Stochastic Modeling and Applications
Vol25 No. 1 (January-June, 2021)
ISSN: 0972-3641

Received: 07th April 2021 *Revised: 15th May 2021* *Selected: 10th July 2021*

RETRIAL QUEUE AND EXTENDED ORBIT SERVICES FROM OPTIONAL SERVICES TO RE-SERVICES BASED ON BERNOULLI'S VACATION

V. RAJAM AND S. UMA

Abstract. This paper observes and controls the evaluation of a batch arrival retrial queue with all possible normal/regular, optional and re-service (on demand basis) using Bernoulli's vacation time which includes a breakdown and crash time. The primary hypothesis is that the repair method does not begin without any delay after a breakdown and there's a crash time which calculates the waiting time to begin the queue after the repair process is over. In addition, it is focused that the clients who identified that the server is still busy with the exiting queue are creates an orbit who's functional method are adhered by FCFS method. Once the initial service is over then the client may depart or the same client may again demand to take the same service as re-service or still to be in a part of some other queue to get the related service. At the end of epoch of every service, if the server didn't find its clients in the queue, the server will look ahead to the subsequent patron to reach with possibility $1-a$ and chooses vacation with opportunity a. Consistent steady state conditions was identified and verified through supplementary variable method and compared with the existing methods.

1. Introduction

In the recent years, the retrial queueing services have been studied extensively, because of its applicability in machine learning, telecommunication networks to all kinds of networks [1]. Extensive surveys of retrial queues are evident in Artalejo (1999) [2] Gomez-Corral A (1999) [8] and Choudhury (2002) [4, 5]. A summary of the literature on retrial queues and its applications are discovered in Falin (1997) [6, 7], and Yang and Templeton (1987) [18].

Most of the latest researches were committed themselves to discuss about batch arrival, vacation, orbits, Bernoulli's retrial queue and so on. When we restrict our search on queueing theory with service and re-service provider, we could be seeing many researchers and their contributions [3]. Authors like Madan et al. (2004) [14, 13], Jeyakumar and Arumuganathan (2011) [9] had studied queueing system which provides the concepts of re-service.

M/G/1 retrial queue with server subject to breakdown are analyzed by Choudhury and Deka (2008) [5]. Wide and more relevant concepts of retrial queues are focused by Krishnakumar and Arivudainambi (2002) [12],

2000 *Mathematics Subject Classification.* Primary 60K25; Secondary 68M20, 90B22.
Key words and phrases. Re-service, Batch arrival, Bernoulli Vacation Retrial queue, optional re-service, Breakdowns.

Krishnakumar and Pavai Madheswari (2017) [15] and the queueing literature wherein the idea of popular retrial time has been taken into consideration in conjunction with the Bernoulli's vacation. On the other hand the retrial queueing system's server might not be aware what kind of service should provide the clients in the existing system [10, 11]. Hence it is literally vital to alter the Bernoulli's system schedule in the retrial context. This unique form of Bernoulli's vacation schedule is known as modified Bernoulli's extension schedule which is taken into consideration through Madan (2004), Pavai Madheswari, Krishnakumar and Suganthi (2017). Wang and Li (2008) [16] studied a comparable version with breakdown and restore the existence session without any delay. In this optionally available re-service case, a client has the selection of choosing any service like optional, regular service or re-service [17]. We expect clients arriving in batches can pick out any service which is presenting in the system like: normal, optional and re - service. Such conditions are ordinarily

The structure of this paper is prepared as follows; the outline of the mathematical structure, assumptions and related variables are given in Section 2. Section 3 focuses probability metrics of service (Normal and Optional), Re-service, Breakdown and repair time. Further it shows that whether the customer is in the queue or orbit, its assures the customers service. Optimal and consistent steady state equations are derived in the Section 4. Finally Section 5 concludes the paper.

2. Mathematical Structure

1. The specific description of the version is given as follows: We take into account of $M^X/G/1$ queueing system model, with an arrival of λ. Here X_i denotes the number of clients belonging to the i^{th} arrival batch wherein $X_i, i = 1, 2, 3, \ldots$ $P\left[X_i = n\right], n = 1, 2, 3, \ldots$ and $X(z)$ denotes the probability generating function of X.
2. If there is no waiting area and consequently if the customer/client arrives to take the service and send the request to the service regarding the service. If the server is busy in providing service to some other queues, then the new customers / clients will join the orbit immediately, since there is no waiting area is there for the customers. Now the clients in the orbit, then try and join any relevant queue to get their service. Arbitrary probability distribution $X(a)$ with the Laplace-Stieljie's transform (LST) $X^*(a)$. Then the retrial queue time to complete the service is $a(x)da = \frac{dX(a)}{l-X(a)}$.
3. A server presents normal/regular, optional and optional re - service to every function and the service related process. The service provider area is function under the concepts of FCFS (First Come First Serve). As quickly as the first server completes its work / service, then it allows the clients/customers from the orbit to get the services as they wants like regular/normal service, optional and optional reservice. Finally the client who chooses optionally available re - service will become a member of the orbit with chance r or might be leaves the queue with the probability $1-r$. The service time follows the random variables Y_1 and Y_2 with distribution characteristic $B_1(t)$ for regular service provider and

$Y_2(t)$ for optional service and LST $Y_1 * (a), Y_2 * (a)$. The conditional rate of service is arrived by

$$\mu_1(a)da = \frac{dY_1(a)}{l - Y_1(a)}, \quad \mu_2(a)da = \frac{dY_2(a)}{l - Y_2(a)}$$

4. At the end of service to every customer, the server may fit for a vacation of random period Z with probability $x(0 \leq x \leq 1)$ or may still wait in the queue to provide service with the probability $l - x$. The vacation time of the server Z has the characteristic function $Z(t)$ and LST is $Z * (x)$. The conditional rate of completion of server's vacation service is estimated by $\in (a)da = \frac{dZ(A)}{l-Z(a)}$.
5. The device may also breakdown at random, and the service channel may fail at any periodical time. The server's time instances are generated by the rate of changes as β_1 and β_2 for normal and optional service respectively.

Once the system is breakdown, then the waiting time of the customers and approximate service time are defined using probability density function. Once the system is breakdown, then the repair should start immediately and reduce the waiting time of the customers in the queue and customers in the orbit. The waiting time follows the distribution with density function which is defined on $Z_1(t)$ and $Z_2(t)$ for normal and optional service respectively and its corresponding LST is defined as $Z_1^*(b), Z_2^*(b)$ respectively. The repair time of the server is denoted by $S_1(t)$ for regular service, $S_2(t)$ for optional service and its corresponding LST is $S_1^*(b)$ and $S_2^*(b)$ respectively. The conditional time with respect to repair (in case of breakdown) on normal service, delayed service, optional service and optional reservice are estimated by

$$\zeta_1(b)db = \frac{dZ_1(b)}{l - Z_1(b)}, \qquad \zeta_2(b)db = \frac{dZ_2(b)}{l - Z_2(b)}$$
$$v_1(b)db = \frac{dS_1(b)}{l - S_1(b)}, \qquad v_2(b)db = \frac{dS_2(b)}{l - S_2(b)}$$

All randomized methods occupied a central position in the stochastic process. Now we introduce few additional notations as a way to be used in the mathematical system as $X^0(t), Y_1^0(t), Y_2^0(t), Z_1^0(t), Z_2^0(t), S_1^0(t), S_2^0(t), S^0(t)$ be the elapsed retrial time, normal service time, optional re - service time, delayed repair time (when the server is in breakdown), normal service during the repair time, optional service during the repair time, optional re-service during the repair time and server's vacation time respectively.

Theorem 2.1. *The embedded Markov chain is ergodic if $\{G/l \in T\}$ and most effective if $\rho < 1$ wherein*

$$\begin{aligned}\rho = E(A)(l - X^*(\lambda)) + \lambda E(A)[E(Y_1)[1 + \beta_1(E(Z_1) + E(S_1))]\\ + uE(Y_2)[1 + \beta_2(E(Z_2) + E(S_2))] + aE(Z)]\end{aligned}$$

Utilizing the above theorem, construct a steady state system in various phases along with the probability as :

$$P_0(t) = P\{D(t) = 0,\ A(t) = 0\}$$
$$P_{ln}(a,t)da = P\left\{D(t) = 0,\ A(t) = 1, a \leq X^0(t) < a + da\right\}, l \geq 1$$

$$M_{1,1}(a,t)da = P\left\{D(t)=1,\ A(t)=1, a \le Y_1^0(t) < a+da\right\}$$
$$\text{for } t \ge 0, a \ge 0, l \ge 0$$
$$M_{2,1}(a,t)da = P\left\{D(t)=2,\ A(t)=1, a \le Y_2^0(t) < a+da\right\}$$
$$\text{for } t \ge 0, a \ge 0, l \ge 0$$
$$G_{1,1}(a,b,t)db = P\left\{D(t)=3,\ A(t)=1,\ b \le Z_1^0(t) < b+db/Y_1^0(t)=a\right\}$$
$$G_{2,1}(a,b,t)db = P\left\{D(t)=4,\ A(t)=1,\ b \le Z_2^0(t) < b+db/Y_2^0(t)=a\right\}$$
$$R_{t,1}(a,b,t)db = P\left\{D(t)=5,\ A(t)=1,\ b \le S_1^0(t) < b+db/Y_1^0(t)=a\right\}$$
$$\text{for } t \ge 0, (a,b) \ge 0, l \ge 0$$
$$R_{2,1}(a,b,t)db = P\left\{D(t)=6,\ A(t)=1,\ b \le S_2^0(t) < b+db/Y_2^0(t)=a\right\}$$
$$\text{for } t \ge 0, (a,b) \ge 0, l \ge 0$$
$$\phi l(a,t)da = P\left\{D(t)=7,\ A(t)=1, a \le Z^0(t) < a+da\right\}, l \ge 0$$

3. Probability Metrics of Service (Normal and Optional), Re-service, Breakdown and repair time

The following chances are utilized in the subsequent sections.

$p_o(t)$ is the probability of getting chance that the system is empty.

$P_1(a,t)$ is the chance that there are precisely l clients in the orbit with the elapsed retrial time of the customers who doesn't stay in the queue.

$M_{1,1}(a,t)$ is the probability of getting chance that there are precisely l clients in the orbit with the elapsed service time and the customer who is getting services at present is denoted by a.

$M_{2,1}(a,t)$ is the probability of getting chance that there are precisely l clients in the orbit with the elapsed re-service time and the customer who is getting services at present is denoted by a.

$G_{1,1}(a,b,t)$ is the probability of getting chance that there are precisely l clients in the orbit with the elapsed service time during the breakdown of the server and the customer who is getting services at present is denoted by a.

$G_{2,1}(a,b,t)$ is the probability of getting chance that there are precisely l clients in the orbit with the elapsed optional re-service time during the breakdown of the server and the customer who is getting services at present is denoted by a.

$R_{t,1}(a,b,t)$ is the probability of getting chance that there are precisely l clients in the orbit with the elapsed service time during the repair time of the server and the customer who is getting services at present is denoted by a.

$R_{2,1}(a,b,t)$ is the probability of getting chance that there are precisely l clients in the orbit with the elapsed re-service time during the repair time of the server and the customer who is getting services at present is denoted by a.

$\phi_1(a,t)$ is the probability of getting chance that there are precisely l clients in the orbit with the elapsed vacation time and the customer who is getting services at present is denoted by a.

In general, the delay is denoted by b and time is denoted as t. The following stability conditions satisfied the conditional probabilities of :

$$P_0 = \lim_{l\to\infty} P_0(t) \text{ for } t \ge 0, Pl(a) = \lim_{l\to\infty} P_l\ (a,t) \text{ for } t \ge 0, a \ge 0, l \ge 1$$

$$
\begin{aligned}
M_{1,l}(a) &= \lim_{l\to\infty} M_{1,l}\ (a,t) \text{ for } t \geq 0, a \geq 0, l \geq 0 \\
M_{2,l}(a) &= \lim_{l\to\infty} M_{2,l}\ (a,t) \text{ for } t \geq 0, a \geq 0, l \geq 0 \\
G_{1,l}(a,b) &= \lim_{l\to\infty} G_{1,l}\ (a,t) \text{ for } t \geq 0, \\
G_{2,l}(a,b) &= \lim_{l\to\infty} G_{2,l}\ (a,t) \text{ for } t \geq 0 \\
R_{1,l}(a,b) &= \lim_{l\to\infty} R_{1,l}\ (a,t) \text{ for } t \geq 0, \\
R_{2,l}(a,b) &= \lim_{l\to\infty} R_{2,}\ (a,t) \text{ for } t \geq 0, \phi_l(a) = \lim_{l\to\infty} \phi_l\ (a,t) \text{ for } t \geq 0
\end{aligned}
$$

exists.

4. Steady State Equations to control the queueing system

Based on supplementary variable techniques, we obtained the following system of equations that govern the existing dynamics of the system and controls its behavior under various steady state conditions and stability conditions as:

$$
\lambda P_0 = (l - x)\left[u - \int_0^\infty M_{1,0}\,(a)\,\mu_1\,(a)\,da + \int_0^\infty M_{2,0}\,(a)\,\mu_2\,(a)\,da\right] + \int_0^\infty \Phi_0(a) \in (a) da \tag{4.1}
$$

$$
\frac{dP_l(a)}{da} + (\lambda + x(a))P,(a) = 0, l \geq 1 \tag{4.2}
$$

$$
\frac{dM_{1,0}(a)}{da} + [\lambda + \beta_1 + \mu_1(a)]M_{1,0}(a) = \int_0^\infty \gamma_1(b) S_{1,0}(a,b) db \tag{4.3}
$$

$$
\frac{dM_{1,l}(a)}{da} + [\lambda + \beta_1 + \mu_1\,(a)]\,M_{1,l}\,(a) = \lambda \sum_{\theta=1}^{n} D_\theta M_{1,l-\theta}\,(a) + \int_0^\infty \gamma\,(b)\,S_{1,l}\,(a,b)\,db,\ \ l \geq 1 \tag{4.4}
$$

$$
\frac{dG_{1,0}(a,b)}{db} + (\lambda + \zeta_1(b))G_{1,0}(a,b) = 0 \tag{4.5}
$$

$$
\frac{dG_{1,l}(a,b)}{db} + (\lambda + \zeta_1(b))G_{1,l}(a,b) = \lambda \sum_{\theta=1}^{n} D_\theta G_{1,l-\theta}(a,b), l \geq 1 \tag{4.6}
$$

$$
\frac{dG_{2,0}(a,b)}{db} + (\lambda + \zeta_2(b))G_{2,0}(a,b) = 0, l = 0 \tag{4.7}
$$

$$
\frac{dG_{2,l}(a,b)}{db} + (\lambda + \zeta_2\,(b))\,G_{2,l}(a,b) = \lambda \sum_{\theta=1}^{n} D_\theta G_{2,l-\theta}(a,b), l \geq 1 \tag{4.8}
$$

$$
\frac{dR_{1,0}(a,b)}{db} + (\lambda + \xi_1(b))R_{1,0}(a,b) = 0, l = 0 \tag{4.9}
$$

$$
\frac{dR_{1,1}(a,b)}{db} + (\lambda + \xi_1(b))\,R_{t,1}(a,b) = \lambda \sum_{\theta=1}^{n} D_\theta R_{1,l-\theta}(a,b), l \geq 1 \tag{4.10}
$$

$$\frac{dR_{2,0}(a,b)}{db} + (\lambda + \xi_2(b))\, R_{2,0}(a,b) = 0 \tag{4.11}$$

$$\frac{dR_{2,1}(a,b)}{db} + (\lambda + \xi_2(b))\, R_{2,1}(a,b) = \lambda \sum_{\theta=1}^{n} D_\theta R_{2,l-\theta}(a,b), 1 \geq 1 \tag{4.12}$$

$$\frac{d\Phi_0(a)}{da} + (\lambda + u(a))\Phi_0(a) = 0, l = 0 \tag{4.13}$$

$$\frac{d\Phi_0(a)}{da} + (\lambda + \varepsilon(a))\Phi_0(a) = 0, l = 0 \tag{4.14}$$

$$\frac{d\Phi_1(a)}{da} + (\lambda + \varepsilon(a))\Phi_1(a) = \lambda \sum_{\theta=1}^{n} D_\theta \Phi_{l-\theta}(a), l \geq 1 \tag{4.15}$$

The steady state boundary conditions are

$$\begin{aligned} P(0,c) = \int_0^\infty \mathit{\Phi}(a,c)\mathcal{E}(a)da + (1-x)\Bigg[\bar{u}\int_0^\infty M_1(a,c)\mu_1(a)da \\ + (1-x)\int_0^\infty M_2(a,c)\mu_2(a)da - \lambda P_0\Bigg], l \geq 1 \end{aligned} \tag{4.16}$$

$$M_{1,1}(0) = \int_0^\infty P_{1+1}(a)x(a)da + \lambda \sum_{\theta=1}^{n} D_\theta \int_0^\infty P_{1-(\theta-1)}(a)da + \lambda D_{1+1}P_0, l \geq 1 \tag{4.17}$$

$$M_{2,1}(0) = u\int_0^\infty M_{1,1}(a)\mu_1(a)da, l \geq 0 \tag{4.18}$$

$$G_{1,1}(a,0) = \beta_1 M_{1,1}(a), l \geq 0 \tag{4.19}$$

$$G_{2,1}(a,0) = \beta_2 M_{2,1}(a), l \geq 0 \tag{4.20}$$

$$R_1(a,0,c) = \int_0^\infty M_1(a,b,c)\varepsilon_1(b)db, l \geq 0 \tag{4.21}$$

$$R_2(a,0,c) = \int_0^\infty M_2(a,b,c)\varepsilon_2(b)db, l \geq 0 \tag{4.22}$$

$$\phi_1(0) = \overline{u}\int_0^\infty M_{1,0}(a)\mu_1(a)da + \int_0^\infty M_{2,0}(a)\mu_2(a)da, l = 0 \tag{4.23}$$

$$\phi_n(0,c) = xu\int_0^\infty M_{1,1}(a,c)\mu_1(a)da + x\int_0^\infty M_{2,1}(a,c)\mu_2(a)da, l \geq 1 \tag{4.24}$$

The normalizing condition is

$$P_0 + \sum_{=1}^{\infty} \int_0^{\infty} P,(a)\,da + \sum_{=0}^{\infty} \left[\begin{array}{l} \int_0^{\infty} M_{1,l}(a)\,da + \int_0^{\infty} M_{2,l}(a)\,da \\ + \int_0^{\infty} \phi_l(a)\,da + \int_0^{\infty}\int_0^{\infty} G_{1,l}(a,b)\,dadb \\ + \int_0^{\infty}\int_0^{\infty} G_{2,l}(a,b)\,dadb \\ + \int_0^{\infty}\int_0^{\infty} R_{1,l}(a,b)\,dadb \\ + \int_0^{\infty}\int_0^{\infty} R_{2,l}(a,b)\,dadb \end{array} \right] = 1 \tag{4.25}$$

$$P(a,\ c) = De^{-\int(1+x(a))da} \tag{4.26}$$

$$P(a,\ c) = P(a,\ c)\,e^{-\lambda a}(1 - X(a)) \tag{4.27}$$

The above values calculated, and Table 1 shows the computational probability values of λ, ρ and averages of M, G, R and ϕ.

λ	ρ	Mean M	Mean G	Mean R	Mean ϕ	P_0
4	1.6	0.00952148	0.519043	0.156776	0.65677	0.115678
4.5	1.8	0.00569019	0.51138	0.144483	0.64443	0.114448
5	2	0.0036	0.5072	0.134293	0.634293	0.113429
5.5	2.2	0.00238434	0.504769	0.125596	0.625596	0.11256
6	2.4	0.00163966	0.503279	0.118033	0.618033	0.111803
6.5	2.6	0.0011635	0.502327	0.11137	0.61137	0.111137
7	2.8	0.000847861	0.501696	0.105442	0.605442	0.110544
7.5	3	0.000632099	0.501264	0.100126	0.600126	0.110013
8	3.2	0.000480652	0.500961	0.0953296	0.59533	0.109533
8.5	3.4	0.000371868	0.500744	0.0909767	0.591	0.109098
9	3.6	0.00029213	0.500584	0.0870073	0.587	0.108701

TABLE 1. Computational probability values of λ, ρ and averages of M, G, R and ϕ

5. Conclusion

This paper discussed about retrial queue and batch arrival (customers) system with various variable sizes which dealt with breakdown and repair. In addition, service time, vacation time, delay time, repair time and breakdown time have calculated through general distribution. The probability generating functions of number of customers in the orbit for different states revealed optimal probability and consistent steady state conditions. Further, we focused that the clients themselves creates an orbit and gets the service before they depart or the same client may again demand to take the same service as re-service or still to be in a part of some other queue to get the related service. At the end of epoch of every service, if the server didn't find its clients in the queue, the server will look ahead to the subsequent patron to reach with probability $1 - a$ and chooses vacation with opportunity a. Consistent steady state conditions was identified and verified through supplementary variable method and compared with the existing methods.

References

[1] Ivo JBF Adan, Bernardo D'Auria, and Offer Kella, *Special volume on 'recent developments in queueing theory'of the third ecqt conference*, 2019, pp. 1–2.

[2] JR Artalejo and Antonio Gómez-Corral, *Retrial queueing systems: A computational approach*, Springer Science & Business Media, 2008.

[3] Monita Baruah, Kailash C Madan, and Tillal Eldabi, *Balking and re-service in a vacation queue with batch arrival and two types of heterogeneous service*, Journal of Mathematics Research **4** (2012), no. 4, 114.

[4] Gautam Choudhury, *A batch arrival queue with a vacation time under single vacation policy*, Computers & Operations Research **29** (2002), no. 14, 1941–1955.

[5] Gautam Choudhury and Kandarpa Deka, *An M/G/1 retrial queueing system with two phases of service subject to the server breakdown and repair*, Performance Evaluation **65** (2008), no. 10, 714–724.

[6] Gennadi Falin and James GC Templeton, *Retrial queues*, vol. 75, CRC Press, 1997.

[7] Gennadij Falin, *A survey of retrial queues*, Queueing systems **7** (1990), no. 2, 127–167.

[8] Antonio Gómez-Corral, *Stochastic analysis of a single server retrial queue with general retrial times*, Naval Research Logistics (NRL) **46** (1999), no. 5, 561–581.

[9] S Jeyakumar and R Arumuganathan, *A non-markovian bulk queue with multiple vacations and control policy on request for re-service*, Quality Technology & Quantitative Management **8** (2011), no. 3, 253–269.

[10] J Keilson and LD Servi, *Oscillating random walk models for GI/G/1 vacation systems with bernoulli schedules*, Journal of Applied probability **23** (1986), no. 3, 790–802.

[11] Vidyadhar G. Kulkarni and Bong Dae Choi, *Retrial queues with server subject to breakdowns and repairs*, Queueing systems **7** (1990), no. 2, 191–208.

[12] B Krishna Kumar and D Arivudainambi, *The M/G/1 retrial queue with bernoulli schedules and general retrial times*, Computers & Mathematics with Applications **43** (2002), no. 1-2, 15–30.

[13] Kailash C Madan and Ayman Baklizi, *An M/G/1 queue with additional second stage service and optional re-service*, International journal of information and Management sciences **13** (2002), no. 4, 13–32.

[14] Kailash C Madan and Gautam Choudhury, *Steady state analysis of an MX/(G1, G2)/1 queue with restricted admissibility of arriving batches and modified bernoulli schedule server vacations*, Journal of Probability and Statistical Science **2** (2005), 167–185.

[15] S Pavai Madheswari, B Krishna Kumar, and Poomalai Suganthi, *Analysis of M/G/1 retrial queues with second optional service and customer balking under two types of bernoulli vacation schedule*, RAIRO-Operations Research **53** (2019), no. 2, 415–443.

[16] Jinting Wang, *An M/G/1 queue with second optional service and server breakdowns*, Computers & Mathematics with Applications **47** (2004), no. 10-11, 1713–1723.

[17] Rachel Wilf-Miron, Ilya Novikov, Arnona Ziv, Avishai Mandelbaum, Yaacov Ritov, and Osnat Luxenburg, *A novel methodology to measure waiting times for community-based specialist care in a public healthcare system*, Health Policy **124** (2020), no. 8, 805–811.

[18] Tao Yang and James G. C. Templeton, *A survey on retrial queues*, Queueing systems **2** (1987), no. 3, 201–233.

V. Rajam: Assistant Professor of Mathematics, Rajah Serfoji Govt. College, Thanjavur -613 005, Bharathidasan University, Tiruchirappalli, Tamil Nadu, India.
Email address: rajamramv@gmail.com

S. Uma: Associate Professor of Mathematics, D. G. Govt. Arts College for Women, Mayiladuthurai-01, Bharathidasan University, Tiruchirappalli, Tamil Nadu, India.
Email address: umamaths@gmail.com