : \Upsilon \to M$ is called transvesal [2, 3] to the submanifold $N \subset M$, if

$$T_{A(\tau)}(M) = T_{A(\tau)}(N) + A_* T_\tau(\Upsilon). \tag{2.1}$$

As the diffeomorphism group $Diff(\mathbb{T}^n)$ naturally acts on a fixed vector field $A \in \Gamma(T(\mathbb{T}^n))$, its orbit $Or(A; Diff(\mathbb{T}^n)) = Ad_{Diff(\mathbb{T}^n)} A \subset \Gamma(T(\mathbb{T}^n))$. Thus, a deformation $A(\tau) \in \Gamma(T(\mathbb{T}^n))$ can be considered as a mapping $A : \Upsilon \to \Gamma(T(\mathbb{T}^n))$ of the deformation base $\Upsilon\{\sigma \in \mathbb{C}^m\}$ into the space of vector fields $\Gamma(T(\mathbb{T}^n))$ on the torus $\mathbb{T}^n$. The following lemma [3, 5] holds.

Lemma 2.2. *A deformation $A(\tau) \in \Gamma(T(\mathbb{T}^n))$ is versal iff the mapping $A : \{\tau \in \mathbb{C}^m\} \to \Gamma(T(\mathbb{T}^n))$ is transversal to the orbit of the corresponding element $A \in \Gamma(T(\mathbb{T}^n))$, that is any deformation $B(\sigma) = Ad^*_{g(\sigma)}(\hat{\varphi} A)(\sigma)$ on the deformation base $\Upsilon\{\sigma \in \mathbb{C}^k\}$ for some mapping $\varphi : \Upsilon\{\sigma \in \mathbb{C}^k\} \to \Upsilon\{\tau \in \mathbb{C}^m\}$.*

Proof. Really, owing to the versality condition (1.2), for any deformation $B(\tau) \in \Gamma(T(\mathbb{T}^n))$ of the vector field $A \in \Gamma(T(\mathbb{T}^n))$ one has

$$B(\tau) = Ad_{g(\tau)}(\hat{\varphi} A)(\tau) \tag{2.2}$$

on the deformation base $\Upsilon\{\tau \in \mathbb{C}^m\}$. Then, upon differentiating (2.2) with respect to $\tau \in \Upsilon$ one obtains that

$$B_*(0)\xi = A_*(0)\hat{\varphi}(0)\xi + [C_*(0)\xi, A] \tag{2.3}$$

for any $\xi \in T(\Upsilon)$,. where $[\cdot,\cdot]$ is the usual commutator of vector fields on $\mathbb{T}^n$ and $\nabla_\tau \varphi(\tau)|_{\tau=0} := \xi \in T_0(\Upsilon)$, $\nabla_\tau g(\tau)|_{\tau=0} := C_*(0) \in \Gamma(T(\mathbb{T}^n))$. Now it is easy to see that (2.3) is equivalent to the transversality condition (2.1), if to put $M := \Gamma(T(\mathbb{T}^n)), N := Or(A; Diff(\mathbb{T}^n) \subset \Gamma(T(\mathbb{T}^n))$.

Consider now a smooth mapping $\alpha : Diff(\mathbb{T}^n) \to \Gamma(T(\mathbb{T}^n))$, where

$$\alpha(g) := Ad_g A, \tag{2.4}$$

which induces the tangent mapping $\alpha_* : diff(\mathbb{T}^n) \to T_A(\ \Gamma(T(\mathbb{T}^n)))$, where $diff(\mathbb{T}^n) := T_{Id}(Diff(\mathbb{T}^n)$ is the Lie algebra of vector fields on the torus $\mathbb{T}^n$ and acts as

$$\alpha_* C = [C, A]. \tag{2.5}$$

The kernel $Ker\alpha_*$ is a Lie subalgebra of vector fields commuting with the vector field $A \in \Gamma(T(\mathbb{T}^n))$ and is called its *centralizer.* It is also interesting to observe that the codimension $co\dim Or(A; Diff(\mathbb{T}^n) = \dim Ker\alpha_*$. As a result from reasoning in [2, 3, 5] for small enough $\tau \in \Upsilon$ there exists an invertible mapping $\beta : V \times \Upsilon\{\tau \in \mathbb{C}^m\} \to \Gamma(T(\mathbb{T}^n))$ for V to be a submanifold of $Diff(\mathbb{T}^n)$, transversal to the centralizer $\dim Ker\alpha_*$ and of maximal dimension $\dim V = \dim Or(A; Diff(\mathbb{T}^n)$, allowing the representation

$$\beta(g, \tau) = Ad_g A(\tau) \tag{2.6}$$

on the deformation base $\Upsilon\{\tau \in \mathbb{C}^m\}$ for some $g \in V$. Let now $B(\sigma) \in \Gamma(T(\mathbb{T}^n))$ be an arbitrary transversal deformation. Then it can be represented as $B(\sigma) = \beta(v, \tau)$, giving rise to the following expression:

$$B(\sigma) = Ad_{g(\sigma)} A(\varphi(\sigma)) \ , \tag{2.7}$$

where $\varphi(\sigma) := \pi_2 \beta^{-1}(B(\sigma)), g(\sigma) := \pi_1 \beta^{-1}(B(\sigma))$ and π_1 and π_2 are projections of $V \times \Upsilon\{\tau \in \mathbb{C}^k\}$ on the first and the second factor, respectively. The obtained expression (2.7) exactly means that this arbitrary deformation $B(\sigma) \in \Gamma(T(\mathbb{T}^n))$ is versal, thus proving the lemma. □

Consider now a 1-form deformation $l(\tau) \in \Lambda^1(\mathbb{T}^n)$ on the deformation base $\Upsilon\{\tau \in \mathbb{C}^m\}$. The same way as above one can prove the following dual to Lemma (2.2) proposition.

Proposition 2.3. *A 1-form deformation $l(\tau) \in \Lambda^1(\mathbb{T}^n)$ is versal iff the mapping $l : \Upsilon\{\tau \in \mathbb{C}^m\} \to \Lambda^1(\mathbb{T}^n)$ is transversal to the orbit of the corresponding element $l \in \Lambda^1(\mathbb{T}^n)$, that is any deformation $p(\sigma) = Ad^*_{g(\sigma)}(\check{\varphi}l)(\sigma)$ on the deformation base $\Upsilon\{\sigma \in \mathbb{C}^k\}$ for some mapping $\varphi : \Upsilon\{\sigma \in \mathbb{C}^k\} \to \Upsilon\{\tau \in \mathbb{C}^m\}$.*

Being interested in describing versal deformations of pencils of differential forms, analytically depending on the *"spectral"* parameter $\lambda \in \mathbb{C}$, we will proceed below first to studying their orbits from the Marsden-Weinstein reduction theory point of view.

3. Torus diffeomorphism group and its orbits

Let us now consider the action of the diffeomorphism group $Diff(\mathbb{T}^n)$ on the space $\mathcal{G} := diff(\mathbb{T}^n) \ltimes diff(\mathbb{T}^n)^*$, being the semidirect product $\Gamma(T(\mathbb{T}^n)) \ltimes \Lambda^1(\mathbb{T}^n) \simeq diff(\mathbb{T}^n) \ltimes diff(\mathbb{T}^n)^*$. It is well known [7, 8] that the semidirect sum $\mathcal{G} = diff(\mathbb{T}^n) \ltimes diff(\mathbb{T}^n)^*$ is a metrized Lie algebra with the Lie structure

$$[a_1 \ltimes l_1, a_2 \ltimes l_2] := [a_1, a_2] \ltimes (ad^*_{a_1} l_2 - ad^*_{a_2} l_1), \tag{3.1}$$

allowing to identify it with its adjoint space $\mathcal{G}^* \simeq \mathcal{G}$ via the nondegenerate and symmetric scalar product

$$(a_1 \ltimes l_1, a_2 \ltimes l_2) = (l_1, a_2) + (l_2, a_1) \tag{3.2}$$

for arbitrary $a_1 \ltimes l_1, a_2 \ltimes l_2 \in \mathcal{G}^* \simeq \mathcal{G}$, where $(\cdot,\cdot) : \Lambda^1(\mathbb{T}^n) \times \Gamma(T(\mathbb{T}^n)) \to \mathbb{C}$ is the standard pairing.

Consider now the point product $\check{\mathcal{G}} := \prod_{z \in \mathbb{S}^1} \tilde{\mathcal{G}}$ of Lie algebra $\mathcal{G}$ and endow it wit the central extension generated by a two-cocycle $\omega_2 : \check{\mathcal{G}} \times \check{\mathcal{G}} \to \mathbb{C}$, where

$$\omega_2(a_1 \ltimes l_1, a_2 \ltimes l_2) := \int_{\mathbb{S}^1} [(l_1, \partial a_2/\partial z) - (l_2, \partial a_1/\partial z)]dz \tag{3.3}$$

for arbitrary $a_1 \ltimes l_1, a_2 \ltimes l_2 \in \check{\mathcal{G}}$. Thus, the adjoint space $\check{\mathcal{G}}^*$ is a Poisson manifold [2, 11, 4, 10] endowed with the canonical Lie-Poisson structure

$$\begin{aligned} \{f,h\}_0 &:= (a \ltimes l, [\nabla f(a \ltimes l), \nabla h(a \ltimes l)]) + \\ &+ \int_{\mathbb{S}^1} [< \nabla f_a(a \ltimes l), \frac{\partial}{\partial z} \nabla h_l(a \ltimes l) \ > - < \nabla h_a(a \ltimes l), \frac{\partial}{\partial z} \nabla f_l(a \ltimes l) >]dz, \end{aligned} \tag{3.4}$$

where $f, h \in \mathcal{D}(\check{\mathcal{G}}^*)$, $\nabla f(a \ltimes l) := \nabla f_l(a \ltimes l) \ltimes \nabla f_a(a \ltimes l) \in \check{\mathcal{G}}$, $\nabla h(a \ltimes l) := \nabla h_l(a \ltimes l) \ltimes \nabla h_a(a \ltimes l) \in \check{\mathcal{G}}$ and $\nabla : \mathcal{D}(\check{\mathcal{G}}^*) \to \check{\mathcal{G}}$ is the usual functional gradient mapping. If to take now a constant vector field $d(a \ltimes l)/ds = J(\alpha) := \sum_{j,k=\overline{1,n}} \alpha_{jk}\ \partial/\partial x_j \ltimes dx_k \in \check{\mathcal{G}}^*$, depending on the constant parameters $\alpha_{kj} \in \mathbb{C}, j,k = \overline{1,n}$, one can construct [6, 9] by means of the Lie differentiation $L_{J(\alpha,\beta)}$ of the bracket (3.4) a new Poisson bracket

$$\begin{aligned} \{f,h\}_1 &:= L_{J(\alpha,\beta)}\{f,h\}_0 - \{L_{J(\alpha,\beta)} f, h\}_0 - \{f, L_{J(\alpha,\beta)} h\}_0 = \\ &= (J(\alpha), [\nabla f(a \ltimes l), \nabla h(a \ltimes l)]), \end{aligned} \tag{3.5}$$

defined for any $f, h \in \mathcal{D}(\check{\mathcal{G}}^*)$ and satisfying the Jacobi condition.

Consider now the infinitesimal $Diff(\mathbb{T}^n)$-actions on the space $\check{\mathcal{G}}^* \simeq \check{\mathcal{G}}$ subject to the Poisson brackets (3.4) and (3.5):

$$d(a \ltimes l)/d\tau = \{h, a \ltimes l\}_0 = (-[\nabla h_l, a] + \frac{\partial}{\partial z} \nabla h_l) \ltimes (ad^*_{\nabla h_l} l - ad^*_a \nabla h_a - \frac{\partial}{\partial z} \nabla h_a) \tag{3.6}$$

subject to any function $h \in \mathcal{D}(\check{\mathcal{G}}^*)$ and

$$d(a \ltimes l)/d\xi = \{f, a \ltimes l\}_1 = -\sum_{j=\overline{1,n}} \alpha_{jk}\ [\nabla f_l, \partial/\partial x_j] \ltimes ad^*_{\nabla f_l} dx_k \tag{3.7}$$

subject to a Casimir function $f \in \mathcal{D}(\check{\mathcal{G}}^*)$, respectively to the evolution parameters τ and $\xi \in \mathbb{C}$. Making use of the vector fields (3.6) and (3.7), one can construct the following integrable on the space $\check{\mathcal{G}}^*$ distributions:

$$\mathcal{D}_0 = \{(-[\nabla h_l, a] + \frac{\partial}{\partial z}\nabla h_l) \ltimes (ad^*_{\nabla h_l} l - ad^*_a \nabla h_a - \frac{\partial}{\partial z}\nabla h_a) : h \in I_1(\check{\mathcal{G}}^*)\}, \tag{3.8}$$

where $I_1(\check{\mathcal{G}}^*)$ is the space of Casimir functions for the Poisson bracket (3.5), and

$$\mathcal{D}_1 = \{-\sum_{j,k=\overline{1,n}} \alpha_{jk} \, [\nabla f_l, \partial/\partial x_j] \ltimes ad^*_{\nabla f_l} dx_k : f \in \mathcal{D}(\check{\mathcal{G}}^*)\}, \tag{3.9}$$

as $[\mathcal{D}_0, \mathcal{D}_0] \subset \mathcal{D}_0$ and $[\mathcal{D}_1, \mathcal{D}_1] \subset \mathcal{D}_1$. The set of maximal integral submanifolds of (3.9) generates the foliation $\check{\mathcal{G}}^*_J \backslash \mathcal{D}_0$, whose leaves are the intersections of fixed integral submanifolds $\check{\mathcal{G}}^*_J \subset \check{\mathcal{G}}^*$ of the distribution $\mathcal{D}_1$ passing through an element $a \ltimes l \in \mathcal{G}^*$ with the leaves of the distribution $\mathcal{D}_0$. If the foliation $\check{\mathcal{G}}^*_J \backslash \mathcal{D}_0$ is sufficiently smooth, one can define the quotient manifold $\check{\mathcal{G}}^*_{red} := \check{\mathcal{G}}^*_J / (\check{\mathcal{G}}^*_J \backslash \mathcal{D}_0)$ with its associated projection mapping $\check{\mathcal{G}}^*_J \to \check{\mathcal{G}}^*_{red}$. The structure of the reduced manifold $\check{\mathcal{G}}^*_{red}$ is characterized by the following theorem.

Theorem 3.1. *On the manifold $\check{\mathcal{G}}^*_{red}$ the pair of Poisson structures $\{\cdot,\cdot\}_0$ and $\{\cdot,\cdot\}_1$ are compatible, that is for any parameter $\lambda \in \mathbb{R}$ the algebraic sum $\{\cdot,\cdot\}_0 + \lambda\{\cdot,\cdot\}_1$ is Poisson too.*

A proof of Theorem 3.1 is strongly based on the classical differential-geometric Marsden-Weinstein reduced space construction.

As a consequence of Theorem 3.1 and reasonings, based on the structure of the distribution (3.8), one can describe its invariants on and a leave $\check{\mathcal{G}}^*_J$ and generate the related coordinates on the reduced manifold $\check{\mathcal{G}}^*_{red} = \check{\mathcal{G}}^*_J / (\check{\mathcal{G}}^*_J \backslash \mathcal{D}_0)$. Thus, the related with (3.6) reduced flow on the manifold $\check{\mathcal{G}}^*_{red}$ will present the canonical representation of the studied versal deformation subject to a metric Lie algebra generated by the semidirect sum $\Gamma(T(\mathbb{T}^n)) \ltimes \Lambda^1(\mathbb{T}^n) \simeq diff(\mathbb{T}^n) \ltimes diff(\mathbb{T}^n)^*$ of the smooth affine vector fields $\Gamma(T(\mathbb{T}^n))$ on the torus $\mathbb{T}^n$ and its adjoint space $\Lambda^1(\mathbb{T}^n)$. Their detailed analytical structure is under preparation and will be presented in other place.

4. Acknowledgements

The authors cordially thank Prof. Yu.B. Zelinsky (IM NANU, Kyiv) for the cooperation and useful discussions of the results obtained.

References

[1] R. Abraham and J. Marsden, Foundations of Mechanics, Second Edition, NY, Benjamin Cummings (1978)
[2] V.I. Arnold. Mathematical methods of classical mechanics. Springer (1989)
[3] V.I. Arnold. On Matrices Depending on Parameters, Russian Math. Surveys, 26:29-43, 1971
[4] D. Blackmore, A.K. Prykarpatsky and V.H. Samoylenko, Nonlinear dynamical systems of mathematical physics, World Scientific Publisher, NJ, USA, 2011
[5] A. Edelman, E. Elmrothy and B. Kagstrom, A Geometric Approach to Perturbation Theory of Matrices and Matrix Pencils, Part I: Versal Deformations, March 23, 1995, Preprint: UMINF-95.09
[6] Magri F., Acta Applicandea Mathematica, 1995, V.41, 247–270
[7] V. Ovsienko, Bi-Hamilton nature of the equation $u_{tx} = u_{xy}u_y - u_{yy}u_x$, arXiv:0802.1818v1 [math-ph] 13 Feb 2008
[8] V. Ovsienko, C. Roger, Looped Cotangent Virasoro Algebra and Non-Linear Integrable Systems in Dimension 2 + 1, Commun. Math. Phys. 273 (2007), 357–378
[9] A. Prykarpatsky, D. Blackmore, Versal Deformations of a Dirac Type Differential Operator, Journal of Nonlinear Mathematical Physics 1999, V.6, N 3, 246–254
[10] A.G. Reyman, M.A. Semenov-Tian-Shansky , Integrable Systems, The Computer Research Institute Publ., Moscow-Izhvek, 2003 (in Russian)
[11] L.A. Takhtadjian, L.D. Faddeev, Hamiltonian Approach in Soliton Theory, Springer, Berlin-Heidelberg, 1987

[1)]Department of Mathematical Sciences, New Jersey Institute of Technology (NJIT), Newark NJ 07102, USA, [2)] AGH University of Science and Technology, Kraków, Poland

E-mail address: deblac@m.njit.edu, abaran@agh.edu.pl, pryk.anat@cybergal.com