Generalized integral guiding functions and periodic solutions for inclusions with causal multioperators

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Abstract

In the present paper the method of generalized integral guiding functions is applied to study the periodic problem for a differential inclusion with a causal multioperator.

Key words: differential inclusion; causal multivalued map; periodic solution; guiding function; integral guiding function

2010 Mathematics Subject Classification: Primary: 34A60 Secondary: 34C25.

1 Introduction

The study of systems governed by differential and functional equations with causal operators, which is due to Tonelli [29] and Tychonov [28], attracts the attention of many researchers. The term causal arises from the engineering and the notion of a causal operator turns out to be a powerful tool for unifying problems in ordinary differential equations, integro-differential equations, functional differential equations with finite or infinite delay, Volterra integral equations, neutral functional equations et al. (see monograph [2]). Various problems for functional differential equations with causal operators were considered in recent
papers [4, 5, 8, 23, 26]. In particular, boundary and periodic problems were studied in [5] and [23]. In the present paper we apply the method of generalized integral guiding functions to the investigation of the periodic problem for a differential inclusion with a multivalued causal operator.

The main ideas of the method of guiding functions were formulated by Krasnoselskii and Perov in the fifties (see [18, 19]). Being geometrically clear, this method was originally applied to the study of periodic and bounded solutions of ordinary differential equations (see, e.g., [20, 24, 25]). Thereafter the method was extended to differential inclusions (see, e.g., [1, 7]), functional differential equations and inclusions (see, e.g., [6, 10, 13, 14, 16]) and other objects. The sphere of applications was extended to the study of qualitative behavior and bifurcations of solutions (see, e.g., [15, 21, 22]) and asymptotics of solutions (see, e.g., [11, 12, 17]). These and other aspects of the method of guiding functions and its applications, as well as the additional bibliography, may be found in the recent monograph [27].

The paper is organized in the following way. After preliminaries (Section 2), we give the notion of a multivalued causal operator (Section 3.1) and formulate the periodic problem for a differential inclusion with a causal multioperator (Section 3.2). Our main existence result (Theorem 2) is presented for the case when the right-hand side of the inclusion is convex-valued and closed.

2 Preliminaries

In what follows we will use some known notions and notation from the theory of multivalued maps (multimaps) (see, e.g., [1, 3, 7, 9]). Recall some of them.

Let \((X, d_X)\) and \((Y, d_Y)\) be metric spaces. By the symbols \(P(Y)\) and \(K(Y)\) we denote the collections of all nonempty and, respectively, nonempty and compact subsets of the space \(Y\). If \(Y\) is a normed space, \(Cv(K)\) and \(Kv(Y)\) denote the collections of all nonempty convex closed [and, respectively, compact] subsets of \(Y\).

**Definition 1** A multimap \(F : X \to P(Y)\) is called upper semicontinuous (u.s.c.) at a point \(x \in X\) if for each open set \(V \subset Y\) such that \(F(x) \subset Y\) there exists \(\delta > 0\) such that \(d_X(x, x') < \delta\) implies \(F(x') \subset V\). A multimap \(F : X \to P(Y)\) is called u.s.c. if it is u.s.c. at each point \(x \in X\).

**Definition 2** A multimap \(F : X \to P(Y)\) is called lower semicontinuous (l.s.c.) at a point \(x \in X\) if for each open set \(V \subset Y\) such that \(F(x) \cap V \neq \emptyset\) there exists \(\delta > 0\) such that \(d_X(x, x') < \delta\) implies \(F(x') \cap V \neq \emptyset\). A multimap \(F : X \to P(Y)\) is called l.s.c. if it is l.s.c. at each point \(x \in X\).

**Definition 3** A multimap \(F : X \to P(Y)\) is called continuous if it is both u.s.c. and l.s.c.

**Definition 4** A multimap \(F : X \to P(Y)\) is called closed if its graph
\[
\Gamma_F = \{(x, y) \mid (x, y) \in X \times Y, \quad y \in F(x)\}
\]
is a closed subset of the space $X \times Y$.

**Definition 5** A multimap $F : X \to P(Y)$ is called compact if its range $F(X)$ is relatively compact in $Y$.

**Remark 1** If multimap $F : X \to P(Y)$ is closed and compact, it is u.s.c.

Let $I$ be a closed subset of $\mathbb{R}$ endowed with the Lebesgue measure.

**Definition 6** A multifunction $F : I \to K(Y)$ is called measurable if, for each open subset $W \subset Y$, its pre-image

$$F^{-1}(W) = \{t \in I : F(t) \subset W\}$$

is a measurable subset of $I$.

**Remark 2** Each measurable multifunction $F : I \to K(Y)$ has a measurable selection, i.e., there exists such measurable function $f : I \to Y$, that $f(t) \in F(t)$ for a.e. $t \in I$.

In the sequel we will use some standard properties of the topological degree theory of single-valued and multivalued vector fields (see, e.g., [3, 7, 9, 18]).

### 3 Periodic problem for inclusions with causal multioperators

#### 3.1 Causal multioperators

Let $T > 0$ and $\sigma \geq 0$ be given numbers. By the symbols $C([kT-\sigma,(k+1)T]; \mathbb{R}^n)$ and $L^1((kT,(k+1)T); \mathbb{R}^n)$, where $k \in \mathbb{Z}$, we will denote the corresponding spaces of continuous and integrable functions with usual norms.

For any subset $\mathcal{N} \subset L^1((kT,(k+1)T); \mathbb{R}^n)$ and $\tau \in (kT,(k+1)T)$ we define the restriction of $\mathcal{N}$ on $(kT,\tau)$ as

$$\mathcal{N}|_{(kT,\tau)} = \{ f|_{(kT,\tau)} : f \in \mathcal{N} \}.$$

**Definition 7** We will say that $Q$ is a causal multioperator if for each $k \in \mathbb{Z}$ a multimap

$$Q : C([kT-\sigma,(k+1)T]; \mathbb{R}^n) \to L^1((kT,(k+1)T); \mathbb{R}^n)$$

is defined in such a way that for each $\tau \in (kT,(k+1)T)$ and for all $u(\cdot), v(\cdot) \in C([kT-\sigma,(k+1)T]; \mathbb{R}^n)$

the condition $u|_{[kT-\sigma,\tau]} = v|_{[kT-\sigma,\tau]}$ implies $Q(u)|_{(kT,\tau)} = Q(v)|_{(kT,\tau)}$.

Let us consider some examples of causal multioperators. Denote by $\mathcal{C}$ the Banach space $C([\sigma,0]; \mathbb{R}^n)$. 
**Example 1** Suppose that a multimap \( F : \mathbb{R} \times \mathcal{C} \to K_v(\mathbb{R}^n) \) satisfies the following conditions:

1. \((F1)\) the multifunction \( F(\cdot, c) : \mathbb{R} \to K_v(\mathbb{R}^n) \) admits a measurable selection for every \( c \in \mathcal{C} \);
2. \((F2)\) the multimap \( F(t, \cdot) : \mathcal{C} \to K_v(\mathbb{R}^n) \) is u.s.c. for a.e. \( t \in \mathbb{R} \);
3. \((F3)\) for every \( r > 0 \) there exists a locally integrable nonnegative function \( \eta_r(\cdot) \in L^1_{loc}(\mathbb{R}) \) such that
   \[ \| F(t, c) \| := \sup \{ \| y \| : y \in F(t, c) \} \leq \eta_r(t) \quad \text{a.e. } t \in \mathbb{R} , \]
   for all \( c \in \mathcal{C} \), \( \| c \| \leq r \).

It is known (see, e.g., [3, 9]) that under conditions \((F1) - (F3)\) for each \( k \in \mathbb{Z} \), the superposition multioperator

\[
\mathcal{P}_F : C([kT - \sigma, (k + 1)T]; \mathbb{R}^n) \to L^1((kT, (k + 1)T); \mathbb{R}^n),
\]

\[
\mathcal{P}_F(u) = \{ f \in L^1((kT, (k + 1)T); \mathbb{R}^n) : f(t) \in F(t, u_t) \quad \text{a.e. } t \in (kT, (k + 1)T) \} \tag{1}
\]

is well defined. Here \( u_t \in \mathcal{C} \) is defined as \( u_t(\theta) = u(t + \theta), \theta \in [-\sigma, 0] \). It is easy to see that the multioperator \( \mathcal{P}_F \) is causal.

**Remark 3** We will say that a multimap \( F : \mathbb{R} \times \mathcal{C} \to K(\mathbb{R}^n) \) obeying \((F1)-(F2)\) satisfies the upper Carathéodory conditions. If \((F2)\) may be replaced with

1. \((F2')\) the multimap \( F(t, \cdot) : \mathcal{C} \to K(\mathbb{R}^n) \) is continuous for a.e. \( t \in \mathbb{R} \)
   we say that \( F \) satisfies the Carathéodory conditions.

**Example 2** Let \( F : \mathbb{R} \times \mathcal{C} \to K_v(\mathbb{R}^n) \) be a multimap satisfying conditions \((F1) - (F3)\) of Example 1. Suppose that \( \{ K(t, s) : -\infty < s \leq t < +\infty \} \) is a continuous (with respect to the norm) family of linear operators in \( \mathbb{R}^n \) and \( m \in L^1_{loc}(\mathbb{R}; \mathbb{R}^n) \) is a given locally integrable function. Consider, for each \( k \in \mathbb{Z} \), the Volterra type integral multioperator \( \mathcal{G} : C([kT - \sigma, (k + 1)T]; \mathbb{R}^n) \to L^1((kT, (k + 1)T); \mathbb{R}^n) \) defined as

\[
\mathcal{G}(u)(t) = m(t) + \int_{kT}^{t} K(t, s) F(s, u_s) ds ,
\]

i.e.,

\[
\mathcal{G}(u) = \{ y \in L^1((kT, (k + 1)T); \mathbb{R}^n) : y(t) = m(t) + \int_{kT}^{t} K(t, s) f(s) ds : f \in \mathcal{P}_F(u) \} .
\tag{2}
\]

It is also obvious that the multioperator \( \mathcal{G} \) is causal.

**Example 3** Suppose that a multimap \( F : \mathbb{R} \times \mathcal{C} \to K(\mathbb{R}^n) \) satisfies the following condition of almost lower semicontinuity:
There exists a sequence of disjoint closed sets \( \{ J_n \} \), \( J_n \subseteq \mathbb{R}^n \) \( n = 1, 2, \ldots \) such that: (i) \( \text{meas} \left( \mathbb{R} \setminus \bigcup_n J_n \right) = 0 \); (ii) the restriction of \( F \) on each set \( J_n \times \mathbb{C} \) is l.s.c.

Then (see, e.g., [3, 9]) under conditions (\( F_L \)), (F3), for each \( k \in \mathbb{Z} \), the superposition multioperator

\[
P_F : C \left( [kT - \sigma, (k + 1)T]; \mathbb{R}^n \right) \rightarrow L^1 \left( [(kT, (k + 1)T); \mathbb{R}^n] \right)
\]

is also well-defined and causal.

### 3.2 Periodic problem

Denote by \( C_T \) the space of continuous \( T \)-periodic functions \( x : \mathbb{R} \rightarrow \mathbb{R}^n \) with the norm \( \| x \|_C = \sup_{t \in [0, T]} \| x(t) \| \). By \( \| x \|_2 \) we denote the norm of function \( x \) in the space \( L^2 \),

\[
\| x \|_2 = \left( \int_0^T \| x(s) \|^2 \, ds \right)^{\frac{1}{2}}.
\]

To define the notion of a periodic causal multioperator, introduce, for \( k \in \mathbb{Z} \), the following shift operator \( j_k : L^1 \left( [(kT, (k + 1)T); \mathbb{R}^n] \rightarrow L^1 \left( [(0, T); \mathbb{R}^n] \right) \right) : \)

\[
j_k(f)(t) = f(t + kT).
\]

**Definition 8** A causal multioperator \( Q \) will be called \( T \)-periodic if, for each \( x \in C_T \) and \( k \in \mathbb{Z} \),

\[
j_k(\mathcal{Q}(x \mid_{[kT - \tau, (k + 1)T]})) = \mathcal{Q}(x \mid_{[-T, T]}).
\]

It is clear that, to provide the periodicity of the causal multioperators in the above examples, it is sufficient to assume that the multimaps \( F \) are \( T \)-periodic in the first argument:

\[
F(t + T, c) = F(t, c)
\]

for all \( (t, c) \in \mathbb{R} \times \mathbb{C} \) and in Example 2, additionally, that function \( m(t) \) and family \( K(t, s) \) are also \( T \)-periodic:

\[
m(t + T) = m(t), \quad \forall t \in \mathbb{R}; \]

\[
K(t + T, s + T) = K(t, s), \quad \forall -\infty < s \leq t < +\infty.
\]

It is clear that the condition of periodicity of the causal multioperator allows to consider it only on the space \( C([-\tau, T]; \mathbb{R}^n) \).

Given a \( T \)-periodic causal multioperator \( \mathcal{Q} \), we will consider the existence of solutions to the following problem:

\[
x' \in \mathcal{Q}(x), \quad (3)
\]
where \( x \in C_T \) is an absolutely continuous function.

Denote by \( L^1_T \) the space of integrable \( T \)-periodic functions \( f : \mathbb{R} \to \mathbb{R}^n \).

In this section we will assume that the \( T \)-periodic causal multioperator \( Q : C_T \to Cv(L^1_T) \) satisfies the following conditions:

\((Q1)\) for each bounded linear operator \( A : L^1_T \to E \), where \( E \) is a Banach space, the composition \( A \circ Q : C_T \to Cv(E) \) is closed;

\((Q2)\) there exists a non-negative \( T \)-periodic integrable function \( \alpha(t) \) such that
\[
\|Q(x)(t)\| \leq \alpha(t)(1 + \|x(t)\|) \quad \text{for a.e. } t \in \mathbb{R}
\]
for each \( x \in C_T \).

To provide condition \((Q1)\) in Examples 1 and 2, it is sufficient to assume, besides the above mentioned periodicity conditions, that the multimap \( F \) satisfies conditions \((F1) – (F3)\) (see, e.g. [1], Theorem 1.5.30) and to fulfil condition \((Q2)\), we can suppose, in Example 1, the following sublinear growth condition: for each \( x \in C_T \) we have, for some non-negative integrable function \( \beta(t) \):
\[
\|F(t,x_t)\| \leq \beta(t)(1 + \|x(t)\|) \quad \text{a.e. } t \in [0, T],
\]
and, in Example 2, the global boundedness condition
\[
\|F(t,c)\| \leq \gamma(t)
\]
for some non-negative integrable function \( \gamma(t) \).

To study periodic problem (3) we will need a coincidence point result for a multivalued perturbation of a linear Fredholm operator. Let us give necessary definitions.

Let \( E_1, E_2 \) be Banach spaces, \( U \subset E_1 \) an open bounded set; \( l : Dom l \subseteq E_1 \to E_2 \) a linear Fredholm operator of zero index such that \( \text{Im } l \subset E_2 \) is closed.

Consider continuous linear projection operators \( p : E_1 \to E_1 \) and \( q : E_2 \to E_2 \) such that \( \text{Im } p = \text{Ker } l, \text{Im } l = \text{Ker } q \). By the symbol \( l_p \) denote the restriction of the operator \( l \) to \( \text{Dom } l \cap \text{Ker } p \).

Further, let the continuous operator \( k_{p,q} : E_2 \to \text{Dom } l \cap \text{Ker } p \) is defined by the relation \( k_{p,q}(y) = l_p^{-1}(y - q(y)), y \in E_2 \); the canonical projection operator \( \pi : E_2 \to E_2/\text{Im } l \) has the form \( \pi(y) = y + \text{Im } l, y \in E_2 \); and \( \phi : \text{Coker } l \to \text{Ker } l \) a continuous linear isomorphism.

Let \( G : \overline{U} \to Kv(E_2) \) be a closed multimap such that

\((a)\) \( G(U) \) is a bounded subset of \( E_2 \);

\((b)\) \( k_{p,q} \circ G : \overline{U} \to Kv(E_1) \) is compact and u.s.c.

The following assertion holds true (see [3], Lemma 13.1).

**Lemma 1** Suppose that:
(i) \( l(x) \notin \lambda \mathcal{G}(x) \) for all \( \lambda \in (0, 1], \ x \in \text{Dom} \ l \cap \partial U; \)

(ii) \( 0 \notin \pi \mathcal{G}(x) \) for all \( x \in \text{Ker} \ l \cap \partial U; \)

(iii) \( \deg_{\text{Ker} \ l}(\phi \pi_{\mathcal{G}|U_{\text{Ker} \ l}}, U_{\text{Ker} \ l}) \neq 0, \) where the symbol \( \deg_{\text{Ker} \ l} \) denotes the topological degree of a multivalued vector field evaluating in the space \( \text{Ker} \ l, \) and \( U_{\text{Ker} \ l} = U \cap \text{Ker} \ l. \)

Then \( l \) and \( \mathcal{G} \) has a coincidence point in \( U, \) i.e., there exists \( x \in U \) such that \( l(x) \in \mathcal{G}(x). \)

Developing notions introduced in [6, 10, 18], let us give the following definition.

**Definition 9** A continuously differentiable function \( V : \mathbb{R}^n \to \mathbb{R} \) is called the integral guiding function for inclusion (3) if there exists \( N > 0 \) such that

\[
\int_0^T \langle \nabla V(x(s)), f(s) \rangle \, ds > 0 \quad \text{for all} \ f \in \mathcal{Q}(x),
\]

for each absolutely continuous function \( x \in C_T \) such that \( \|x\| \geq N \) and \( \|x'(t)\| \leq \|\mathcal{Q}(x)(t)\| \) a.e. \( t \in [0, T]. \)

From the definition it immediately follows that the integral guiding function \( V \) is a non-degenerate potential in the sense that

\[ \nabla V(x) \neq 0, \]

for all \( x \in \mathbb{R}^n, \|x\| \geq K = \frac{N}{\sqrt{T}}. \) Therefore, on each closed ball \( B_{\tilde{K}} \subset \mathbb{R}^n \) centered at the origin of the radius \( \tilde{K} \geq K, \) the topological degree of the gradient \( \deg(\nabla V; B_{\tilde{K}}) \) is well defined and, moreover, it does not depend on the radius \( \tilde{K} \) (see, e.g., [18, 20]). This generic value of the degree will be called the index \( \text{Ind} V \) of an integral guiding function \( V. \)

**Definition 10** A non-degenerate potential \( V : \mathbb{R}^n \to \mathbb{R} \) is called the generalized integral guiding function for inclusion (3) if there exists \( N > 0 \) such that

\[
\int_0^T \langle \nabla V(x(s)), f(s) \rangle \, ds \geq 0 \quad \text{for some} \ f \in \mathcal{Q}(x),
\]

for each absolutely continuous function \( x \in C_T \) such that \( \|x\| \geq N \) and \( \|x'(t)\| \leq \|\mathcal{Q}(x)(t)\| \) a.e. \( t \in [0, T]. \)

Now we are in position to formulate the main result of this section.

**Theorem 1** Let \( V : \mathbb{R}^n \to \mathbb{R} \) be an generalized integral guiding function for problem (3) such that

\[ \text{Ind} V \neq 0. \]

Then problem (3) has a solution.
Remark 4 Notice that conditions of the theorem are fulfilled if, for example, the function $V$ is even or satisfies the coercivity condition: $\lim_{\|x\| \to +\infty} V(x) = \pm \infty$.

Proof Step 1. Let us consider the case of the strict integral guiding function for inclusion (3). Let us justify the solvability of the following operator inclusion

$$lx \in Q(x), \quad (8)$$

where $l : \text{Dom } l := \{x \in C_T : x \text{ is absolutely continuous} \} \subset C_T \to L^1_T$ is the linear Fredholm operator of zero index. It is easy to see that $\ker l = \mathbb{R}^n$, projection $\pi : L^1_T \to \mathbb{R}^n$ may be given by the formula

$$\pi f = \int_0^T f(s) \, ds$$

and the multioperators $\pi Q$ and $k_{p,q} Q$ are convex-valued and compact on bounded subsets.

Now, let, for some $\lambda \in (0, 1]$ a function $x \in \text{Dom } l$ is the solution of the inclusion

$$lx \in \lambda Q(x).$$

It means that $x(\cdot)$ is an absolutely continuous function such that $x'(t) = \lambda f(t)$ a.e. $t \in [0, T]$, for some $f \in Q(x)$.

Then

$$\int_0^T \langle \nabla V(x(s)), f(s) \rangle \, ds = \frac{1}{\lambda} \int_0^T \langle \nabla V(x(s)), x'(s) \rangle \, ds =$$

$$= \frac{1}{\lambda} \int_0^T V'(x(s)) \, ds = \frac{1}{\lambda} (V(x(T)) - V(x(0))) = 0,$$

yielding

$$\|x\|_2 < N.$$

From condition (Q2) it follows that $\|x'\|_2 < M'$, where $M' > 0$. But then there exists also $M > 0$ such that

$$\|x\|_C < M.$$

Now, take as $U$ the ball $B_r \subset C_T$ of the radius $r = \max\{M, NT^{-1/2}\}$. Then we have

$$lx \notin \lambda Q(x)$$

for all $x \in \partial U$.

Take an arbitrary $u \in \partial U \cap \ker l$. We have $\|u\| \geq NT^{-1/2}$ and considering $u$ as a constant function, from the definition of the strict integral guiding function we obtain

$$\int_0^T \langle \nabla V(u), f(s) \rangle \, ds > 0$$

for each $f \in Q(u)$. But

$$\int_0^T \langle \nabla V(u), f(s) \rangle \, ds = \langle \nabla V(u), \int_0^T f(s) \, ds \rangle = T\langle \nabla V(u), \pi f \rangle > 0,$$
and, therefore
\[ \langle \nabla V(u), y \rangle > 0 \]
for each \( y \in \pi Q(u) \).

It means that \( 0 \notin \pi Q(u) \) and, moreover, the multifield \( \pi Q(u) \) and the field \( \nabla V(u) \) do not admit opposite directions for \( u \in \partial U \cap \text{Ker } l \). It means that they are homotopic and, hence,
\[ \text{deg}(\pi Q\mid_{\overline{U} \cap \text{Ker } l}) = \text{deg}(\nabla V, \overline{U} \cap \text{Ker } l) \neq 0, \]
where \( \overline{U} \cap \text{Ker } l = \overline{U} \cap \text{Ker } l \). Therefore, all conditions of Lemma 1 are fulfilled and problem (8), and, hence (3) have a solution.

**Step 2.** Now we consider the case of the generalized integral guiding function for inclusion (3). Consider a multimap \( B : C_T \to P(L^1_T) \) defined as
\[
B(x) = \left\{ \varphi : |\varphi(t)| \leq \alpha(t)(1 + \|x(t)\|) \text{ and } \gamma(x) \int_0^T \langle \nabla V(x(s)), \varphi(s) \rangle \, ds \geq 0 \right\},
\]
where the first relation holds true for a.e. \( t \in [0, T] \), \( \alpha(\cdot) \) is a function from the condition (Q2), and
\[
\gamma(x) = \begin{cases} 
0, & \text{if } \|x\|_2 \leq N, \\
1, & \text{if } \|x\|_2 > N.
\end{cases}
\]
It is easy to verify that \( B \) is a closed multimap.

Let us consider a multimap \( Q_B : C_T \to P(L^1_T) \) given as
\[
Q_B(x) = Q(x) \cap B(x).
\]
Obviously, the multimap \( Q_B \) is closed and the condition (7) is satisfied for all \( f \in Q_B(x) \).

For the non-degenerate potential \( V \) we define a map \( Y_V : \mathbb{R}^n \to \mathbb{R}^n \) as follows
\[
Y_V(x) = \begin{cases} 
\nabla V(x), & \text{if } \|\nabla V(x)\| \leq 1, \\
\nabla V(x) / \|\nabla V(x)\|, & \text{if } \|\nabla V(x)\| > 1.
\end{cases}
\]
It is easy to see that the map \( Y \) is continuous.

For any \( \varepsilon_m > 0 \) we define a multimap \( Q_m : C_T \to P(L^1_T) \) as following
\[
Q_m(x) = Q_B(x) + \varepsilon_m Y_V(x).
\]
The multimap \( Q_m \) is closed and for each \( \varepsilon_m > 0 \) the condition (6) is fulfilled. By applying results of Step 1 we can prove the solvability of the following operator inclusion
\[ lx \in Q_m(x), \]
for each \( \varepsilon_m > 0 \). From which follows the existence of a solution for problem (3).

**Acknowledgements.** The work was supported by the Ministry of Education and Science of the Russian Federation, Project No. 1.3464.2017/4.6 (S. Kornev and V. Obukhovskii) and the Agreement No 02.A03.21.0008 (V. Obukhovskii). The work of P. Zecca is partially supported by INdAM and by the Universita di Firenze.
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