Stochastic Modeling and Applications
Vol. 22 No.2 (December, 2018)

KURATOWSKI'S CONTRIBUTION IN THE TOPOLOGY OF HYPERSPACES

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To the memory of Yuri Zelinskii

Abstract. This note is devoted to Kazimierz Kuratowski's contribution to the theory of hyperspaces. Some modern applications of his results, in particular, to the theory of absorbing sets in the infinite-dimensional manifolds, are presented.

Received: 13th August 2017 Revised: 24th December 2017 Accepted: 10th March

"... his personal copy of that book,
the red one, what was it?"
"The Topology of Hyperspace."
"Right. And he said,
'It's too difficult for you ... ' "
Stanisław Lem, *"Return From The Stars"*

1. Introduction

Kazimierz Kuratowski was born in Warsaw, which was then under Russian occupation, on on 2 February 1896. He learned in a gymnasium with instruction in Polish and took part in the school strike in 1905 (they struggled for universal instruction in Polish). The certificate of a Polish gymnasium did not allow for education at a university, therefore Kuratowski obtained his maturity certificate in Moscow.

He started a course of studies at the University of Warsaw during World War I (which was allowed by Germans; the Russians just relocated the Imperial University of Warsaw to Rostov-on-Don). He got his PhD already in the Second Republic of Poland. During 1927-34 Kuratowski was a professor of the important General Faculty of the Lvov Polytechnic School, where, among others, he supervised Stanisław Ulam. He belonged to the famous group of Polish mathematicians that met in the Scottish café in Lvov.

He returned to Warsaw in 1934 and was active as a professor and an organizer of scientific life until 1939. During the war Kuratowski took part in the activities of the clandestine university and clandestine education. After the war his activities were based in a different reality: the universities of Lvov and Vilnius were outside of Poland, which was in the sphere of influence of the Soviet Union. Kuratowski made a great contribution in rebuilding mathematical life in Poland. In particular,

2000 *Mathematics Subject Classification.* Primary 01A60; Secondary 54B20.
Key words and phrases. Kazimierz Kuratowski, hyperspace, history of mathematics.

he was the Editor-in-Chief of the "Fundamenta mathematicae", published the influential two-volume treatise "Topology".

More information about K.Kuratowski can be found in [12].

Several mathematical notions and results are named after Kuratowski, in particular, Kuratowski closure axioms, Kuratowski-Zorn lemma, Kuratowski's planar graph theorem, Tarski-Kuratowski algorithm, Kuratowski's 14 set theorem, Kuratowski convergence etc.

The present note is devoted to Kuratowski's contribution to the topology of hyperspaces; we focus on some modern applications of his results. The paper is partially based on the author's talk on the International conference dedicated to 120th anniversary of K. Kuratowski (Lviv, 2016).

2. Preliminaries

Let X be a topological space. By $\exp X$ we denote the set of all nonempty compact subsets in X. A base of the Vietoris topology in $\exp X$ is formed by the sets of the form

$$\langle U_1, \dots, U_n\rangle = \{A \in \exp X \mid A \subset \cup_{i=1}^n U_i,\ A \cap U_i \neq \emptyset,\ i = 1, \dots, n\},$$

where $U_1, \dots, U_n$ are open subsets in X. The obtained topological space $\exp X$ (sometimes also denoted by 2^X) is called the hyperspace of X or, more precisely, the hyperspace of compact subsets in X. Some other hyperspaces, e.g., the hyperspaces of closed, finite etc. subsets, are also considered.

If (X, d) is a metric space, the Vietoris topology on $\exp X$ is induced by the Hausdorff metric d_H:

$$d_H(A, B) = \inf\{r > 0 \mid A \subset O_r(B),\ B \subset O_r(A)\}.$$

Let $I = [0, 1]$. The Hilbert cube Q is the countable power of I, $Q = I^\omega$. A Q-manifold is a manifold modeled on the Hilbert cube Q, i.e., a metrizable space locally homeomorphic to Q. The pseudoboundary of the Hilbert cube is the set

$$B(Q) = \{(x_i) \mid x_j \in \{0, 1\},\ \text{for some } j \in \mathbb{N}\}.$$

The space $B(Q)$ is known to be homeomorphic to the linear hull Σ of the standard Hilbert cube in the Hilbert space ℓ^2. One can find more necessary information concerning the topology of infinite-dimensional manifolds in the books [3, 10, 28].

A metrizable space X is called an absolute neighborhood retract (ANR-space) if for any closed embedding of X into a metrizable space Y there is a neighborhood U of $i(X)$ such that $i(X)$ is a retract of U. See, e.g., [5, 18, 28] for backgrounds of the theory of retracts.

3. Kuratowski convergence

Given a sequence (A_n) of closed subsets in a metric space (X, d), its limit inferior (resp. limit superior) is defined as follows:

$$\mathrm{Li}_{n\to\infty} A_n = \{x \in X \mid \lim\sup_{n\to\infty} d(x, A_n) = 0\}$$

(resp.

$$\mathrm{Ls}_{n\to\infty} A_n = \{x \in X \mid \lim\inf_{n\to\infty} d(x, A_n) = 0\}).$$

If $A = \mathrm{Li}_{n\to\infty} A_n = \mathrm{Ls}_{n\to\infty} A_n$, then A is called the Kuratowski limit of (A_n). In [26], Kuratowski mentioned P. Painlevé who defined the limit inferior and limit superior (note that the term Painlevé-Kuratowski convergence is also used). Some simple properties of the the limit inferior and limit superior were established in [20]. In [26], some applications of the Kuratowski limit to the semi-continuous decompositions of topological spaces are given.

The Kuratowski convergence plays an important role in different areas of mathematics. It is compatible with the Fell topology on the hyperspaces of closed sets [13]. Also, in general, the Kuratowski convergence is weaker than convergence in Vietoris topology (Hausdorff metric). However, it is compactible with the Vietoris topology (Hausdorff metric) in the case of compact metric space.

Among the numerous applications of the Kuratowski convergence we mention [2]: here, the epi-convergence of functions is described in terms of Kuratowski convergence of the level sets.

An analogue of Painlevé-Kuratowski convergence for sequences of closed convex sets in a Banach space as well as sequences of lower semicontinuous convex functions (identified with their epigraphs) is introduced by Mosco [29]. The Mosco convergence is widely used in non-linear analysis.

4. Applications to the topology of infinite-dimensional manifolds

Bestvina and Mogilski [4] created a powerful method of absorbing sets in the Hilbert manifolds and Q-manifolds. This method found numerous applications for recognizing some classes of infinite-dimensional manifolds in the hyperspaces.

Let $C(X)$ denote the hyperspace of all nonempty subcontinua of a metric space X. Let $L(X)$ denote the subspace of C(X) consisting of all nonempty Peano continua (locally connected continua). Kuratowski [19] proved that the hyperspace $L(X)$ is an $F_{\sigma\delta}$-subset in X. The main result of [17] (announced in [16]) states that the hyperspace $L(\mathbb{R}^n)$, $n \geq 3$, is homeomorphic to Σ^ω.

It is also proved in [19] that the set of convergent sequences is $F_{\sigma\delta}$ in the hyperspace of compact sets. Note that [19] is cited in [9], where the hyperspaces of ANR-spaces are investigated from the point of view of infinite-dimensional topology.

It is shown in [20] that the hyperspace $a(M)$ of (simple) arcs is an $F_{\sigma\delta}$-subset in $C(X)$. R. Cauty [6] characterized the topology of this hyperspace as follows: $a(M)$ is homeomorphic to the space $M \times \Sigma^\omega$.

Let dim denote the covering dimension. Theorem IV.45.4 of [21] states that the hyperspace of compacta X with $\dim X \leq n$ (denoted $\dim_{\leq n}(2^X)$) is an absolute G_δ-set (equivalently: a G_δ-subset of 2^Q).

Using this result R. Cauty [8] described the topology of the sequences

$$\{\dim_{\geq n}(2^X)\}_{n=0}^\infty \text{ and } \{\dim_{\geq n+1}(C(X))\}_{n=0}^\infty,$$

for any Peano continuum X satisfying the property: every nonempty subset of X contains compacta of arbitrary dimension. Both the sequences are homeomorphic to the following sequence:

$$(\prod_{i=1}^{\infty} Q_i, B_1 \times \prod_{i=2}^{\infty} Q_i, \dots, \prod_{i=1}^{k} B_i \times \prod_{i=k+1}^{\infty} Q_i, \dots),$$

where $Q_i = Q$ and $B_i = B$ for every i. This result is a generalization of some results of H. Gladdines [14, 15]. Dobrowolski and Rubin [11] applied similar arguments to the case of cohomological dimension.

5. Miscellanea

Let $\mathrm{ANR}(Z)$ denote the hyperspace of all ANR-subspaces of a space Z. In [25], a complete metric on the hyperspace $\mathrm{ANR}(Z)$ is constructed for the case of finite-dimensional complete metric space Z. In the same paper, a complete metric is constructed on another hyperspaces, namely, $\mathrm{LC}^n(Z)$ of locally homotopically n-connected subsets of Z and $\mathrm{lc}^n(Z)$ of locally homologically n-connected subsets of Z.

Given a subset A of the segment $I = [0, 1]$, let $\mathcal{N}_A$ denote the subset in 2^I consisting of all closed neighborhoods of A. Banach and Kuratowski [1] showed that if A is an analytic subset of I neither dense nor Borelian, then $\mathcal{N}_A$ is a coanalytic space which is not Borelian. R. Cauty [7] proved a stronger result. Let $\mathcal{D}$ denote the set of all differentiable functions on I.

Theorem 5.1. *Let A be an analytic subset of I which is neither dense nor F_σ. Then $\mathcal{N}_A$ is homeomorphic to $\mathcal{D}$.*

In the paper [20], multivalued maps with closed point images are considered. It is proved, in particular, that the semi-continuous maps are of the 1st Baire class. As a consequence, the set of discontinuity points of a map is an F_σ-set of the 1st category.

Some results concern the notion of upper semi-continuous map. Recall that a map is upper semi-continuous if and only if its graph is closed.

(1) The union of two upper semi-continuous functions is upper semi-continuous.
(2) The inverse of a continuous map is upper semi-continuous.
(3) (Non-empty) intersection of sets an upper semi-continuous operation.

It is also proved in [20] that the boundary and the derivative operations are of the 2nd Baire class and, in general, are not of the 1st class.

A continuum X is called uni-coherent if for any decomposition $X = A \cup B$, where A, B are closed connected, the set $A \cap B$ is also connected. It is proved in [20] that the family of all uni-coherent continua is G_δ. Also, it is proved that the family of continua whose every subcontinuum is uni-coherent, is G_δ. The latter result allows for proving that the family of dendrites is $F_{\sigma\delta}$.

In the paper [23], monotone families in hyperspaces are considered. Some applications to the theory of connected spaces are given.

The paper [24] contains some selection theorems for the set-valued maps. As a special case of the main result, the following result of Kuratowski and Ryll-Nardzewski [27] can be obtained.

Theorem 5.2. *For each Polish space X there is a choice-function $2^X \to X$ of the first Baire class.*

The latter selection theorem has many applications in mathematical economics and optimal control.

References

1. Banach, S. et Kuratowski, C.: Sur la structure des ensembles linéaires *Studia Math.* **4** (1933) 95–99.
2. Beer, G., Rockafellar, R. T., Wets, R. J.-B.: A characterization of epi-convergence in terms of convergence of level sets *Proc. Amer. Math. Soc.* **116** (1992) 753–761.
3. Bessaga, C., Pelczynski, A.: *Selected Topics in Infinite-Dimensional Topology* Polish Scientific Publishers, Warsaw, 1975.
4. Bestvina, M., Mogilski, J.: Characterizing certain incomplete infinite-dimensional absolute retracts *Michigan Math. J.* **33** (1986) 291–313.
5. Borsuk, K. Theory of Retracts. Warszawa, Poland: PWN, 1967.
6. R. Cauty, L'espace des arcs d'une surface. [The space of arcs of a surface] *Trans. Amer. Math. Soc.* **332** (1992), no. 1, 193–209. (French. English summary)
7. Cauty, R.: Sur un exemple de Banach et Kuratowski *Fund. Math.* **144** (1994), 195–207.
8. Cauty, R.: Suites F_σ-absorbantes en théorie de la dimension *Fund. Math.* **159** (1999), 115–126.
9. Cauty, R., Dobrowolski, T., Gladdines, H. et van Mill, J.: Les hyperespaces des rétractes absolus et des rétractes absolus de voisinage du plan *Fund. math.* **148** (1995), 257–282.
10. Chapman, T.A.: *Lectures on Hilbert cube manifolds* CBMS Regional Conf. Series in Math. no. 28, Amer. Math. Soc, Providence, R.I (1976).
11. Dobrowolski, T., Rubin, L.: The hyperspaces of infinite-dimensional compacta for covering and cohomological dimension are homeomorphic *Pacific J. Math.* **164** (1994) 15–39.
12. Engelking, R.: Kazimierz Kuratowski (1896-1980). His life and work in topology, in: *C.E. Aull, R. Lowen (ed.), Handbook of the History of General Topology* **2** (1998) 431–452, Dordrecht.
13. Fell, J.: A Hausdorff topology for the closed subsets of a locally compact non-Hausdorff space *Proc. Amer. Math. Soc.* **13** (1962), 472-476.
14. Gladdines, H.: F_σ-absorbing sequences in hyperspaces of compact sets *Bull. Polish Acad. Sci.* **40** (1992) 175–184.
15. Gladdines, H.: F_σ-absorbing sequences in hyperspaces of subcontinua *Comment. Math. Univ. Carolin.* **34** (1993) 729745.
16. Gladdines, H., van Mill J.: Hyperspaces of locally connected continua of Euclidean spaces *Topology Proc.* **17** (1992) 343–349.
17. Gladdines, H., van Mill J.: Hyperspaces of Peano continua of Euclidean spaces *Fund. math.* **142** (1993) 173–188.
18. Hu, S. T.: *Theory of Retracts* Detroit, Wayne State University Press, 1965.
19. Kuratowski, C.: Evaluation de la classe borélienne ou projective d'un ensemble de points à l'aide des symboles logiques *Fund. Math.* **17** (1931) 249–272.
20. Kuratowski, C.: Les fonctions semi-continues dans l'espace des ensembles fermés *Fund. Math.* **18** 148-159 (1932).
21. Kuratowski, K.: *Topology, Volume II* Academic Press, New York, 1968.
22. Kuratowski, K.: Some remarks on the relation of classical set-valued mappings to the Baire classification *Colloq. Math.* **42** (1979) 273–277.
23. Kuratowski, C.: Sur les familles monotones d'ensembles fermés et leurs applications à la théorie des espaces connexes *Fund. Math.* **30** (1938) 17–33.
24. Kuratowski, K.: On the selector problems for the partitions of Polish spaces and for the compact-valued mappings *Ann. Polon. Math.* **29** (1974-1975) 421–427.
25. Kuratowski, K.: Sur une méthode de métrisation complète de certains espaces d'ensembles compacts *Fund. Math.* **43** (1956) 114–138.

26. Kuratowski, C.: Sur les décompositions semi-continues d'espaces métriques compacts *Fund. Math.* **1928** tom 11, nr 1, 169–185.
27. Kuratowski, K., Ryll-Nardzewski, C.: A general theorem on selectors *Bull. Acad. Polon. Sci. Sér. Sci. Math. Astronom. Phys.* **13** (1965) 397–403.
28. van Mill, J.: *Infinite-Dimensional Topology. Prerequisites and Introduction* North-Holland Math. Library, Vol.43, North-Holland, Amsterdam (1989).
29. Mosco, U.: Convergence of convex sets and of solutions of variational inequalities *Adv. in Math.* **3** (1969) 510–585.

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