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ON BIVARIATE COM-POISSON THOMAS DISTRIBUTION

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Abstract. In this research paper, bivariate COM-Poisson Thomas distribution is introduced. It is a generalization of the bivariate Thomas distribution. The characteristics of the distribution such as, joint probability generating function, joint probability mass function, mean, variance, covariance and correlation coefficient are derived. The simulation study is executed for computing the probabilities, expectation, variance, covariance and correlation coefficients. An illustrative example is presented using real-life data containing the number of dragonflies and damselflies in the wetlands.

1. Introduction

In 1949, Thomas introduced the compound Poisson distribution with compounding shifted Poisson distribution. It is called double Poisson distribution. Thomas applied this distribution to observed distributions of plants per quadrant and obtained marked improvements over fits with Poisson distributions. In 2011, Gamze Ozel introduced, bivariate versions of the Neyman type A, Neyman type B, geometric-Poisson, Thomas distributions and the usefulness of these distributions are illustrated in the analysis of earthquake data.

In 2019, Priyadharshini and Saavithri introduced, COM-Poisson Thomas distribution which is a compound COM-Poisson distribution with shifted-Poisson compounding distribution. It is used to analyse the traffic accident data.

In this research paper, bivariate COM-Poisson Thomas distribution is introduced and its properties are derived. The simulation study is carried out for computing the probabilities, mean, variance, covariance and correlation coefficient. Also, the real-life data is analysed.

This paper is organised as follow, In section 2, the univariate COM-Poisson Thomas model is given. In section 3, the bivariate COM-Poisson Thomas model is introduced. Properties of bivariate COM-Poisson Thomas distribution is given in section 4. In section 5, simulation study is given. In section 6, data analysis is given. Section 7, concludes this paper.

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2. Univariate COM-Poisson Thomas distribution

Suppose there are Y independent random variables of form X, and N denotes the sum of these random variables, i.e.,

$$N = X_1 + X_2 + \cdots + X_Y \tag{2.1}$$

Then, the COM-Poisson Thomas model [5] is derived by supposing that

(i) $X_i, i = 1, 2, ..., Y$ denotes the number of objects within a cluster and it follows Shifted-Poisson distribution.
(ii) Y denotes the number of clusters and it follows COM-Poisson distribution.
(iii) Then the random variable N, formed by compounding in this fashion, gives rise to the univariate COM-Poisson Thomas distribution.
(iv) Assume that $X_i, i = 1, 2, ..., Y$ are independent and identically distributed random variables and independent of Y

The probability generating function (PGF) of the random variable N is

$$G_N(s) = \frac{1}{Z(\lambda, \nu)} \sum_{j=0}^{\infty} \frac{[\lambda s e^{\phi(s-1)}]^j}{(j!)^\nu} \tag{2.2}$$

The probability mass function (PMF) of the random variable N is

$$P(N = n) = \frac{1}{Z(\lambda, \nu)} \sum_{j=1}^{n} \frac{(\lambda e^{-\phi})^j (j\phi)^{n-j}}{(j!)^\nu (n-j)!} \tag{2.3}$$

This is the PMF of univariate COM-Poisson Thomas distribution.

3. Bivariate compound COM-Poisson Thomas distribution (BVCPTD)

In this section, the bivariate COM-Poisson Thomas distribution is derived. It is a generalization of the bivariate Thomas distribution [2].
Let N be a COM-Poisson distribution with parameters $\lambda > 0$ and $\nu \geq 0$ and the pairs X_i, Y_i, $i = 1, 2, ...$ be mutually i.i.d. discrete random variables, independent of N. Then the bivariate compound COM-Poisson distribution is defined as

$$S_1 = \sum_{i=1}^{N} X_i, \quad S_2 = \sum_{i=1}^{N} Y_i \tag{3.1}$$

where the random variables X_i, Y_i, $i = 1, 2, ...$ be shifted Poisson random variables with parameters μ_1 and μ_2.

The probabilities of X and Y is

$$P(X = x) = \frac{e^{-\mu_1} \mu_1^{x-1}}{(x-1)!}, \quad x = 0, 1, 2, ...$$
$$P(Y = y) = \frac{e^{-\mu_2} \mu_2^{y-1}}{(y-1)!}, \quad y = 0, 1, 2, ...$$

The probability generating function of X and Y is

$$G_X(z_1) = z_1 e^{\mu_1(z_1-1)}, \quad |z_1| \leq 1 \tag{3.2}$$

$$G_Y(z_2) = z_2 e^{\mu_2(z_2-1)}, \quad |z_2| \leq 1 \tag{3.3}$$

The probability mass function of N [1] is

$$P(N = n) = \frac{\lambda^n}{(n!)^\nu} \frac{1}{Z(\lambda, \nu)} \tag{3.4}$$

where $Z(\lambda, \nu) = \sum_{j=0}^{\infty} \frac{\lambda^j}{(j!)^\nu}$, $\lambda > 0$ and $\nu \geq 0$
The probability generating function of N is

$$G_N(z) = \frac{Z(\lambda z, \nu)}{Z(\lambda, \nu)}, \quad |z| \leq 1 \tag{3.5}$$

The joint probability generating functions of S_1 and S_2 [3] is given by

$$\begin{aligned} G_{S_1,S_2}(z_1, z_2) &= G_N\big(G_X(z_1)G_Y(z_2)\big) \\ &= \frac{Z(\lambda G_X(z_1)G_Y(z_2), \nu)}{Z(\lambda, \nu)} \\ &= \frac{Z(\lambda z_1 z_2 e^{\mu_1(z_1-1)} e^{\mu_2(z_2-1)}, \nu)}{Z(\lambda, \nu)} \end{aligned}$$

$$G_{S_1,S_2}(z_1, z_2) = \frac{1}{Z(\lambda, \nu)} \sum_{j=0}^{\infty} \frac{\left(\lambda z_1 z_2 e^{\mu_1(z_1-1)+\mu_2(z_2-1)}\right)^j}{(j!)^\nu} \tag{3.6}$$

This is the joint probability generating function of bivariate COM-Poisson Thomas distribution. Now, expanding the joint probability generating function

$$\begin{aligned} \frac{1}{Z(\lambda, \nu)} \sum_{j=0}^{\infty} \frac{\left(\lambda z_1 z_2 e^{\mu_1(z_1-1)+\mu_2(z_2-1)}\right)^j}{(j!)^\nu} &= \frac{1}{Z(\lambda, \nu)} \Bigg[1 + \frac{\lambda z_1 z_2}{(1!)^\nu} e^{\mu_1(z_1-1)} e^{\mu_2(z_2-1)} + \frac{(\lambda z_1 z_2)^2}{(2!)^\nu} e^{2\mu_1(z_1-1)} e^{2\mu_2(z_2-1)} \\ &\quad + \frac{(\lambda z_1 z_2)^3}{(3!)^\nu} e^{3\mu_1(z_1-1)} e^{3\mu_2(z_2-1)} + \dots \Bigg] \\ &= \frac{1}{Z(\lambda, \nu)} \Bigg[1 + \frac{\lambda z_1 z_2 e^{-(\mu_1+\mu_2)}}{(1!)^\nu} \bigg[\Big(1 + \frac{\mu_1 z_1}{1!} + \frac{(\mu_1 z_1)^2}{2!} + ...\Big) \\ &\quad \Big(1 + \frac{\mu_2 z_2}{1!} + \frac{(\mu_2 z_2)^2}{2!} + ...\Big)\bigg] + \frac{(\lambda z_1 z_2)^2 e^{-2(\mu_1+\mu_2)}}{(2!)^\nu} \\ &\quad \bigg[\Big(1 + \frac{2\mu_1 z_1}{1!} + \frac{(2\mu_1 z_1)^2}{2!} + ...\Big)\Big(1 + \frac{2\mu_2 z_2}{1!} + \frac{(2\mu_2 z_2)^2}{2!} + ...\Big)\bigg] \\ &\quad + \frac{(\lambda z_1 z_2)^3 e^{-3(\mu_1+\mu_2)}}{(3!)^\nu} \bigg[\Big(1 + \frac{3\mu_1 z_1}{1!} + \frac{(3\mu_1 z_1)^2}{2!} + ...\Big) \\ &\quad \Big(1 + \frac{3\mu_2 z_2}{1!} + \frac{(3\mu_2 z_2)^2}{2!} + ...\Big)\bigg] + \dots \Bigg] \end{aligned}$$

Now collecting the coefficient of $z_1^0 z_2^0$, $z_1^1 z_2^1$,..., $z_1^{n_1} z_2^{n_2}$, we get the probability mass function of BVCPTD.

$$P(0,0) = \frac{1}{Z(\lambda,\nu)}$$
$$P(1,1) = \frac{1}{Z(\lambda,\nu)}\left[\frac{\lambda e^{-(\mu_1+\mu_2)}}{(1!)^\nu}\right]$$
$$P(2,2) = \frac{1}{Z(\lambda,\nu)}\sum_{j=1}^{2}\frac{(\lambda e^{-(\mu_1+\mu_2)})^j}{(j!)^\nu}\frac{(\mu_1)^{2-j}(\mu_2)^{2-j}}{(2-j)!(2-j)!}$$
$$P(3,3) = \frac{1}{Z(\lambda,\nu)}\sum_{j=1}^{3}\frac{(\lambda e^{-(\mu_1+\mu_2)})^j}{(j!)^\nu}\frac{(j\mu_1)^{3-j}(j\mu_2)^{3-j}}{(3-j)!(3-j)!}$$
$$P(4,4) = \frac{1}{Z(\lambda,\nu)}\sum_{j=1}^{4}\frac{(\lambda e^{-(\mu_1+\mu_2)})^j}{(j!)^\nu}\frac{(j\mu_1)^{4-j}(j\mu_2)^{4-j}}{(4-j)!(4-j)!}$$

In general,

$$P(n_1,n_2) = \frac{1}{Z(\lambda,\nu)}\sum_{j=0}^{n}\frac{\left(\lambda e^{-(\mu_1+\mu_2)}\right)^j}{(j!)^\nu}\frac{(j\mu_1)^{n_1-j}\,(j\mu_2)^{n_2-j}}{(n_1-j)!(n_2-j)!} \tag{3.7}$$

where $\lambda > 0$, $\mu_1 > 0$, $\mu_2 > 0$ and $\nu \geq 0$. This is the joint probability mass function of bivariate COM-Poisson Thomas distribution.

4. Properties

Differentiating the joint probability generating function given by Equation (3.6) with respect to z_1 and z_2 and setting $z_1 = z_2 = 1$, then the mean, variance, covariance and correlation coefficient of $(N_1 N_2)$ are obtained (See in Appendix)

$$E(S_1) = \frac{\lambda(1+\mu_1)Z_1(\lambda,\nu)}{Z(\lambda,\nu)} \tag{4.1}$$
$$E(S_2) = \frac{\lambda(1+\mu_2)Z_1(\lambda,\nu)}{Z(\lambda,\nu)} \tag{4.2}$$
$$E(S_1S_2) = \frac{\lambda(1+\mu_1)(1+\mu_2)}{Z(\lambda,\nu)}\Big(\lambda Z_2(\lambda,\nu) + Z_1(\lambda,\nu)\Big) \tag{4.3}$$

$$Var(S_1) = \frac{\lambda^2\,(1+\mu_1)^2}{Z(\lambda,\nu)}\left[Z_2(\lambda,\nu) - \frac{(Z_1(\lambda,\nu))^2}{Z(\lambda,\nu)}\right] + \frac{\lambda Z_1(\lambda,\nu)}{Z(\lambda,\nu)}\Big(\mu_1^2 + 3\mu_1 + 1\Big) \tag{4.4}$$
$$Var(S_2) = \frac{\lambda^2\,(1+\mu_2)^2}{Z(\lambda,\nu)}\left[Z_2(\lambda,\nu) - \frac{(Z_1(\lambda,\nu))^2}{Z(\lambda,\nu)}\right] + \frac{\lambda Z_1(\lambda,\nu)}{Z(\lambda,\nu)}\Big(\mu_2^2 + 3\mu_2 + 1\Big) \tag{4.5}$$

The covariance of S_1 and S_2 is given by

$$\mathrm{Cov}(S_1S_2) = E(S_1S_2) - E(S_1)E(S_2)$$

$$\mathrm{Cov}(S_1S_2) = \frac{\lambda(1+\mu_1)(1+\mu_2)}{Z(\lambda,\nu)}\left[\lambda Z_2(\lambda,\nu) + Z_1(\lambda,\nu) - \frac{\lambda(Z_1(\lambda,\nu))^2}{Z(\lambda,\nu)}\right] \quad (4.6)$$

Then, the coefficient of correlation for S_1 and S_2 is given by

$$\rho = Corr(S_1,S_2) = \frac{Cov(S_1,S_2)}{\sqrt{Var(S_1)Var(S_2)}}$$

$$= \frac{\frac{\lambda(1+\mu_1)(1+\mu_2)}{Z(\lambda,\nu)}\left[\lambda Z_2(\lambda,\nu) + Z_1(\lambda,\nu) - \frac{\lambda(Z_1(\lambda,\nu))^2}{Z(\lambda,\nu)}\right]}{\sqrt{\left[\frac{\lambda^2(1+\mu_1)^2}{Z(\lambda,\nu)}\left[Z_2(\lambda,\nu) - \frac{(Z_1(\lambda,\nu))^2}{Z(\lambda,\nu)}\right] + \frac{\lambda Z_1(\lambda,\nu)}{Z(\lambda,\nu)}\left(\mu_1^2 + 3\mu_1 + 1\right)\right]} \times \sqrt{\left[\frac{\lambda^2(1+\mu_2)^2}{Z(\lambda,\nu)}\left[Z_2(\lambda,\nu) - \frac{(Z_1(\lambda,\nu))^2}{Z(\lambda,\nu)}\right] + \frac{\lambda Z_1(\lambda,\nu)}{Z(\lambda,\nu)}\left(\mu_2^2 + 3\mu_2 + 1\right)\right]}}$$

5. Simulation Study

The simulation considered two sample sizes: n = 50 and n = 100, and various parameter settings.

Table.**1. Parameter settings for the simulation study**

Parameters	Setting
Sample size (n)	50, 100
μ_2	0.5
μ_1	0.4
λ	0.4, 0.2, 0.3, 0.25,0.5
ν	0.1, 0.3, 0.4, 0.2, 0.5

The mean and variance, covariance and correlation coefficient for different parameter settings are given in the following table.

Table**. 2. Simulation results for sample sizes 50 and 100.**

parameter settings $\lambda = 0.4$, $\nu = 0.1$, $\mu_1 = 0.5$, $\mu_1 = 0.4$							
	$E(N_1)$	$E(N_2)$	$Var(N_1)$	$Var(N_2)$	$E(N_1N_2)$	$Cov(N_1,N_2)$	$Corr(N_1,N_2)$
Theoretical value	0.9066	0.8462	2.3361	2.0135	2.6655	1.8983	0.8753
sample size 50	0.8091	0.7371	1.8061	1.3915	1.9582	1.3617	0.8590
sample size 100	0.8944	0.8356	2.2162	1.9219	2.5479	1.8007	0.8590
parameter settings $\lambda = 0.2$, $\nu = 0.3$, $\mu_1 = 0.5$, $\mu_1 = 0.4$							
Theoretical value	0.3416	0.3188	0.6955	0.5977	0.6517	0.5428	0.8419
sample size 50	0.3350	0.3118	0.6636	0.5576	0.6210	0.5164	0.8490
sample size 100	0.3394	0.3173	0.6790	0.5892	0.6509	0.5433	0.8589

parameter settings $\lambda = 0.3,\ \nu = 0.4,\ \mu_1 = 0.5,\ \mu_1 = 0.4$							
	$E(N_1)$	$E(N_2)$	$Var(N_1)$	$Var(N_2)$	$E(N_1N_2)$	$Cov(N_1,N_2)$	$Corr(N_1,N_2)$
Theoretical value	0.5280	0.4928	1.0987	0.9446	1.1214	0.8612	0.8454
sample size 50	0.5132	0.4748	1.0248	0.8438	1.0327	0.7890	0.8485
sample size 100	0.5237	0.4889	1.0682	0.9182	1.1180	0.8620	0.8701
parameter settings $\lambda = 0.25,\ \nu = 0.2,\ \mu_1 = 0.5,\ \mu_1 = 0.4$							
Theoretical value	0.4578	0.4273	0.9869	0.8489	0.9743	0.7787	0.8508
sample size 50	0.4439	0.4107	0.9138	0.7528	0.8901	0.7078	0.8534
sample size 100	0.4531	0.4231	0.9531	0.8201	0.9701	0.7785	0.8805
parameter settings $\lambda = 0.2,\ \nu = 0.3,\ \mu_1 = 0.5,\ \mu_1 = 0.4$							
Theoretical value	0.9257	0.8640	1.9927	1.7139	2.3717	1.5718	0.8505
sample size 50	0.8699	0.7979	1.7366	1.3720	2.0013	1.3071	0.8468
sample size 100	0.9199	0.8587	1.9477	1.6767	2.3459	1.5560	0.8610

From the above table, it is clear that for the sample size 50 and 100, the values of mean, variance, covariance and correlation coefficients are approximately the same for a different set of parameters.

The probability values of the bivariate COM-Poisson Thomas distribution is given in the following tables.

Table**. 3. Probability value for BCPT.**
parameter settings $\lambda = 0.4,\ \nu = 0.1,\ \mu_1 = 0.5,\ \mu_1 = 0.4$

$n_1 \backslash n_2$	0	1	2	3	4	5	6	7	8	9	10	**Total**
0	0.6132	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.6131
1	0.0000	0.0997	0.0399	0.0080	0.0011	0.0001	0.0000	0.0000	0.0000	0.0000	0.0000	0.1487
2	0.0000	0.0499	0.0351	0.0161	0.0054	0.0013	0.0003	0.0000	0.0000	0.0000	0.0000	0.1080
3	0.0000	0.0125	0.0201	0.0153	0.0076	0.0029	0.0009	0.0002	0.0001	0.0000	0.0000	0.0595
4	0.0000	0.0021	0.0084	0.0095	0.0067	0.0035	0.0015	0.0005	0.0002	0.0000	0.0000	0.0324
5	0.0000	0.0003	0.0026	0.0045	0.0044	0.0030	0.0016	0.0007	0.0003	0.0001	0.0000	0.0176
6	0.0000	0.0000	0.0006	0.0017	0.0023	0.0021	0.0014	0.0008	0.0004	0.0001	0.0001	0.0094
7	0.0000	0.0000	0.0001	0.0006	0.0010	0.0011	0.0010	0.0006	0.0004	0.0002	0.0001	0.0050
8	0.0000	0.0000	0.0000	0.0002	0.0004	0.0005	0.0006	0.0004	0.0003	0.0002	0.0001	0.0026
9	0.0000	0.0000	0.0000	0.0000	0.0001	0.0002	0.0003	0.0003	0.0002	0.0001	0.0001	0.0013
10	0.0000	0.0000	0.0000	0.0000	0.0000	0.0001	0.0001	0.0001	0.0001	0.0001	0.0001	0.0006
Total	0.0000	0.1644	0.1069	0.0559	0.0291	0.0150	0.0076	0.0038	0.0018	0.0009	0.0004	**0.9989**

Table**. 4. Probability value BCPT.**
parameter settings $\lambda = 0.4,\ \nu = 0.1,\ \mu_1 = 0.5,\ \mu_1 = 0.4$

$n_1 \backslash n_2$	0	1	2	3	4	5	6	7	8	9	10	**Total**
0	0.8078	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.80784
1	0.0000	0.0657	0.0263	0.0053	0.0007	0.0001	0.0000	0.0000	0.0000	0.0000	0.0000	0.09800
2	0.0000	0.0328	0.0175	0.0061	0.0017	0.0004	0.0001	0.0000	0.0000	0.0000	0.0000	0.05865
3	0.0000	0.0082	0.0076	0.0044	0.0018	0.0006	0.0001	0.0000	0.0000	0.0000	0.0000	0.02275
4	0.0000	0.0014	0.0027	0.0022	0.0012	0.0005	0.0002	0.0000	0.0000	0.0000	0.0000	0.00820
5	0.0000	0.0002	0.0008	0.0009	0.0006	0.0003	0.0001	0.0000	0.0000	0.0000	0.0000	0.00295
6	0.0000	0.0000	0.0002	0.0003	0.0003	0.0002	0.0001	0.0000	0.0000	0.0000	0.0000	0.00105
7	0.0000	0.0000	0.0000	0.0001	0.0001	0.0001	0.0000	0.0000	0.0000	0.0000	0.0000	0.00029
8	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.00000
9	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.00000
10	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.00000
Total	0.0000	0.1083	0.0551	0.0192	0.0064	0.0021	0.0006	0.0002	0.0000	0.0000	0.0000	**0.9997**

Table. **5. Probability value for BCPT.**
parameter settings $\lambda = 0.3$, $\nu = 0.4$, $\mu_1 = 0.5$, $\mu_1 = 0.4$

$n_1 \backslash n_2$	0	1	2	3	4	5	6	7	8	9	10	**Total**
0	0.7225	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.72250
1	0.0000	0.0881	0.0352	0.0070	0.0009	0.0001	0.0000	0.0000	0.0000	0.0000	0.0000	0.13147
2	0.0000	0.0441	0.0258	0.0100	0.0031	0.0007	0.0001	0.0000	0.0000	0.0000	0.0000	0.08386
3	0.0000	0.0110	0.0126	0.0080	0.0035	0.0012	0.0003	0.0001	0.0000	0.0000	0.0000	0.03667
4	0.0000	0.0018	0.0048	0.0044	0.0025	0.0011	0.0004	0.0001	0.0000	0.0000	0.0000	0.01517
5	0.0000	0.0002	0.0014	0.0018	0.0014	0.0008	0.0004	0.0001	0.0000	0.0000	0.0000	0.00616
6	0.0000	0.0000	0.0003	0.0006	0.0006	0.0004	0.0002	0.0001	0.0000	0.0000	0.0000	0.00240
7	0.0000	0.0000	0.0001	0.0002	0.0002	0.0002	0.0001	0.0001	0.0000	0.0000	0.0000	0.00092
8	0.0000	0.0000	0.0000	0.0000	0.0001	0.0001	0.0001	0.0000	0.0000	0.0000	0.0000	0.00027
9	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.00000
10	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.00000
Total	0.0000	0.1453	0.0802	0.0321	0.0124	0.0046	0.0017	0.0006	0.0000	0.0000	0.0000	**0.9994**

Table. **6. Probability value for BCPT.**
parameter settings $\lambda = 0.25$, $\nu = 0.2$, $\mu_1 = 0.5$, $\mu_1 = 0.4$

$n_1 \backslash n_2$	0	1	2	3	4	5	6	7	8	9	10	**Total**
0	0.7588	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.75881
1	0.0000	0.0771	0.0309	0.0062	0.0008	0.0001	0.0000	0.0000	0.0000	0.0000	0.0000	0.11506
2	0.0000	0.0386	0.0223	0.0085	0.0026	0.0006	0.0001	0.0000	0.0000	0.0000	0.0000	0.07272
3	0.0000	0.0096	0.0107	0.0068	0.0030	0.0010	0.0003	0.0001	0.0000	0.0000	0.0000	0.03140
4	0.0000	0.0016	0.0041	0.0037	0.0022	0.0010	0.0004	0.0001	0.0000	0.0000	0.0000	0.01294
5	0.0000	0.0002	0.0012	0.0016	0.0012	0.0007	0.0003	0.0001	0.0000	0.0000	0.0000	0.00530
6	0.0000	0.0000	0.0003	0.0005	0.0006	0.0004	0.0002	0.0001	0.0000	0.0000	0.0000	0.00211
7	0.0000	0.0000	0.0001	0.0002	0.0002	0.0002	0.0001	0.0001	0.0000	0.0000	0.0000	0.00083
8	0.0000	0.0000	0.0000	0.0000	0.0001	0.0001	0.0001	0.0000	0.0000	0.0000	0.0000	0.00022
9	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.00000
10	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.00000
Total	0.0000	0.1271	0.0694	0.0275	0.0106	0.0040	0.0015	0.0005	0.0000	0.0000	0.0000	**0.9994**

Table. **7. Probability value for BCPT.**
parameter settings $\lambda = 0.5$, $\nu = 0.5$, $\mu_1 = 0.5$, $\mu_1 = 0.4$

$n_1 \backslash n_2$	0	1	2	3	4	5	6	7	8	9	10	**Total**
0	0.5734	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.57335
1	0.0000	0.1166	0.0466	0.0093	0.0012	0.0001	0.0000	0.0000	0.0000	0.0000	0.0000	0.17387
2	0.0000	0.0583	0.0401	0.0181	0.0060	0.0015	0.0003	0.0000	0.0000	0.0000	0.0000	0.12417
3	0.0000	0.0146	0.0226	0.0165	0.0079	0.0029	0.0009	0.0002	0.0000	0.0000	0.0000	0.06549
4	0.0000	0.0024	0.0093	0.0098	0.0064	0.0032	0.0012	0.0004	0.0001	0.0000	0.0000	0.03304
5	0.0000	0.0003	0.0029	0.0045	0.0040	0.0025	0.0012	0.0005	0.0002	0.0001	0.0000	0.01611
6	0.0000	0.0000	0.0007	0.0017	0.0020	0.0015	0.0009	0.0005	0.0002	0.0001	0.0000	0.00754
7	0.0000	0.0000	0.0001	0.0005	0.0008	0.0008	0.0006	0.0003	0.0002	0.0001	0.0000	0.00344
8	0.0000	0.0000	0.0000	0.0001	0.0003	0.0004	0.0003	0.0002	0.0001	0.0001	0.0000	0.00149
9	0.0000	0.0000	0.0000	0.0000	0.0001	0.0001	0.0001	0.0001	0.0001	0.0000	0.0000	0.00061
10	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0001	0.0001	0.0000	0.0000	0.0000	0.00018
Total	0.0000	0.1921	0.1224	0.0606	0.0286	0.0130	0.0057	0.0023	0.0009	0.0004	0.0000	**0.9993**

6. Data Analysis

Model Formation

(i) Let Z denotes the number of wetlands in a particular place and it follows COM-Poisson distribution.
(ii) Let X_i, Y_i denotes the number of dragonflies and damselflies at the i^{th} wetland, for $i = 1, 2, 3, ..., Z$ and it follows Shifted-Poisson distribution.
(iii) Then N_1 and N_2 denote the total number of dragonflies and damselfly respectively.
(iv) Assume that X_i and Y_i, $i = 1, 2, ..., Z$ are separately independent and identically distributed random variables and independent of Z.

Numerical Illustration:

The data is taken from Tamil Nadu state ENVIS Hub, Department of Environment, Govt of Tamil Nadu, Chennai [4].

Table. 8

District	No.of Wetlands	No.of Dragonflies	No.of damselflies	District	No.of Wetlands	No.of Dragonflies	No.of damselflies
Ariyalur	3	6	4	Puthukottai	3	7	1
Chennai	5	6	4	Ramanathapuram	6	4	1
Coimbatore	6	7	3	Salem	3	11	6
Cuddalore	4	8	3	Sivagangai	4	9	3
Dharmapuri	3	5	1	Thanjavur	3	4	1
Dindigul	3	8	1	Theni	3	10	3
Erode	4	12	5	Thoothukudi	5	10	5
Kancheepuram	15	9	6	Tiruchirappalli	4	7	5
Kanniyakumari	6	8	6	Tirunelveli	6	10	3
Karur	3	3	0	Tiruppur	3	9	5
Krishnagiri	3	7	1	Tiruvallur	7	8	6
Madurai	4	6	1	Tiruvannamalai	8	11	7
Nagapattinam	3	6	1	Tiruvarur	5	5	4
Namakkal	3	6	0	Vellore	3	7	3
Nilgiris	2	6	1	Villupuram	5	11	5
Perambalur	3	5	2	Virudhunagar	3	7	1

The following table gives the estimated parameters, log-likelihood, AIC and BIC values for bivariate COM-Poisson Thomas and bivariate Thomas distributions.

Table 9

Distribution	Parameters	Log-Likelihood	AIC	BIC
BCPT	$\lambda = 31.9069$ $\mu_1 = 2.2428$ $\mu_2 = 0.2$ $\nu = 3.9571$	-55.6414	**119.2828**	**125.1457**
BT	$\lambda = 3.1206$ $\mu_1 = 1.2700$ $\mu_2 = 0.2000$	-59.9040	125.8081	130.2053

From the above table, it is clear that the bivariate COM-Poisson Thomas distribution is a better fit than the bivariate Thomas distribution.

The following table gives the mean, variance, covariance and correlation coefficient of bivariate COM-Poisson Thomas and bivariate Thomas distributions.

Table 10

Distribution	$E(N_1)$	$E(N_2)$	$E(N_1, N_2)$	$Var(N_1)$	$Var(N_2)$	$Cov(N_1, N_2)$	$Corr(N_1, N_2)$
BCPT	6.5089	2.4086	18.0590	10.9375	1.2827	2.3816	0.6358
BT	7.0838	3.7447	35.0272	20.0433	5.1178	8.5005	0.8393

The following figure shows estimated probabilities for bivariate COM-Poisson Thomas and bivariate Thomas distributions.

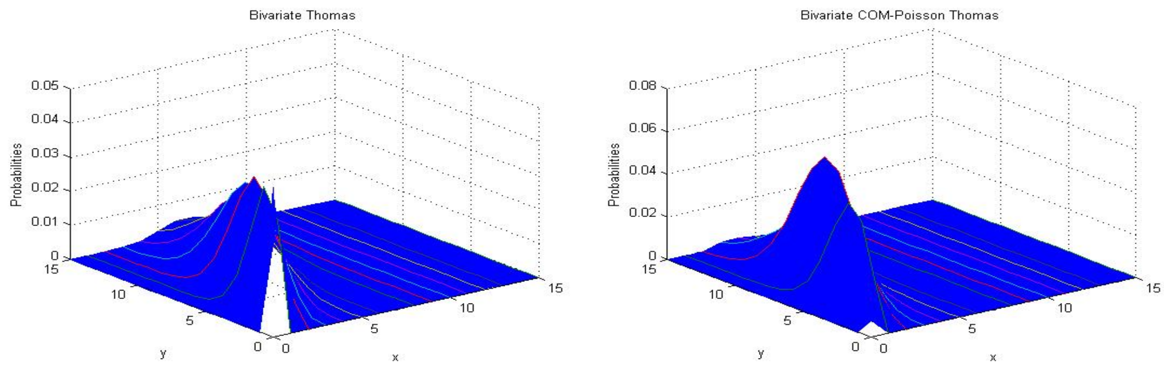

FIGURE 1

7. Conclusion

In this work, we have studied bivariate COM-Poisson Thomas distribution. The characteristics of the distribution such as, joint probability generating function, joint probability mass function, mean, variance, covariance and correlation coefficients are derived. The simulation study considered sample sizes 50 and 100 and five sets of parameters setting. It is observed that the values obtained match the theoretical values. An illustrative example is presented using real-life data containing the number of dragonflies and damselflies in the wetlands. It is proved that the bivariate COM-Poisson Thomas distribution is a better fit than the bivariate Thomas distribution.

Appendix

The probability generating function of bivariate COM-Poisson Thomas distribution is

$$G_{S_1,S_2}(z_1, z_2) = \frac{1}{Z(\lambda, \nu)} \sum_{j=0}^{\infty} \frac{\left(\lambda z_1 z_2 e^{\mu_1(z_1-1)+\mu_2(z_2-1)}\right)^j}{(j!)^\nu}$$

$$E(S_1) = \left.\frac{\partial G_{S_1,S_2}(z_1, z_2)}{\partial z_1}\right|_{z_1=z_2=1}$$

$$= \frac{1}{Z(\lambda,\nu)}\left[\sum_{j=0}^{\infty}\frac{j\Big(\lambda z_1 z_2 e^{\mu_1(z_1-1)+\mu_2(z_2-1)}\Big)^{j-1}}{(j!)^{\nu}}\right.$$
$$\left.\Big(\lambda z_2 e^{\mu_1(z_1-1)+\mu_2(z_2-1)} + \lambda z_1 z_2 e^{\mu_1(z_1-1)+\mu_2(z_2-1)}\mu_1\Big)\right]\Bigg|_{z_1=z_2=1}$$
$$= \frac{(\lambda+\lambda\mu_1)}{Z(\lambda,\nu)}\sum_{j=0}^{\infty}\frac{j\,(\lambda)^{j-1}}{(j!)^{\nu}}$$
$$E(S_1) = \frac{\lambda(1+\mu_1)Z_1(\lambda,\nu)}{Z(\lambda,\nu)}$$

Similarly

$$E(S_2) = \frac{\lambda(1+\mu_2)Z_1(\lambda,\nu)}{Z(\lambda,\nu)}$$

$$E(S_1S_2) = \frac{\partial^2 G_{S_1,S_2}(z_1,z_2)}{\partial z_1 \partial z_2}\Bigg|_{z_1=z_2=1}$$
$$= \frac{1}{Z(\lambda,\nu)}\left[\sum_{j=0}^{\infty}\frac{j(j-1)\Big(\lambda z_1 z_2 e^{\mu_1(z_1-1)+\mu_2(z_2-1)}\Big)^{j-2}}{(j!)^{\nu}}\right.$$
$$\Bigg(\Big(\lambda z_1 e^{\mu_1(z_1-1)+\mu_2(z_2-1)} + \lambda z_1 z_2 \mu_2 e^{\mu_1(z_1-1)+\mu_2(z_2-1)}\Big)\times$$
$$\Big(\lambda z_2 e^{\mu_1(z_1-1)+\mu_2(z_2-1)} + \lambda z_1 z_2 \mu_1 e^{\mu_1(z_1-1)+\mu_2(z_2-1)}\Big)\Bigg)$$
$$+\sum_{j=0}^{\infty}\frac{j\Big(\lambda z_1 z_2 e^{\mu_1(z_1-1)+\mu_2(z_2-1)}\Big)^{j-1}}{(j!)^{\nu}}\Big(\lambda e^{\mu_1(z_1-1)+\mu_2(z_2-1)}$$
$$\left.+\lambda z_2 \mu_2 e^{\mu_1(z_1-1)+\mu_2(z_2-1)} + \lambda z_1 \mu_1 e^{\mu_1(z_1-1)+\mu_2(z_2-1)} + \lambda z_1 z_2 \mu_1 \mu_2 e^{\mu_1(z_1-1)+\mu_2(z_2-1)}\Big)\right]\Bigg|_{z_1=z_2=1}$$
$$= \frac{\lambda^2 Z_2(\lambda,\nu)(1+\mu_1)(1+\mu_2)}{Z(\lambda,\nu)} + \frac{\lambda Z_1(\lambda,\nu)(1+\mu_1)(1+\mu_2)}{Z(\lambda,\nu)}$$
$$E(S_1S_2) = \frac{\lambda(1+\mu_1)(1+\mu_2)}{Z(\lambda,\nu)}\Big(\lambda Z_2(\lambda,\nu) + Z_1(\lambda,\nu)\Big) \tag{7.1}$$

$$Var(S_1) = G''_(N_1,N_2)(1,1) + G'_(N_1,N_2)(1,1) - [G'_(N_1,N_2)(1,1)]^2$$
$$= \frac{\lambda^2\,(1+\mu_1)^2}{Z(\lambda,\nu)}\left[Z_2(\lambda,\nu) - \frac{(Z_1(\lambda,\nu))^2}{Z(\lambda,\nu)}\right] + \frac{\lambda Z_1(\lambda,\nu)}{Z(\lambda,\nu)}\Big(\mu_1^2 + 3\mu_1 + 1\Big) \tag{7.2}$$

Similarly (7.3)

$$Var(S_2) = \frac{\lambda^2 \left(1+\mu_2\right)^2}{Z(\lambda,\nu)}\left[Z_2(\lambda,\nu) - \frac{(Z_1(\lambda,\nu))^2}{Z(\lambda,\nu)}\right] + \frac{\lambda Z_1(\lambda,\nu)}{Z(\lambda,\nu)}\left(\mu_2^2 + 3\mu_2 + 1\right) \tag{7.4}$$

The covariance of S_1 and S_2 is obtained by

$$\begin{aligned}\mathrm{Cov}(S_1S_2) &= E(S_1S_2) - E(S_1)E(S_2)\\ &= \frac{\lambda(1+\mu_1)(1+\mu_2)}{Z(\lambda,\nu)}\Big(\lambda Z_2(\lambda,\nu) + Z_1(\lambda,\nu)\Big) - \frac{\lambda^2(1+\mu_1)(1+\mu_2)\big(Z_1(\lambda,\nu)\big)^2}{\big(Z(\lambda,\nu)\big)^2}\end{aligned}$$

$$\mathrm{Cov}(S_1S_2) = \frac{\lambda(1+\mu_1)(1+\mu_2)}{Z(\lambda,\nu)}\left[\lambda Z_2(\lambda,\nu) + Z_1(\lambda,\nu) - \frac{\lambda(Z_1(\lambda,\nu))^2}{Z(\lambda,\nu)}\right] \tag{7.5}$$

Then, the coefficient of correlation for S_1 and S_2 is given by

$$\rho = Corr(S_1,S_2) = \frac{Cov(S_1,S_2)}{\sqrt{Var(S_1)Var(S_2)}}$$

$$= \frac{\dfrac{\lambda(1+\mu_1)(1+\mu_2)}{Z(\lambda,\nu)}\left[\lambda Z_2(\lambda,\nu) + Z_1(\lambda,\nu) - \dfrac{\lambda(Z_1(\lambda,\nu))^2}{Z(\lambda,\nu)}\right]}{\sqrt{\left[\dfrac{\lambda^2\left(1+\mu_1\right)^2}{Z(\lambda,\nu)}\left[Z_2(\lambda,\nu) - \dfrac{(Z_1(\lambda,\nu))^2}{Z(\lambda,\nu)}\right] + \dfrac{\lambda Z_1(\lambda,\nu)}{Z(\lambda,\nu)}\left(\mu_1^2+3\mu_1+1\right)\right]} \times \sqrt{\left[\dfrac{\lambda^2\left(1+\mu_2\right)^2}{Z(\lambda,\nu)}\left[Z_2(\lambda,\nu) - \dfrac{(Z_1(\lambda,\nu))^2}{Z(\lambda,\nu)}\right] + \dfrac{\lambda Z_1(\lambda,\nu)}{Z(\lambda,\nu)}\left(\mu_2^2+3\mu_2+1\right)\right]}}$$

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