Stochastic Modeling and Applications
Vol25 No. 1 (January-June, 2021)
ISSN: 0972-3641

Received: 02nd April 2021 *Revised: 10th May 2021* *Selected: 07th July 2021*

ON BLOCK CIRCULANT POLYNOMIAL MATRICES

R. MUTHAMILSELVAM AND G. RAMESH

ABSTRACT. The characterization of block circulant polynomial matrices are derived as a generalization of the block circulant matrices.

1. Introduction

Let $(a_1(\alpha), a_2(\alpha), ...a_n(\alpha))$ be an ordered n-tuple of polynomial complex numbers and let them generate the circulant polynomial matrix [2] [3] [5] [7] of order n:

$$A(\alpha) = \begin{pmatrix} a_1(\alpha) & a_2(\alpha) & \dots & a_n(\alpha) \\ a_n(\alpha) & a_1(\alpha) & \dots & a_2(\alpha) \\ \dots & \dots & \dots & \dots \\ a_2(\alpha) & a_3(\alpha) & \dots & a_1(\alpha) \end{pmatrix} \tag{1.1}$$

We shall often denote this circulant polynomial matrix as

$$A(\alpha) = Circ(a_1(\alpha), a_2(\alpha), ..., a_n(\alpha)) \tag{1.2}$$

It is well known that all circulant polynomial matrices of order n are simultaneously diagonalizable by the polynomial matrix $F(\alpha)$ associated with the finite Fourier transform.

Specifically, let

$$\omega(\alpha) = exp(\frac{2\pi i}{n}(\alpha)), i = \sqrt{-1} \tag{1.3}$$

and set

$$F^*(\alpha) = n^{(\frac{-1}{2})} \begin{pmatrix} 1 & 1 & 1 & \dots & 1 \\ 1 & \omega(\alpha) & \omega^2(\alpha) & \dots & \omega^{n-1}(\alpha) \\ 1 & \omega^2(\alpha) & \omega^4(\alpha) & \dots & \omega^{2(n-1)}(\alpha) \\ \dots & \dots & \dots & \dots & \\ 1 & \omega^{(n-1)}(\alpha) & \omega^{(n-2)}(\alpha) & \dots & \omega(\alpha) \end{pmatrix} \tag{1.4}$$

The Fourier polynomial matrix $F(\alpha)$ depends only on n. This matrix is also symmetric polynomial and unitary polynomial $F(\alpha)F^*(\alpha) = F^*(\alpha)F(\alpha) = I(\alpha)$ and we have

$$A(\alpha) = F^*(\alpha) \wedge (\alpha) F(\alpha) \tag{1.5}$$

2000 *Mathematics Subject Classification.* 15A09, 15A15,15A57.

Key words and phrases. Circulant polynomial matrices, Block Circulant polynomial matrices.

* This research is supported by G. Ramesh.

where $\wedge(\alpha) = diag(\alpha_1, \alpha_2, \dots \alpha_n)$

The symbol * designates the conjugate transpose.

From the spectral mapping theorem, we may represent $A(\alpha)$ in the form

$$A(\alpha) = a_1(\alpha) + a_2(\alpha)\pi(\alpha) + a_3(\alpha)\pi^2(\alpha) + \cdots + a_n(\alpha)\pi^{n-1}(\alpha) \tag{1.6}$$

where $\pi(\alpha)$ is the permutation matrix $circ(0, 1, 0, 0, \dots)$.

Also, let $A(\alpha)$ be an $n \times n$ polynomial matrix. Then $A(\alpha)$ is a circulant polynomial matrix if and only if

$$A(\alpha)\pi(\alpha) = \pi(\alpha)A(\alpha) \tag{1.7}$$

The matrix $\pi(\alpha) = circ(0, 1, 0, \dots, 0)$

This paper is devoted to the study of block circulant polynomial matrices.

2. Block Circulant Polynomial Matrices

In this section we define block circulant polynomial matrices and we extend some of the properties of block circulant matrices found in [1], [4], [6], [8], [9] to block circulant polynomial matrices.

Definition 2.1. A block circulant polynomial matrix is a polynomial matrix in the following form

$$\text{b circ}(A_1(\alpha), A_2(\alpha), \dots, A_n(\alpha)) = \begin{pmatrix} A_1(\alpha) & A_2(\alpha) & \dots & A_m(\alpha) \\ A_m(\alpha) & A_1(\alpha) & \dots & A_{m-1}(\alpha) \\ \dots & \dots & \dots & \dots \\ A_2(\alpha) & A_3(\alpha) & \dots & A_1(\alpha) \end{pmatrix}$$

We denote the set of all block circulant polynomial matrices of order $m \times n$ as $\mathbb{BC}_{m,n}(\alpha)$.

Example 2.2. The polynomial matrix

$$\begin{pmatrix} 1-\alpha^2 & \alpha^3 & 2+\alpha^2 & -11\alpha \\ \alpha+3\alpha^2 & 1+\alpha & 4+6\alpha^2 & -8+\alpha \\ 2+\alpha^2 & -11\alpha & 1-\alpha^2 & \alpha^3 \\ 4+6\alpha^2 & -8+\alpha & \alpha+3\alpha^2 & 1+\alpha \end{pmatrix}$$

is a block circulant polynomial matrix.

Theorem 2.3. *$A(\alpha) \in \mathbb{BC}_{m,n}(\alpha)$ iff $A(\alpha)$ commutes with the unitary polynomial matrix*
$\pi_m(\alpha) \otimes I_n(\alpha) : A(\alpha)(\pi_m(\alpha) \otimes I_n(\alpha)) = (\pi_m(\alpha) \otimes I_n(\alpha))A(\alpha)$

Proof. Assume that $A(\alpha)$ is a block circulant polynomial matrix. That is □

$$\text{b circ}(A_1(\alpha), A_2(\alpha), \dots A_n(\alpha)) = \begin{pmatrix} A_1(\alpha) & A_2(\alpha) & \dots & A_m(\alpha) \\ A_m(\alpha) & A_1(\alpha) & \dots & A_{m-1}(\alpha) \\ \dots & \dots & \dots & \dots \\ A_2(\alpha) & A_3(\alpha) & \dots & A_1(\alpha) \end{pmatrix}$$

We have to prove that $A(\alpha)(\pi_m(\alpha) \otimes I_n(\alpha)) = (\pi_m(\alpha) \otimes I_n(\alpha))A(\alpha)$.
Now the polynomial matrix $\pi_m(\alpha) \otimes I_n(\alpha) \in \mathbb{BC}_{m,n}(\alpha)$ is given by

$$\pi_m(\alpha) \otimes I_n(\alpha) = \begin{pmatrix} O_n(\alpha) & I_n(\alpha) & O_n(\alpha) & \dots & O_n(\alpha) \\ O_n(\alpha) & O_n(\alpha) & I_n(\alpha) & \dots & O_n(\alpha) \\ \dots & \dots & \dots & \dots & \dots \\ O_n(\alpha) & O_n(\alpha) & O_n(\alpha) & \dots & I_n(\alpha) \\ I_n(\alpha) & O_n(\alpha) & O_n(\alpha) & \dots & O_n(\alpha) \end{pmatrix}$$

$$A(\alpha)(\pi_m(\alpha) \otimes I_n(\alpha)) = \begin{pmatrix} A_m(\alpha) & A_1(\alpha) & A_2(\alpha) & \dots & A_{m-1}(\alpha) \\ A_{m-1}(\alpha) & A_m(\alpha) & A_1(\alpha) & \dots & A_{m-2}(\alpha) \\ \dots & \dots & \dots & \dots & \dots \\ A_1(\alpha) & A_2(\alpha) & A_3(\alpha) & \dots & A_m(\alpha) \end{pmatrix} \tag{2.1}$$

$$(\pi_m(\alpha) \otimes I_n(\alpha))A(\alpha) = \begin{pmatrix} A_m(\alpha) & A_1(\alpha) & A_2(\alpha) & \dots & A_{m-1}(\alpha) \\ A_{m-1}(\alpha) & A_m(\alpha) & A_1(\alpha) & \dots & A_{m-2}(\alpha) \\ \dots & \dots & \dots & \dots & \dots \\ A_1(\alpha) & A_2(\alpha) & A_3(\alpha) & \dots & A_m(\alpha) \end{pmatrix} \tag{2.2}$$

From (8) and (9), we get $A(\alpha)(\pi_m(\alpha) \otimes I_n(\alpha)) = (\pi_m(\alpha) \otimes I_n(\alpha))A(\alpha)$.

Conversely, assume that $A(\alpha)(\pi_m(\alpha) \otimes I_n(\alpha)) = (\pi_m(\alpha) \otimes I_n(\alpha))A(\alpha)$.

We have to prove that $A(\alpha)$ is a block circulant polynomial matrix.

$$I_m(\alpha) \otimes A_1(\alpha) = \begin{pmatrix} A_1(\alpha) & 0 & \dots & 0 \\ 0 & A_1(\alpha) & \dots & 0 \\ \dots & \dots & \dots & \\ 0 & 0 & \dots & A_1(\alpha) \end{pmatrix}$$

$$\pi_m(\alpha) \otimes A_2(\alpha) = \begin{pmatrix} 0 & A_2(\alpha) & 0 & \dots & 0 \\ 0 & 0 & A_2(\alpha) & \dots & 0 \\ \dots & \dots & \dots & \dots & \\ A_2(\alpha) & 0 & 0 & \dots & 0 \end{pmatrix}$$

$$\pi_m^2(\alpha) \otimes A_3(\alpha) = \begin{pmatrix} 0 & 0 & A_3(\alpha) & 0 & \dots & 0 \\ 0 & 0 & 0 & A_3(\alpha) & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & A_3(\alpha) & 0 & 0 & \dots & 0 \end{pmatrix}$$

etc

$(I_m(\alpha) \otimes A_1(\alpha) + (\pi_m(\alpha) \otimes A_2(\alpha)) + \dots + (\pi_m^{m-1}(\alpha) \otimes A_m(\alpha)) = \text{b circ}(A_1(\alpha), A_2(\alpha), \dots, A_m(\alpha).$ ∎

Hence, $A(\alpha)$ is a block circulant polynomial matrix.

Theorem 2.4. *b circ*$(A_1(\alpha), A_2(\alpha), \dots, A_n(\alpha)) = \sum_{k=0}^{m-1} [\pi_m^k(\alpha) \otimes A_{K+1}(\alpha)]$.

Proof. Given that $A(\alpha) = \text{b circ}(A_1(\alpha), A_2(\alpha), \dots, A_n(\alpha))$ is a block circulant polynomial matrix.

That is,

$$A(\alpha) = \begin{pmatrix} A_1(\alpha) & A_2(\alpha) & \dots & A_n(\alpha) \\ A_n(\alpha) & A_1(\alpha) & \dots & A_{n-1}(\alpha) \\ \dots & \dots & \dots & \dots \\ A_2(\alpha) & A_3(\alpha) & \dots & A_1(\alpha) \end{pmatrix}$$

Now

$$I_m(\alpha) \otimes A_1(\alpha) = \begin{pmatrix} A_1(\alpha) & 0 & \dots & 0 \\ 0 & A_1(\alpha) & \dots & 0 \\ \dots & \dots & \dots & \\ 0 & 0 & \dots & A_1(\alpha) \end{pmatrix}$$

$$\pi_m(\alpha) \otimes A_2(\alpha) = \begin{pmatrix} 0 & A_2(\alpha) & 0 & \dots & 0 \\ 0 & 0 & A_2(\alpha) & \dots & 0 \\ \dots & \dots & \dots & \dots & \\ A_2(\alpha) & 0 & 0 & \dots & 0 \end{pmatrix}$$

$$\pi_m^{m-1}(\alpha) \otimes A_3(\alpha) = \begin{pmatrix} 0 & 0 & A_3(\alpha) & 0 & \dots & 0 \\ 0 & 0 & 0 & A_3(\alpha) & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots & \\ 0 & A_3(\alpha) & 0 & 0 & \dots & 0 \end{pmatrix}$$

Since the pre direct of any $n \times n$ polynomial matrix by $\pi_m(\alpha)$ shifts the columns of the matrix one place to the right. Therefore, we find that

$$\pi_{m-1}^2(\alpha) \otimes A_n(\alpha) = \begin{pmatrix} 0 & 0 & 0 & \dots & 0 & A_n(\alpha) \\ A_n(\alpha) & 0 & 0 & \dots & 0 & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & \dots & A_n(\alpha) & 0 \end{pmatrix}$$

$$\text{b circ}(A_1(\alpha), A_2(\alpha), \dots, A_n(\alpha) = \sum_{k=0}^{m-1} [\pi_m^k(\alpha) \otimes A_{K+1}(\alpha)].$$ □

Remark 2.5. Block circulant polynomial matrix of the same type do not necessarily commute.

Example 2.6.

$$\begin{pmatrix} A(\alpha) & O(\alpha) \\ O(\alpha) & A(\alpha) \end{pmatrix} \begin{pmatrix} B(\alpha) & O(\alpha) \\ O(\alpha) & B(\alpha) \end{pmatrix} = \begin{pmatrix} A(\alpha)B(\alpha) & O(\alpha) \\ O(\alpha) & A(\alpha)B(\alpha) \end{pmatrix}$$

$$\begin{pmatrix} B(\alpha) & O(\alpha) \\ O(\alpha) & B(\alpha) \end{pmatrix} \begin{pmatrix} A(\alpha) & O(\alpha) \\ O(\alpha) & A(\alpha) \end{pmatrix} = \begin{pmatrix} B(\alpha)A(\alpha) & O(\alpha) \\ O(\alpha) & B(\alpha)A(\alpha) \end{pmatrix}$$

Theorem 2.7. *Let* $A(\alpha) = b\ circ(A_1(\alpha), A_2(\alpha), \dots, A_m(\alpha))$*,*
$B(\alpha) =\ b\ circ(B_1(\alpha), B_2(\alpha), \dots, B_m(\alpha)) \in \mathbb{BC}_{m\times n}(\alpha)$*.*

Then, if the $A_j(\alpha)^{'}s$ *commutes with the* $B_K(\alpha)^{'}s$*,* $A(\alpha)$ *and* $B(\alpha)$ *commute.*

Proof. By theorem (2.4), we have

$$A(\alpha) = \sum_{j=0}^{m-1} [\pi^j(\alpha) \otimes A_{j+1}(\alpha)],\ B(\alpha) = \sum_{k=0}^{m-1} [\pi^k(\alpha) \otimes B_{k+1}(\alpha)]$$

$$\begin{aligned}
A(\alpha)B(\alpha) &= [\sum_{j=0}^{m-1}[\pi^{j}(\alpha)\otimes A_{j+1}(\alpha)]][\sum_{k=0}^{m-1}[\pi^{k}(\alpha)\otimes B_{k+1}(\alpha)]] \\
&= \sum_{j=0}^{m-1}\sum_{k=0}^{m-1}[\pi^{j+k}(\alpha)\otimes A_{j+1}(\alpha)B_{k+1}(\alpha)] \\
&= \sum_{k=0}^{m-1}\sum_{j=0}^{m-1}[\pi^{k+j}(\alpha)\otimes B_{k+1}(\alpha)A_{j+1}(\alpha)] \\
&= [\sum_{k=0}^{m-1}[\pi^{k}(\alpha)\otimes B_{k+1}(\alpha)]][\sum_{j=0}^{m-1}[\pi^{j}(\alpha)\otimes A_{j+1}(\alpha)]] \\
&= B(\alpha)A(\alpha)
\end{aligned}$$

□

Theorem 2.8. *$A(\alpha)\in \mathbb{BC}_{m\times n}(\alpha)$ if and only if it is of the form*
$A(\alpha)=[F_m(\alpha)\otimes F_n(\alpha)^*]diag[M_1(\alpha),M_2(\alpha),\dots,M_n(\alpha)][F_m(\alpha)\otimes F_n(\alpha)]$
where the $M_k(\alpha)$ are arbitrary polynomial square matrices of order n.

Proof. Assume that $A(\alpha)$ is a block circulant polynomial matrix.
From theorem (2.4), we have

$A(\alpha)=\text{b circ}(A_1(\alpha),A_2(\alpha),\dots,A_m(\alpha))=\sum_{k=0}^{m-1}[\pi_m^k(\alpha)\otimes A_{k+1}(\alpha)]$ for some $A_k(\alpha)$.

Now $\pi_m^k(\alpha)\otimes A_{k+1}(\alpha)=[F_m^*(\alpha)\Omega^k(\alpha)F_m(\alpha)]\otimes[F_n^*(\alpha)(F_n(\alpha)A_{k+1}(\alpha)F_n^*(\alpha)F_n(\alpha))]$
Let $B_K(\alpha)=(F_n(\alpha)A_{k+1}(\alpha)F_n^*(\alpha)$
$\pi_m^k(\alpha)\otimes A_{k+1}(\alpha)=[F_m^*(\alpha)\Omega^k(\alpha)F_m(\alpha)]\otimes[F_n^*(\alpha)B_K(\alpha)F_n(\alpha))]$
$=(F_m^*(\alpha)\otimes F_n^*(\alpha))(\Omega^k(\alpha)\otimes B_K(\alpha))(F_m(\alpha)\otimes F_n(\alpha))$
$\sum_{k=0}^{m-1}\pi_m^k(\alpha)\otimes A_{k+1}(\alpha)=\sum_{k=0}^{m-1}(F_m(\alpha)\otimes F_n(\alpha))^*(\Omega^k(\alpha)\otimes B_K(\alpha))(F_m(\alpha)\otimes F_n(\alpha))$
$A(\alpha)=(F_m(\alpha)\otimes F_n(\alpha))^*\sum_{k=0}^{m-1}(\Omega^k(\alpha)\otimes B_K(\alpha))(F_m(\alpha)\otimes F_n(\alpha))$
$=(F_m(\alpha)\otimes F_n(\alpha))^*diag(M_1(\alpha),M_2(\alpha),\dots,M_n(\alpha))(F_m(\alpha)\otimes F_n(\alpha))$
where

$$\begin{pmatrix} M_1(\alpha)\\ M_2(\alpha)\\ \dots\\ M_m(\alpha)\end{pmatrix}=(m^{\frac{1}{2}}F_m^*(\alpha)\otimes I_m(\alpha))\begin{pmatrix} B_0(\alpha)\\ B_1(\alpha)\\ \dots\\ B_{(m-1)}(\alpha)\end{pmatrix} \tag{2.3}$$

Thus, $A(\alpha)=(F_m(\alpha)\otimes F_n(\alpha))^*diag(M_1(\alpha),M_2(\alpha),\dots,M_n(\alpha))(F_m(\alpha)\otimes F_n(\alpha))$
From (10),

$$\begin{pmatrix} B_0(\alpha)\\ B_1(\alpha)\\ \dots\\ B_{(m-1)}(\alpha)\end{pmatrix}=(m^{\frac{-1}{2}}F_m^*(\alpha)\otimes I_m(\alpha))\begin{pmatrix} M_1(\alpha)\\ M_2(\alpha)\\ \dots\\ M_m(\alpha)\end{pmatrix}$$

Since $A_{(k+1)}(\alpha)=F_n^*(\alpha)B_k(\alpha)F_n(\alpha)$
$M_k(\alpha)$ arbitrary $\Leftrightarrow B_k(\alpha)$ are arbitrary.
$\Leftrightarrow A_k(\alpha)$ are arbitrary.
Hence, $A(\alpha)\in\mathbb{BC}_{(m,n)}(\alpha)$.

□

3. Conclusion

some of the characterization of block circulant polynomial matrices are discussed here. Further we can study the circulant block polynomial matrices.

References

1. Ahlberg,J.H.:— Block Circulant of level K.Division of applied Mathematics, *Brown University.*, Providence, R.I, 1976.
2. Ahlberg,J.H.: Doubly Periodic Initial value problems. Lecture notes, 1997.
3. Chao, Chong -Yun: On a Type of Circulant: *Linear Algebra and its Appl.* **6**, (1973) 241–248.
4. Chao, Chong -Yun: A note on Block Circulant Matrices: *Kyungpook,Math. J.,* **14**, (1974).
5. Davis.P.J: Circulant Matrices, New Yark, 1979.
6. Horn,R.A., and C.R.Johnson.: Matrix Analysis, *Cambridge University press*, Newyork, 2013.
7. Ramesh. G., and Muthamilselvam.R: A study on g-circulant polynomial matrices: *The International Journal of analytical and experimental modal analysis,* **11 (12)**, (2019), 3284–3297.
8. Smith, R.L.Moore-Penrose: Inverse of Block Circulant and Block K-Circulant Matrices: *Linear Algebra and its Appl,* **16**, (1997), 237–245.
9. Trapp. G.E: Inverses of Circulant Matrices and Block Circulant Matrices: *Kyungpook Math.J.,* **13** (1973), PP.11–20.

R. MUTHAMILSELVAM :Research Scholar,Department of Mathematics,Government Arts College (Autonomous), Kumbakonam., Tamil Nadu,India. Affiliated to Bharathidasan University, Trichirappalli - 620024.,

E-mail address: mtamilmaths@gmail.com

G. RAMESH:Associate Professor and Head, Department of Mathematics, Government Arts College (Autonomous), Kumbakonam., Tamil Nadu,India. Affiliated to Bharathidasan University, Trichirappalli - 620024.

E-mail address: grameshgac@gmail.com