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CERTAIAN FRACTIONAL INTEGRAL AND INTEGRAL TRANSFORM FORMULAS ASSOCIATED WITH THE PRODUCT OF GENERAL CLASS OF POLYNOMIAL, MULTIVARIABLE MITTAGE-LEFFLER FUNCTION AND GENERALIZED LAURICELLA'S HYPERGEOMETRIC FUNCTIONS

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Abstract: In this paper, first we establish two theorems by applying the Marichev-Saigo-Maeda fractional integral operators involving the product of general class of polynomial, multivariable Mittage-Leffler function and generalized Lauricella's hypergeometric function. Some interesting special cases of these two theorems are also given for Saigo, Erdelyi-Kober, Riemann-Liouville and Weyl type fractional integral operators. Next we establish certain composition formulas by employing some integral transform like Beta transform, Laplace transform and Verma transform on the result of these two theorems.

Keywords and phrases: Generalized fractional integral operators, extended beta functions, generalized Lauricella's hypergeometric function, Srivastava's polynomial, multivariable Mittage–Leffler function, Beta transform, Laplace transform, and Verma transform.

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1. Introduction and Preliminaries

In 1903, Mittage-Leffler [8] introduced the function $E_\alpha(y)$ in the following manner:

$$E_\alpha(y)=\sum_{k=0}^{\infty}\frac{y^k}{\Gamma(\alpha k+1)} \quad \text{where } \alpha, y\in\mathbb{C}, \Re(\alpha)>0 \tag{1}$$

In 1905, Wiman [20] generalized the function $E_\alpha(y)$ and gave the function $E_{\alpha,\beta}(y)$ in the following manner:

$$E_{\alpha,\beta}(y)=\sum_{k=0}^{\infty}\frac{y^k}{\Gamma(\alpha k+\beta)} \quad \text{where } \alpha,\beta, y\in\mathbb{C}, \Re(\alpha)>0, \Re(\beta)>0 \tag{2}$$

In 1971, Prabhakar [11] generalized the function $E_{\alpha,\beta}(y)$ and gave the function $E^\lambda_{\alpha,\beta}(y)$ in the following manner:

$$E^\lambda_{\alpha,\beta}(y)=\sum_{k=0}^{\infty}\frac{(\lambda)_k}{\Gamma(\alpha k+\beta)}\frac{y^k}{k!} \tag{3}$$

where $\alpha,\beta,\lambda, y\in C, \Re(\alpha)>0, \Re(\beta)>0, \Re(\lambda)>0$

Saxena et al. [15] gave the multivariable analogue of multivariable Mittage-Leffler function in the following manner:

$$E_{\mu_j,c}^{\lambda_j}(y_1,\ldots,y_l)=E_{\mu_1,\ldots,\mu_l,c}^{\lambda_1,\ldots,\lambda_l}(y_1,\ldots,y_l)=\sum_{k_1,\ldots,k_l=0}^{\infty}\frac{(\lambda_1)_{k_1},\ldots,(\lambda_r)_{k_r}}{\Gamma(c+\sum_{j=1}^{l}\mu_j k_j)}\frac{y_1^{k_1}}{k_1!}\cdots\frac{y_l^{k_l}}{k_l!} \tag{4}$$

where $c,\lambda_j,\mu_j \in \square, \Re(\mu_j)>0, \forall j=1,2\cdots l$

For the sake of convenience, let $F_{\mu_j,\nu}^{\lambda_j}=\dfrac{(\lambda_1)_{k_1},\ldots,(\lambda_l)_{k_l}}{\Gamma(\nu+\sum_{j=1}^{l}\mu_j k_j)}$ (5)

Srivastava [18] introduced the general class of polynomials in the following manner:

$$S_N^M(y)=\sum_{k=0}^{[N/M]}\frac{(-N)_{Mk}}{k!}A_{N,k}y^k \quad N=0,1,2,\cdots \tag{6}$$

where M is an arbitrary positive integer and the coefficients $A_{N,k}\,(N,k\geq 0)$ are arbitrary constants, real or complex and $(\lambda)_N$ is the pochhammer symbol.

Recently Cetinkaya et al. [1] defined the generalized beta function in the following manner:

$$B_{p_1,p_2}^{(\gamma,\delta,\tau,\nu)}(x,y)=\int_0^1 t^{x-1}(1-t)^{y-1}\,{}_1F_1\left(\gamma;\delta;-\frac{p_1}{t^\tau}-\frac{p_2}{(1-t)^\nu}\right)dt \tag{7}$$

where $\min\{\Re(p_1),\Re(p_2)\}\geq 0,\ \min\{\Re(x),\Re(y)\}>0,\ \min\{\Re(\gamma),\Re(\delta),\Re(\tau),\Re(\nu)\}>0$

when $\tau=\nu=1$, the generalized beta function in equation (7) reduces to the following generalized beta function defined by Goswami et al. [4]

$$B_{p_1,p_2}^{(\gamma,\delta)}(x,y)=\int_0^1 t^{x-1}(1-t)^{y-1}\,{}_1F_1\left(\gamma;\delta;-\frac{p_1}{t}-\frac{p_2}{(1-t)}\right)dt \tag{8}$$

where $\min\{\Re(p_1),\Re(p_2)\}\geq 0,\ \min\{\Re(x),\Re(y)\}>0,\ \min\{\Re(\gamma),\Re(\delta)\}>0$

when $\tau=\nu=1$ and $p_1=p_2$, the generalized beta function in equation (7) reduces to the following generalized beta function defined by Ozergin et al. [9]

$$B_{p_1}^{(\gamma,\delta)}(x,y)=\int_0^1 t^{x-1}(1-t)^{y-1}\,{}_1F_1\left(\gamma;\delta;-\frac{p_1}{t(1-t)}\right)dt \tag{9}$$

where $\Re(p_1)\geq 0,\ \min\{\Re(x),\Re(y)\}>0,\ \min\{\Re(\gamma),\Re(\delta)\}>0$

when $\gamma=\delta=\tau=\nu=1,$ the generalized beta function in equation (7) reduces to the following generalized beta function defined by J. Choi et al. [3]

$$B_{p_1,p_2}(x,y)=\int_0^1 t^{x-1}(1-t)^{y-1}\exp\left(-\frac{p_1}{t}-\frac{p_2}{1-t}\right)dt \tag{10}$$

where $\min\{\Re(p_1),\Re(p_2)\}\geq 0,\ \min\{\Re(x),\Re(y)\}>0$

when $\gamma=\delta=\tau=v=1$ and $p_1=p_2$, the generalized beta function in equation (7) reduces to the following generalized beta function defined by M.A. Chaudhry et al. [2]

$$B_{p_1}(x,y)=\int_0^1 t^{x-1}(1-t)^{y-1}\exp\left(-\frac{p_1}{t(1-t)}\right)dt \tag{11}$$

where $\Re(p_1)\geq 0,\ \min\{\Re(x),\Re(y)\}>0$

when $p_1=p_2=0$ the generalized beta function in equation (7) reduces to the following classical beta function [19]

$$B(x,y)=\int_0^1 t^{x-1}(1-t)^{y-1}dt \tag{12}$$

where $\min\{\Re(x),\Re(y)\}>0$

Recently Cetinkaya et al. [1] defined the generalized Lauricella's hypergeometric function F_D^3 in the following manner:

$$F_D^{3(\gamma,\delta,\tau,v,p_1,p_2)}(a,a_1,a_2,a_3;b;y_1,y_2,y_3;n_1,n_2)=$$

$$\sum_{m_1,m_2,m_3=0}^{\infty}\frac{(a)_{m_1+m_2+m_3}(a_1)_{m_1}(a_2)_{m_2}(a_3)_{m_3}}{(b)_{m_1+m_2+m_3}}\frac{B_{p_1,p_2}^{(\gamma,\delta,\tau,v)}(a-n_1+m_1+m_2+m_3,b-a+n_2)}{B(a-n_1+m_1+m_2+m_3,b-a+n_2)}\frac{y_1^{m_1}}{m_1!}\frac{y_2^{m_2}}{m_2!}\frac{y_3^{m_3}}{m_3!} \tag{13}$$

where $|y_1|<1, |y_2|<1, |y_3|<1$

2. Generalized Fractional Integral Operators

The generalized fractional integral operators (called the Marichev-Saigo-Maeda operators) including the Saigo operators and involving the Appell's function $F_3(.)$ of the third kind as the kernel, introduced by Marichev (see [7], [13]) are defined in the following manner:

Definition: Let $\alpha_1,\alpha_2,\beta_1,\beta_2,\eta\in\mathbb{C}$ and $x>0$, then for $\Re(\eta)>0$ we have (see [4], [10])

$$(I_{0,x}^{\alpha_1,\alpha_2,\beta_1,\beta_2,\eta}f)(x)=\frac{x^{-\alpha_1}}{\Gamma(\eta)}\int_0^x(x-t)^{\eta-1}t^{-\alpha_2}F_3\left(\alpha_1,\alpha_2,\beta_1,\beta_2;\eta;1-\frac{t}{x},1-\frac{x}{t}\right)f(t)dt \tag{14}$$

and

$$(I_{x,\infty}^{\alpha_1,\alpha_2,\beta_1,\beta_2,\eta}f)(x)=\frac{x^{-\alpha_2}}{\Gamma(\eta)}\int_x^{\infty}(x-t)^{\eta-1}t^{-\alpha_1}F_3\left(\alpha_1,\alpha_2,\beta_1,\beta_2;\eta;1-\frac{x}{t},1-\frac{t}{x}\right)f(t)dt \tag{15}$$

If we put $\alpha_2=\beta_2=0,\ \beta_1=-\eta,\ \alpha_1=\alpha_1+\beta_1$ and $\eta=\alpha_1,$ then the above equations (14) and (15) reduce to the following Saigo fractional integral operators (see [12])

$$(I_{0,x}^{\alpha_1,\beta_1,\eta}f)(x)=\frac{x^{-\alpha_1-\beta_1}}{\Gamma(\alpha_1)}\int_0^{x}(x-t)^{\alpha_1-1}\,{}_2F_1\left(\alpha_1+\beta_1,-\eta;\alpha_1;1-\frac{t}{x}\right)f(t)dt \tag{16}$$

and

$$(I_{x,\infty}^{\alpha_1,\beta_1,\eta}f)(x)=\frac{1}{\Gamma(\alpha_1)}\int_x^{\infty}(t-x)^{\alpha_1-1}t^{-\alpha_1-\beta_1}\,{}_2F_1\left(\alpha_1+\beta_1,-\eta;\alpha_1;1-\frac{x}{t}\right)f(t)dt \tag{17}$$

If we put $\beta_1=0,$ then the Saigo fractional integral operators given in equations (16) and (17) reduce to the following Erdelyi-Kober fractional integral operators (see [5])

$$(E_{0,x}^{\alpha_1,\eta}f)(x)=\frac{x^{-\alpha_1-\eta}}{\Gamma(\alpha_1)}\int_0^{x}t^{\eta}(x-t)^{\alpha_1-1}f(t)dt \tag{18}$$

and

$$(K_{x,\infty}^{\alpha_1,\eta}f)(x)=\frac{x^{\eta}}{\Gamma(\alpha_1)}\int_x^{\infty}(t-x)^{\alpha_1-1}t^{-\alpha_1-\eta}f(t)dt \tag{19}$$

If we put $\beta_1=-\alpha_1,$ then the Saigo fractional integral operators given in equations (16) and (17) reduce to the following Riemann-Liouville and Weyl fractional integral operators respectively (see [10])

$$(I_{0,x}^{\alpha_1}f)(x)=\frac{1}{\Gamma(\alpha_1)}\int_0^{x}(x-t)^{\alpha_1-1}f(t)dt \tag{20}$$

and

$$(W_{x,\infty}^{\alpha_1}f)(x)=\frac{1}{\Gamma(\alpha_1)}\int_x^{\infty}(t-x)^{\alpha_1-1}f(t)dt \tag{21}$$

Now we are recalling the following lemma which gives the power function formulas for the above fractional integral operators. These power function formulas (see [12], [13]) are required for our present study.

Lemma: Let $\alpha_1,\alpha_2,\beta_1,\beta_2,\eta\in\mathbb{C}$ and $x>0.$ Then the following formulas hold (see [12], [13])

(a) Let $\Re(\eta)>0$ and $\Re(\rho)>\max\{0,\Re(\alpha_1+\alpha_2+\beta_1-\eta),\Re(\alpha_2-\beta_2)\},$ then

$$(I_{0,x}^{\alpha_1,\alpha_2,\beta_1,\beta_2,\eta}t^{\rho-1})(x)=\frac{\Gamma(\rho)\Gamma(\rho+\eta-\alpha_1-\alpha_2-\beta)\Gamma(\rho+\beta_2-\alpha_2)}{\Gamma(\rho+\eta-\alpha_1-\alpha_2)\Gamma(\rho+\eta-\alpha_2-\beta_1)\Gamma(\rho+\beta_2)}x^{\rho+\eta-\alpha_1-\alpha_2-1} \tag{22}$$

(b) Let $\Re(\eta) > 0$ and $\Re(\rho) < 1 + \min\{\Re(-\beta_1), \Re(\alpha_1+\alpha_2-\eta), \Re(\alpha_1+\beta_2-\eta)\}$, then

$$(I_{x,\infty}^{\alpha_1,\alpha_2,\beta_1,\beta_2,\eta} t^{\rho-1})(x) = \frac{\Gamma(1-\rho-\beta_1)\Gamma(1-\rho-\eta+\alpha_1+\alpha_2)\Gamma(1-\rho-\eta+\alpha_1+\beta_2)}{\Gamma(1-\rho)\Gamma(1-\rho-\eta+\alpha_1+\alpha_2+\beta_2)\Gamma(1-\rho+\alpha_1-\beta_1)} x^{\rho+\eta-\alpha_1-\alpha_2-1} \tag{23}$$

3. Fractional integration of the product of Srivastava polynomial, multivariable Mittage-Leffler function and generalized Lauricella's hypergeometric function.

In this section, we shall give some fractional integral formulas involving the product of Srivastava polynomial, multivariable Mittage-Leffler function and generalized Lauricella's hypergeometric function by using the fractional integral operators.

Theorem 1: Let $x > 0,\ \alpha_1,\ \alpha_2,\ \beta_1,\ \beta_2,\ \eta,\ \rho,\ \lambda_j,\ \mu_j,\ c \in \mathbb{C},\ d,\ d_j \in \mathbb{R}^+$ where $j = 1, 2, \cdots l$; $|t| \le 1$ be such that $\Re(\eta) > 0,\ \Re(\mu_j) > 0$. Further, let
$\Re(\rho + dk + \sum_{j=1}^{l} d_j k_j + m_1 + m_2 + m_3) > \max\{0, \Re(\alpha_1+\alpha_2+\beta_1-\eta), \Re(\alpha_2-\beta_2)\}$,
then the following fractional integral formula holds true:

$$\left\{I_{0,x}^{\alpha_1,\alpha_2,\beta_1,\beta_2,\eta}\left(t^{\rho-1} S_N^M(\sigma t^d) E_{\mu_j,c}^{\lambda_j}(\sigma_1 t^{d_1},\ldots,\sigma_l t^{d_l}) F_D^{3(\gamma,\delta,\tau,v,p_1,p_2)}(a,a_1,a_2,a_3;b;y_1 t,y_2 t,y_3 t;n_1,n_2)\right)\right\}(x)$$

$$= x^{\rho-\alpha_1-\alpha_2+\eta-1} \frac{\Gamma(\rho+dk+\sum_{j=1}^{l} d_j k_j)\Gamma(\rho+dk+\sum_{j=1}^{l} d_j k_j+\eta-\alpha_1-\alpha_2-\beta_1)}{\Gamma(\rho+dk+\sum_{j=1}^{l} d_j k_j+\beta_2)\Gamma(\rho+dk+\sum_{j=1}^{l} d_j k_j+\eta-\alpha_1-\alpha_2)}$$

$$\times \frac{\Gamma(\rho+dk+\sum_{j=1}^{l} d_j k_j+\beta_2-\alpha_2)}{\Gamma(\rho+dk+\sum_{j=1}^{l} d_j k_j+\eta-\alpha_2-\beta_1)} \sum_{k=0}^{[N/M]} \sum_{k_1,\cdots,k_l=0}^{\infty} \frac{(-N)_{Mk}}{k!} A_{N,k} F_{\mu_j,c}^{\lambda_j} (\sigma x^d)^k \prod_{j=1}^{l} \frac{(\sigma_j x^{d_j})^{k_j}}{k_j!}$$

$$\times {}_3F_{D,3}^{3(\gamma,\delta,\tau,v,p_1,p_2)} \left[\begin{matrix} a, \rho+dk+\sum_{j=1}^{l} d_j k_j, \rho+dk+\sum_{j=1}^{l} d_j k_j+\eta-\alpha_1-\alpha_2-\beta_1, \\ b, \rho+dk+\sum_{j=1}^{l} d_j k_j+\beta_2, \rho+dk+\sum_{j=1}^{l} d_j k_j+\eta-\alpha_1-\alpha_2, \end{matrix}\right.$$

$$\left.\begin{matrix} \rho+dk+\sum_{j=1}^{l} d_j k_j+\beta_2-\alpha_2, a_1, a_2, a_3; \\ \rho+dk+\sum_{j=1}^{l} d_j k_j+\eta-\alpha_2-\beta_1,; \end{matrix} \; y_1 x, y_2 x, y_3 x \right] \tag{24}$$

Proof: let Δ denote the left hand side of the equation (24). Then using equations (4), (6) and (13) in the equation (24) and changing the order of integration and summation, which is valid under the given conditions, we have

$$\Delta=\sum_{k=0}^{[N/M]}\sum_{k_1,\cdots,k_l=0}^{\infty}\frac{(-N)_{Mk}}{k!}A_{N,k}F_{\mu_j,c}^{\lambda_j}\sigma^k\frac{\sigma_1^{k_1}}{k_1!}\cdots\frac{\sigma_l^{k_l}}{k_l!}\sum_{m_1,m_2,m_3=0}^{\infty}\frac{(a)_{m_1+m_2+m_3}(a_1)_{m_1}(a_2)_{m_2}(a_3)_{m_3}}{(b)_{m_1+m_2+m_3}}$$

$$\times\frac{B_{p_1,p_2}^{(\gamma,\delta,\tau,v)}(a-n_1+m_1+m_2+m_3,b-a+n_2)}{B(a-n_1+m_1+m_2+m_3,b-a+n_2)}\frac{y_1^{m_1}}{m_1!}\frac{y_2^{m_2}}{m_2!}\frac{y_3^{m_3}}{m_3!}$$

$$\times\left(I_{0,x}^{\alpha_1,\alpha_2,\beta_1,\beta_2,\eta}t^{\rho+dk+\sum_{j=1}^{l}d_jk_j+m_1+m_2+m_3-1}\right)(x)$$

(25)

Now using the equation (22), then the above equation (25) reduces to

$$\Delta=\sum_{k=0}^{[N/M]}\sum_{k_1,\cdots,k_l=0}^{\infty}\frac{(-N)_{Mk}}{k!}A_{N,k}F_{\mu_j,c}^{\lambda_j}\sigma^k\frac{\sigma_1^{k_1}}{k_1!}\cdots\frac{\sigma_l^{k_l}}{k_l!}\sum_{m_1,m_2,m_3=0}^{\infty}\frac{(a)_{m_1+m_2+m_3}(a_1)_{m_1}(a_2)_{m_2}(a_3)_{m_3}}{(b)_{m_1+m_2+m_3}}$$

$$\times\frac{B_{p_1,p_2}^{(\gamma,\delta,\tau,v)}(a-n_1+m_1+m_2+m_3,b-a+n_2)}{B(a-n_1+m_1+m_2+m_3,b-a+n_2)}\frac{y_1^{m_1}}{m_1!}\frac{y_2^{m_2}}{m_2!}\frac{y_3^{m_3}}{m_3!}\frac{\Gamma(\rho+dk+\sum_{j=1}^{l}d_jk_j+m_1+m_2+m_3)}{\Gamma(\rho+dk+\sum_{j=1}^{l}d_jk_j+\beta_2+m_1+m_2+m_3)}$$

$$\times\frac{\Gamma(\rho+dk+\sum_{j=1}^{l}d_jk_j+\eta-\alpha_1-\alpha_2-\beta_1+m_1+m_2+m_3)\Gamma(\rho+dk+\sum_{j=1}^{l}d_jk_j+\beta_2-\alpha_2+m_1+m_2+m_3)}{\Gamma(\rho+dk+\sum_{j=1}^{l}d_jk_j+\eta-\alpha_1-\alpha_2+m_1+m_2+m_3)\Gamma(\rho+dk+\sum_{j=1}^{l}d_jk_j+\eta-\alpha_2-\beta_1+m_1+m_2+m_3)}$$

$$\times x^{\rho+dk+\sum_{j=1}^{l}d_jk_j+m_1+m_2+m_3+\eta-\alpha_1-\alpha_2-1} \tag{26}$$

Now using the result $\Gamma(\lambda+n)=(\lambda)_n\Gamma(\lambda)$, the above equation (26) reduces to

$$=x^{\rho-\alpha_1-\alpha_2+\eta-1}\frac{\Gamma(\rho+dk+\sum_{j=1}^{l}d_jk_j)\Gamma(\rho+dk+\sum_{j=1}^{l}d_jk_j+\eta-\alpha_1-\alpha_2-\beta_1)}{\Gamma(\rho+dk+\sum_{j=1}^{l}d_jk_j+\beta_2)\Gamma(\rho+dk+\sum_{j=1}^{l}d_jk_j+\eta-\alpha_1-\alpha_2)}$$

$$\times\frac{\Gamma(\rho+dk+\sum_{j=1}^{l}d_jk_j+\beta_2-\alpha_2)}{\Gamma(\rho+dk+\sum_{j=1}^{l}d_jk_j+\eta-\alpha_2-\beta_1)}\sum_{k=0}^{[N/M]}\sum_{k_1,\cdots,k_l=0}^{\infty}\frac{(-N)_{Mk}}{k!}A_{N,k}F_{\mu_j,c}^{\lambda_j}(\sigma x^d)^k\prod_{j=1}^{l}\frac{(\sigma_jx^{d_j})^{k_j}}{k_j!}$$

$$\times\sum_{m_1,m_2,m_3=0}^{\infty}\frac{(a)_{m_1+m_2+m_3}(\rho+dk+\sum_{j=1}^{l}d_jk_j)_{m_1+m_2+m_3}(\rho+dk+\sum_{j=1}^{l}d_jk_j+\eta-\alpha_1-\alpha_2-\beta_1)_{m_1+m_2+m_3}}{(b)_{m_1+m_2+m_3}(\rho+dk+\sum_{j=1}^{l}d_jk_j+\beta_2)_{m_1+m_2+m_3}\Gamma(\rho+dk+\sum_{j=1}^{l}d_jk_j+\eta-\alpha_1-\alpha_2)_{m_1+m_2+m_3}}$$

$$\times\frac{\Gamma(\rho+dk+\sum_{j=1}^{l}d_jk_j+\beta_2-\alpha_2)_{m_1+m_2+m_3}(a_1)_{m_1}(a_2)_{m_2}(a_3)_{m_3}}{\Gamma(\rho+dk+\sum_{j=1}^{l}d_jk_j+\eta-\alpha_2-\beta_1)_{m_1+m_2+m_3}}$$

$$\times\frac{B_{p_1,p_2}^{(\gamma,\delta,\tau,v)}(a-n_1+m_1+m_2+m_3,b-a+n_2)}{B(a-n_1+m_1+m_2+m_3,b-a+n_2)}\frac{(xy_1)^{m_1}}{m_1!}\frac{(xy_2)^{m_2}}{m_2!}\frac{(xy_3)^{m_3}}{m_3!} \tag{27}$$

The above equation (27) in view of the equation (13) gives the required result (24).

Theorem 2: Let $x>0,\ \alpha_1,\ \alpha_2,\ \beta_1,\ \beta_2,\ \eta,\ \rho,\ \lambda_j,\ \mu_j,\ c\in\mathbb{C},\ d,\ d_j\in\mathbb{R}^+$ where $j=1,2,\cdots l;$ $|1/t|\le 1$ be such that $\Re(\eta)>0,\ \Re(\mu_j)>0.$ Further, let $\Re(\rho+dk+\sum_{j=1}^{l}d_jk_j-m_1-m_2-m_3)<1+\min\{\Re(-\beta_1),\Re(\alpha_1+\alpha_2-\eta),\Re(\alpha_1+\beta_2-\eta)\}$, then the following fractional integral formula holds true:

$$\left\{I_{x,\infty}^{\alpha_1,\alpha_2,\beta_1,\beta_2,\eta}\left(t^{\rho-1}S_N^M(\sigma t^d)E_{\mu_j,c}^{\lambda_j}(\sigma_1t^{d_1},\ldots,\sigma_lt^{d_l})F_D^{3(\gamma,\delta,\tau,v,p_1,p_2)}(a,a_1,a_2,a_3;b;y_1/t,y_2/t,y_3/t;n_1,n_2)\right)\right\}(x)$$

$$=x^{\rho-\alpha_1-\alpha_2+\eta-1}\frac{\Gamma(1-\rho-dk-\sum_{j=1}^{l}d_jk_j-\beta_1)\Gamma(1-\rho-dk-\sum_{j=1}^{l}d_jk_j-\eta+\alpha_1+\alpha_2)}{\Gamma(1-\rho-dk-\sum_{j=1}^{l}d_jk_j)\Gamma(1-\rho-dk-\sum_{j=1}^{l}d_jk_j-\eta+\alpha_1+\alpha_2+\beta_2)}$$

$$\times\frac{\Gamma(1-\rho-dk-\sum_{j=1}^{l}d_jk_j-\eta+\alpha_1+\beta_2)}{\Gamma(1-\rho-dk-\sum_{j=1}^{l}d_jk_j+\alpha_1-\beta_1)}\sum_{k=0}^{[N/M]}\sum_{k_1,\cdots,k_l=0}^{\infty}\frac{(-N)_{Mk}}{k!}A_{N,k}F_{\mu_j,c}^{\lambda_j}(\sigma x^d)^k\prod_{j=1}^{l}\frac{(\sigma_jx^{d_j})^{k_j}}{k_j!}$$

$$\times{}_3F_{D,3}^{3(\gamma,\delta,\tau,v,p_1,p_2)}\left[\begin{matrix}a,1-\rho-dk-\sum_{j=1}^{l}d_jk_j-\beta_1,1-\rho-dk-\sum_{j=1}^{l}d_jk_j-\eta+\alpha_1+\alpha_2,\\ b,1-\rho-dk-\sum_{j=1}^{l}d_jk_j,1-\rho-dk-\sum_{j=1}^{l}d_jk_j-\eta+\alpha_1+\alpha_2+\beta_2,\end{matrix}\right.$$

$$\left.\begin{matrix}1-\rho-dk-\sum_{j=1}^{l}d_jk_j-\eta+\alpha_1+\beta_2,a_1,a_2,a_3;\\ 1-\rho-dk-\sum_{j=1}^{l}d_jk_j+\alpha_1-\beta_1,;\end{matrix}\ \frac{y_1}{x},\frac{y_2}{x},\frac{y_3}{x}\right] \tag{28}$$

4. Special Cases

Now we shall give some special cases by taking suitable values of parameters $\alpha_1,\alpha_2,\beta_1,\beta_2$ and η. If put $\alpha_2=\beta_2=0,\ \beta_1=-\eta,\ \alpha_1=\alpha_1+\beta_1$ and $\eta=\alpha_1$ in the theorem 1 and theorem 2 we shall get following fractional integral formulas for the Saigo fractional integral operators.

Corollary 1: Let $x>0,\ \alpha_1,\ \beta_1,\ \eta,\ \rho,\ \lambda_j,\ \mu_j,\ c\in\mathbb{C},\ d,\ d_j\in\mathbb{R}^+$ where $j=1,2,\cdots l;\ |t|\leq 1$ be such that $\Re(\alpha_1)>0,\ \Re(\mu_j)>0.$ Further, let $\Re(\rho+dk+\sum_{j=1}^{l}d_jk_j+m_1+m_2+m_3)>\max\{0,\Re(\beta_1-\eta)\}$, then the following fractional integral formula holds true:

$$\left\{I_{0,x}^{\alpha_1,\beta_1,\eta}\left(t^{\rho-1}S_N^M(\sigma t^d)E_{\mu_j,c}^{\lambda_j}(\sigma_1t^{d_1},\ldots,\sigma_lt^{d_l})F_D^{3(\gamma,\delta,\tau,v,p_1,p_2)}(a,a_1,a_2,a_3;b;y_1t,y_2t,y_3t;n_1,n_2)\right)\right\}(x)$$

$$=x^{\rho-\beta_1-1}\frac{\Gamma(\rho+dk+\sum_{j=1}^{l}d_jk_j)\Gamma(\rho+dk+\sum_{j=1}^{l}d_jk_j+\eta-\beta_1)}{\Gamma(\rho+dk+\sum_{j=1}^{l}d_jk_j-\beta_1)\Gamma(\rho+dk+\sum_{j=1}^{l}d_jk_j+\eta+\alpha_1)}\sum_{k=0}^{[N/M]}\sum_{k_1,\cdots,k_l=0}^{\infty}\frac{(-N)_{Mk}}{k!}A_{N,k}F_{\mu_j,c}^{\lambda_j}(\sigma x^d)^k\prod_{j=1}^{l}\frac{(\sigma_jx^{d_j})^{k_j}}{k_j!}$$

$$\times{}_2F_{D,2}^{3(\gamma,\delta,\tau,v,p_1,p_2)}\left[\begin{matrix}a,\rho+dk+\sum_{j=1}^{l}d_jk_j,\rho+dk+\sum_{j=1}^{l}d_jk_j+\eta-\beta_1,a_1,a_2,a_3;\\ b,\rho+dk+\sum_{j=1}^{l}d_jk_j-\beta_1,\rho+dk+\sum_{j=1}^{l}d_jk_j+\eta+\alpha_1;\end{matrix}\ y_1x,y_2x,y_3x\right]\tag{29}$$

Corollary 2: Let $x>0,\ \alpha_1,\ \beta_1,\ \eta,\ \rho,\ \lambda_j,\ \mu_j,\ c\in\mathbb{C},\ d,\ d_j\in\mathbb{R}^+$ where $j=1,2,\cdots l;$ $|1/t|\leq 1$ be such that $\Re(\alpha_1)>0,\ \Re(\mu_j)>0.$ Further, let $\Re(\rho+dk+\sum_{j=1}^{l}d_jk_j-m_1-m_2-m_3)<1+\min\{\Re(\beta_1),\Re(\eta)\}$, then the following fractional integral formula holds true:

$$\left\{I_{x,\infty}^{\alpha_1,\beta_1,\eta}\left(t^{\rho-1}S_N^M(\sigma t^d)E_{\mu_j,c}^{\lambda_j}(\sigma_1t^{d_1},\ldots,\sigma_lt^{d_l})F_D^{3(\gamma,\delta,\tau,v,p_1,p_2)}(a,a_1,a_2,a_3;b;y_1/t,y_2/t,y_3/t;n_1,n_2)\right)\right\}(x)$$

$$=x^{\rho-\beta-1}\frac{\Gamma(1-\rho-dk-\sum_{j=1}^{l}d_jk_j+\beta_1)\Gamma(1-\rho-dk-\sum_{j=1}^{l}d_jk_j+\eta)}{\Gamma(1-\rho-dk-\sum_{j=1}^{l}d_jk_j)\Gamma(1-\rho-dk-\sum_{j=1}^{l}d_jk_j+\alpha_1+\beta_1+\eta)}\sum_{k=0}^{[N/M]}\sum_{k_1,\cdots,k_l=0}^{\infty}\frac{(-N)_{Mk}}{k!}A_{N,k}F_{\mu_j,c}^{\lambda_j}(\sigma x^d)^k\prod_{j=1}^{l}\frac{(\sigma_jx^{d_j})^{k_j}}{k_j!}$$

$$\times{}_2F_{D,2}^{3(\gamma,\delta,\tau,v,p_1,p_2)}\left[\begin{matrix}a,1-\rho-dk-\sum_{j=1}^{l}d_jk_j+\beta_1,1-\rho-dk-\sum_{j=1}^{l}d_jk_j+\eta,a_1,a_2,a_3;\\ b,1-\rho-dk-\sum_{j=1}^{l}d_jk_j,1-\rho-dk-\sum_{j=1}^{l}d_jk_j+\alpha_1+\beta_1+\eta;\end{matrix}\ \frac{y_1}{x},\frac{y_2}{x},\frac{y_3}{x}\right]\tag{30}$$

If we $\beta_1=0$ in the equations (30) and (31), then we shall get the following fractional integral formulas for the Erdelyi-Kober fractional integral operators.

Corollary 3: Let $x>0,\ \alpha_1,\eta,\rho,\lambda_j,\mu_j,c\in\mathbb{C},\ d,d_j\in\mathbb{R}^+$ where $j=1,2,\cdots l;\ |t|\leq 1$ be such that $\Re(\alpha_1)>0,\ \Re(\mu_j)>0.$ Further, let

$\Re(\rho+dk+\sum_{j=1}^{l}d_jk_j+m_1+m_2+m_3)>-\Re(\eta),$ then the following fractional integral formula holds true:

$$\left\{E_{0,x}^{\alpha_1,\eta}\left(t^{\rho-1}S_N^M(\sigma t^d)E_{\mu_j,c}^{\lambda_j}(\sigma_1 t^{d_1},\ldots,\sigma_l t^{d_l})F_D^{3(\gamma,\delta,\tau,v,p_1,p_2)}(a,a_1,a_2,a_3;b;y_1t,y_2t,y_3t;n_1,n_2)\right)\right\}(x)$$

$$=x^{\rho-1}\frac{\Gamma(\rho+dk+\sum_{j=1}^{l}d_jk_j+\eta)}{\Gamma(\rho+dk+\sum_{j=1}^{l}d_jk_j+\eta+\alpha_1)}\sum_{k=0}^{[N/M]}\sum_{k_1,\cdots,k_l=0}^{\infty}\frac{(-N)_{Mk}}{k!}A_{N,k}F_{\mu_j,c}^{\lambda_j}(\sigma x^d)^k\prod_{j=1}^{l}\frac{(\sigma_j x^{d_j})^{k_j}}{k_j!}$$

$$\times_1F_{D,1}^{3(\gamma,\delta,\tau,v,p_1,p_2)}\left[\begin{matrix}a,\rho+dk+\sum_{j=1}^{l}d_jk_j+\eta,a_1,a_2,a_3; \\ b,\rho+dk+\sum_{j=1}^{l}d_jk_j+\eta+\alpha_1;\end{matrix}\; y_1x,y_2x,y_3x\right] \tag{31}$$

Corollary 4: Let $x>0,\ \alpha_1,\eta,\rho,\lambda_j,\mu_j,c\in\mathbb{C},\ d,d_j\in\mathbb{R}^+$ where $j=1,2,\cdots l;\ |1/t|\le 1$ be such that $\Re(\alpha_1)>0,\ \Re(\mu_j)>0.$ Further, let

$\Re(\rho+dk+\sum_{j=1}^{l}d_jk_j-m_1-m_2-m_3)<1+\Re(\eta),$ then the following fractional integral formula holds true:

$$\left\{K_{x,\infty}^{\alpha_1,\eta}\left(t^{\rho-1}S_N^M(\sigma t^d)E_{\mu_j,c}^{\lambda_j}(\sigma_1 t^{d_1},\ldots,\sigma_l t^{d_l})F_D^{3(\gamma,\delta,\tau,v,p_1,p_2)}(a,a_1,a_2,a_3;b;y_1/t,y_2/t,y_3/t;n_1,n_2)\right)\right\}(x)$$

$$=x^{\rho-1}\frac{\Gamma(1-\rho-dk-\sum_{j=1}^{l}d_jk_j+\eta)}{\Gamma(1-\rho-dk-\sum_{j=1}^{l}d_jk_j+\alpha_1+\eta)}\sum_{k=0}^{[N/M]}\sum_{k_1,\cdots,k_l=0}^{\infty}\frac{(-N)_{Mk}}{k!}A_{N,k}F_{\mu_j,c}^{\lambda_j}(\sigma x^d)^k\prod_{j=1}^{l}\frac{(\sigma_j x^{d_j})^{k_j}}{k_j!}$$

$$\times_1F_{D,1}^{3(\gamma,\delta,\tau,v,p_1,p_2)}\left[\begin{matrix}a,1-\rho-dk-\sum_{j=1}^{l}d_jk_j+\eta,a_1,a_2,a_3; \\ b,1-\rho-dk-\sum_{j=1}^{l}d_jk_j+\alpha_1+\eta;\end{matrix}\; \frac{y_1}{x},\frac{y_2}{x},\frac{y_3}{x}\right] \tag{32}$$

If we put $\beta_1=-\alpha_1$ in the equations (30) and (31), then we shall get the following fractional integral formulas for the Riemann-Liouville and Weyl fractional integral operators respectively.

Corollary5: Let $x>0,\ \alpha_1,\rho,\lambda_j,\mu_j,c\in\mathbb{C},\ d,d_j\in\mathbb{R}^+$ where $j=1,2,\cdots l;\ |t|\le 1$ be such that $\Re(\alpha_1)>0,\ \Re(\mu_j)>0.$ Further, let

$\Re(\rho+dk+\sum_{j=1}^{l}d_jk_j+m_1+m_2+m_3)>0,$ then the following fractional integral formula holds true:

$$\left\{I_{0,x}^{\alpha_1}\left(t^{\rho-1}S_N^M(\sigma t^d)E_{\mu_j,c}^{\lambda_j}(\sigma_1 t^{d_1},\ldots,\sigma_l t^{d_l})F_D^{3(\gamma,\delta,\tau,v,p_1,p_2)}(a,a_1,a_2,a_3;b;y_1t,y_2t,y_3t;n_1,n_2)\right)\right\}(x)$$

$$=x^{\rho-1}\frac{\Gamma(\rho+dk+\sum_{j=1}^{l}d_jk_j)}{\Gamma(\rho+dk+\sum_{j=1}^{l}d_jk_j+\alpha_1)}\sum_{k=0}^{[N/M]}\sum_{k_1,\cdots,k_l=0}^{\infty}\frac{(-N)_{Mk}}{k!}A_{N,k}F_{\mu_j,c}^{\lambda_j}(\sigma x^d)^k\prod_{j=1}^{l}\frac{(\sigma_j x^{d_j})^{k_j}}{k_j!}$$

$$\times_1 F_{D,1}^{3(\gamma,\delta,\tau,v,p_1,p_2)}\left[\begin{matrix}a,\rho+dk+\sum_{j=1}^{l}d_jk_j,a_1,a_2,a_3; \\ b,\rho+dk+\sum_{j=1}^{l}d_jk_j+\alpha_1;\end{matrix}\ y_1x,y_2x,y_3x\right] \tag{33}$$

Corollary 6: Let $x>0,\ \alpha_1,\rho,\lambda_j,\mu_j,c\in\mathbb{C},\ d,d_j\in\mathbb{R}^+$ where $j=1,2,\cdots l;\ |1/t|\le 1$ be such that $\Re(\alpha_1)>0,\ \Re(\mu_j)>0.$ Further, let
$1-\Re(\rho+dk+\sum_{j=1}^{l}d_jk_j-m_1-m_2-m_3)>\Re(\alpha_1)>0,$ then the following fractional integral formula holds true:

$$\left\{I_{x,\infty}^{\alpha_1}\left(t^{\rho-1}S_N^M(\sigma t^d)E_{\mu_j,c}^{\lambda_j}(\sigma_1 t^{d_1},\ldots,\sigma_l t^{d_l})F_D^{3(\gamma,\delta,\tau,v,p_1,p_2)}(a,a_1,a_2,a_3;b;y_1/t,y_2/t,y_3/t;n_1,n_2)\right)\right\}(x)$$

$$=x^{\rho-1}\frac{\Gamma(1-\rho-dk-\sum_{j=1}^{l}d_jk_j-\alpha_1)}{\Gamma(1-\rho-dk-\sum_{j=1}^{l}d_jk_j)}\sum_{k=0}^{[N/M]}\sum_{k_1,\cdots,k_l=0}^{\infty}\frac{(-N)_{Mk}}{k!}A_{N,k}F_{\mu_j,c}^{\lambda_j}(\sigma x^d)^k\prod_{j=1}^{l}\frac{(\sigma_j x^{d_j})^{k_j}}{k_j!}$$

$$\times_1 F_{D,1}^{3(\gamma,\delta,\tau,v,p_1,p_2)}\left[\begin{matrix}a,1-\rho-dk-\sum_{j=1}^{l}d_jk_j-\alpha_1,a_1,a_2,a_3; \\ b,1-\rho-dk-\sum_{j=1}^{l}d_jk_j;\end{matrix}\ \frac{y_1}{x},\frac{y_2}{x},\frac{y_3}{x}\right] \tag{34}$$

5. Integral transform of the product of Srivastava polynomial, multivariable Mittage-Leffler function and generalized Lauricella's hypergeometric function.

In this section we shall obtain integral transforms like Beta transform, Laplace transform and Verma transform involving the results obtained in the previous section.

5.1 Beta Transform

Definition: The beta transform of a function $f(z)$ is defined as (see [17])

$$B\{f(z);s,p\}=\int_0^1 z^{s-1}(1-z)^{p-1}f(z)dz \tag{35}$$

Theorem 3: Let $x>0,\ \alpha_1,\ \alpha_2,\ \beta_1,\ \beta_2,\ \eta,\ \rho,\ \lambda_j,\ \mu_j,\ c\in\mathbb{C},\ d,\ d_j\in\mathbb{R}^+$ where $j=1,2,\cdots l;$ $|t|\le 1$ be such that $\Re(\eta)>0,\ \Re(\mu_j)>0.$ Further, let

$$\Re(\rho + dk + \sum_{j=1}^{l} d_j k_j + m_1 + m_2 + m_3) > \max\{0, \Re(\alpha_1 + \alpha_2 + \beta_1 - \eta), \Re(\alpha_2 - \beta_2)\},$$

then the following fractional integral formula holds true:

$$B\left\{I_{0,x}^{\alpha_1,\alpha_2,\beta_1,\beta_2,\eta}\left(t^{\rho-1}S_N^M(\sigma z^\theta t^d)E_{\mu_j,c}^{\lambda_j}(\sigma_1 z^{\theta_1}t^{d_1},\ldots,\sigma_l z^{\theta_l}t^{d_l})F_D^{3(\gamma,\delta,\tau,v,p_1,p_2)}(a,a_1,a_2,a_3;b;y_1zt,y_2zt,y_3zt;n_1,n_2)\right)(x):s,p\right\}$$

$$= x^{\rho-\alpha_1-\alpha_2+\eta-1}\Gamma(p)\frac{\Gamma(s+\theta k+\sum_{j=1}^{l}\theta_j k_j)}{\Gamma(s+p+\theta k+\sum_{j=1}^{l}\theta_j k_j)}\frac{\Gamma(\rho+dk+\sum_{j=1}^{l}d_jk_j)\Gamma(\rho+dk+\sum_{j=1}^{l}d_jk_j+\eta-\alpha_1-\alpha_2-\beta_1)}{\Gamma(\rho+dk+\sum_{j=1}^{l}d_jk_j+\beta_2)\Gamma(\rho+dk+\sum_{j=1}^{l}d_jk_j+\eta-\alpha_1-\alpha_2)}$$

$$\times\frac{\Gamma(\rho+dk+\sum_{j=1}^{l}d_jk_j+\beta_2-\alpha_2)}{\Gamma(\rho+dk+\sum_{j=1}^{l}d_jk_j+\eta-\alpha_2-\beta_1)}\sum_{k=0}^{[N/M]}\sum_{k_1,\cdots,k_l=0}^{\infty}\frac{(-N)_{Mk}}{k!}A_{N,k}F_{\mu_j,c}^{\lambda_j}(\sigma x^d)^k\prod_{j=1}^{l}\frac{(\sigma_j x^{d_j})^{k_j}}{k_j!}$$

$$\times{}_4F_{D,4}^{3(\gamma,\delta,\tau,v,p_1,p_2)}\left[\begin{matrix}a,\rho+dk+\sum_{j=1}^{l}d_jk_j,\rho+dk+\sum_{j=1}^{l}d_jk_j+\eta-\alpha_1-\alpha_2-\beta_1,\\ b,\rho+dk+\sum_{j=1}^{l}d_jk_j+\beta_2,\rho+dk+\sum_{j=1}^{l}d_jk_j+\eta-\alpha_1-\alpha_2,\end{matrix}\right.$$

$$\left.\begin{matrix}\rho+dk+\sum_{j=1}^{l}d_jk_j+\beta_2-\alpha_2,s+\theta k+\sum_{j=1}^{l}\theta_jk_j,a_1,a_2,a_3;\\ \rho+dk+\sum_{j=1}^{l}d_jk_j+\eta-\alpha_2-\beta_1,s+p+\theta k+\sum_{j=1}^{l}\theta_jk_j;\end{matrix}\quad y_1x,y_2x,y_3x\right] \tag{36}$$

Proof: To prove the equation (36), using the definition (35) of Beta transform, we get

$$B\left\{I_{0,x}^{\alpha_1,\alpha_2,\beta_1,\beta_2,\eta}\left(t^{\rho-1}S_N^M(\sigma z^\theta t^d)E_{\mu_j,c}^{\lambda_j}(\sigma_1 z^{\theta_1}t^{d_1},\ldots,\sigma_l z^{\theta_l}t^{d_l})F_D^{3(\gamma,\delta,\tau,v,p_1,p_2)}(a,a_1,a_2,a_3;b;y_1zt,y_2zt,y_3zt;n_1,n_2)\right)(x):s,p\right\}$$

$$=\int_0^1 z^{s-1}(1-z)^{p-1}\left\{I_{0,x}^{\alpha_1,\alpha_2,\beta_1,\beta_2,\eta}\left(t^{\rho-1}S_N^M(\sigma z^\theta t^\lambda)E_{\mu_j,c}^{\lambda_j}(\sigma_1 z^{\theta_1}t^{d_1},\ldots,\sigma_l z^{\theta_l}t^{d_l})\right.\right.$$

$$\left.\left.\times F_D^{3(\gamma,\delta,\tau,v,p_1,p_2)}(a,a_1,a_2,a_3;b;y_1zt,y_2zt,y_3zt;n_1,n_2)\right)(x)\right\}dz \tag{37}$$

Now using the equation (22), then the above equation (37) reduces to

$$=\int_0^1 z^{s+\theta k+\sum_{j=1}^{l}\theta_jk_j+m_1+m_2+m_3-1}(1-z)^{p-1}\left\{x^{\rho-\alpha_1-\alpha_2+\eta-1}\sum_{k=0}^{[N/M]}\sum_{k_1,\cdots,k_l=0}^{\infty}\frac{(-N)_{Mk}}{k!}A_{N,k}F_{\mu_j,c}^{\lambda_j}(\sigma x^d)^k\right.$$

$$\times\prod_{j=1}^{l}\frac{(\sigma_j x^{d_j})^{k_j}}{k_j!}\sum_{m_1,m_2,m_3=0}^{\infty}\frac{(a)_{m_1+m_2+m_3}(a_1)_{m_1}(a_2)_{m_2}(a_3)_{m_3}}{(b)_{m_1+m_2+m_3}}\frac{B_{p_1,p_2}^{(\gamma,\delta,\tau,v)}(a-n_1+m_1+m_2+m_3,b-a+n_2)}{B(a-n_1+m_1+m_2+m_3,b-a+n_2)}$$

$$\times\frac{(y_1x)^{m_1}}{m_1!}\frac{(y_2x)^{m_2}}{m_2!}\frac{(y_3x)^{m_3}}{m_3!}\frac{\Gamma(\rho+dk+\sum_{j=1}^{l}d_jk_j+\eta-\alpha_1-\alpha_2-\beta_1+m_1+m_2+m_3)}{\Gamma(\rho+dk+\sum_{j=1}^{l}d_jk_j+\eta-\alpha_1-\alpha_2+m_1+m_2+m_3)}$$

$$\left.\times\frac{\Gamma(\rho+dk+\sum_{j=1}^{l}d_jk_j+m_1+m_2+m_3)}{\Gamma(\rho+dk+\sum_{j=1}^{l}d_jk_j+\beta_2+m_1+m_2+m_3)}\frac{\Gamma(\rho+dk+\sum_{j=1}^{l}d_jk_j+\beta_2-\alpha_2+m_1+m_2+m_3)}{\Gamma(\rho+dk+\sum_{j=1}^{l}d_jk_j+\eta-\alpha_2-\beta_2+m_1+m_2+m_3)}\right\}dz \quad (38)$$

After interchanging the order of integration and summation and using $\Gamma(\lambda+n)=(\lambda)_n\Gamma(\lambda)$, we get

$$=x^{\rho-\alpha_1-\alpha_2+\eta-1}\frac{\Gamma(\rho+dk+\sum_{j=1}^{l}d_jk_j)\Gamma(\rho+dk+\sum_{j=1}^{l}d_jk_j+\eta-\alpha_1-\alpha_2-\beta_1)}{\Gamma(\rho+dk+\sum_{j=1}^{l}d_jk_j+\beta_2)\Gamma(\rho+dk+\sum_{j=1}^{l}d_jk_j+\eta-\alpha_1-\alpha_2)}$$

$$\times\frac{\Gamma(\rho+dk+\sum_{j=1}^{l}d_jk_j+\beta_2-\alpha_2)}{\Gamma(\rho+dk+\sum_{j=1}^{l}d_jk_j+\eta-\alpha_2-\beta_1)}\sum_{k=0}^{[N/M]}\sum_{k_1,\cdots,k_l=0}^{\infty}\frac{(-N)_{Mk}}{k!}A_{N,k}F_{\mu_j,c}^{\lambda_j}(\sigma x^d)^k\prod_{j=1}^{l}\frac{(\sigma_j x^{d_j})^{k_j}}{k_j!}$$

$$\times\sum_{m_1,m_2,m_3=0}^{\infty}\frac{(a)_{m_1+m_2+m_3}(\rho+dk+\sum_{j=1}^{l}d_jk_j)_{m_1+m_2+m_3}(\rho+dk+\sum_{j=1}^{l}d_jk_j+\eta-\alpha_1-\alpha_2-\beta_1)_{m_1+m_2+m_3}}{(b)_{m_1+m_2+m_3}(\rho+dk+\sum_{j=1}^{l}d_jk_j+\beta_2)_{m_1+m_2+m_3}\Gamma(\rho+dk+\sum_{j=1}^{l}d_jk_j+\eta-\alpha_1-\alpha_2)_{m_1+m_2+m_3}}$$

$$\times\frac{\Gamma(\rho+dk+\sum_{j=1}^{l}d_jk_j+\beta_2-\alpha_2)_{m_1+m_2+m_3}(a_1)_{m_1}(a_2)_{m_2}(a_3)_{m_3}}{\Gamma(\rho+dk+\sum_{j=1}^{l}d_jk_j+\eta-\alpha_2-\beta_1)_{m_1+m_2+m_3}}\frac{B_{p_1,p_2}^{(\gamma,\delta,\tau,v)}(a-n_1+m_1+m_2+m_3,b-a+n_2)}{B(a-n_1+m_1+m_2+m_3,b-a+n_2)}$$

$$\times\frac{(xy_1)^{m_1}}{m_1!}\frac{(xy_2)^{m_2}}{m_2!}\frac{(xy_3)^{m_3}}{m_3!}\int_0^1 z^{s+\theta k+\sum_{j=1}^{l}\theta_jk_j+m_1+m_2+m_3-1}(1-z)^{p-1}dz \quad (39)$$

Now evaluating the z-integral and after a little simplification, we get

$$=x^{\rho-\alpha_1-\alpha_2+\eta-1}\Gamma(p)\frac{\Gamma(s+\theta k+\sum_{j=1}^{l}\theta_jk_j)}{\Gamma(s+p+\theta k+\sum_{j=1}^{l}\theta_jk_j)}\frac{\Gamma(\rho+dk+\sum_{j=1}^{l}d_jk_j)\Gamma(\rho+dk+\sum_{j=1}^{l}d_jk_j+\eta-\alpha_1-\alpha_2-\beta_1)}{\Gamma(\rho+dk+\sum_{j=1}^{l}d_jk_j+\beta_2)\Gamma(\rho+dk+\sum_{j=1}^{l}d_jk_j+\eta-\alpha_1-\alpha_2)}$$

$$\times\frac{\Gamma(\rho+dk+\sum_{j=1}^{l}d_jk_j+\beta_2-\alpha_2)}{\Gamma(\rho+dk+\sum_{j=1}^{l}d_jk_j+\eta-\alpha_2-\beta_1)}\sum_{k=0}^{[N/M]}\sum_{k_1,\cdots,k_l=0}^{\infty}\frac{(-N)_{Mk}}{k!}A_{N,k}F_{\mu_j,c}^{\lambda_j}(\sigma x^d)^k\prod_{j=1}^{l}\frac{(\sigma_j x^{d_j})^{k_j}}{k_j!}$$

$$\times\sum_{m_1,m_2,m_3=0}^{\infty}\frac{(a)_{m_1+m_2+m_3}(\rho+dk+\sum_{j=1}^{l}d_jk_j)_{m_1+m_2+m_3}(\rho+dk+\sum_{j=1}^{l}d_jk_j+\eta-\alpha_1-\alpha_2-\beta_1)_{m_1+m_2+m_3}}{(b)_{m_1+m_2+m_3}(\rho+dk+\sum_{j=1}^{l}d_jk_j+\beta_2)_{m_1+m_2+m_3}\Gamma(\rho+dk+\sum_{j=1}^{l}d_jk_j+\eta-\alpha_1-\alpha_2)_{m_1+m_2+m_3}}$$

$$\times\frac{(\rho+dk+\sum_{j=1}^{l}d_jk_j+\beta_2-\alpha_2)_{m_1+m_2+m_3}(s+\theta k+\sum_{j=1}^{l}\theta_jk_j)_{m_1+m_2+m_3}(a_1)_{m_1}(a_2)_{m_2}(a_3)_{m_3}}{(\rho+dk+\sum_{j=1}^{l}d_jk_j+\eta-\alpha_2-\beta_1)_{m_1+m_2+m_3}(s+p+\theta k+\sum_{j=1}^{l}\theta_jk_j)_{m_1+m_2+m_3}}$$

$$\times\frac{B_{p_1,p_2}^{(\gamma,\delta,\tau,v)}(a-n_1+m_1+m_2+m_3,b-a+n_2)}{B(a-n_1+m_1+m_2+m_3,b-a+n_2)}\frac{(xy_1)^{m_1}}{m_1!}\frac{(xy_2)^{m_2}}{m_2!}\frac{(xy_3)^{m_3}}{m_3!} \tag{40}$$

The above equation (40) in view of the equation (13) gives the required result (36).

Theorem 4: Let $x>0,\ \alpha_1,\ \alpha_2,\ \beta_1,\ \beta_2,\ \eta,\ \rho,\ \lambda_j,\ \mu_j,\ c\in\mathbb{C},\ d,\ d_j\in\mathbb{R}^+$ where $j=1,2,\cdots l;$ $|1/t|\le 1$ be such that $\Re(\eta)>0,\ \Re(\mu_j)>0.$ Further let
$\Re(\rho+dk+\sum_{j=1}^{l}d_jk_j-m_1-m_2-m_3)<1+\min\{\Re(-\beta_1),\Re(\alpha_1+\alpha_2-\eta),\Re(\alpha_1+\beta_2-\eta)\},$
then the following fractional integral formula holds true:

$$B\left\{I_{x,\infty}^{\alpha_1,\alpha_2,\beta_1,\beta_2,\eta}\left(t^{\rho-1}S_N^M(\sigma z^\theta t^d)E_{\mu_j,c}^{\lambda_j}(\sigma_1 z^{\theta_1}t^{d_1},\ldots,\sigma_l z^{\theta_l}t^{d_l})F_D^{3(\gamma,\delta,\tau,v,p_1,p_2)}(a,a_1,a_2,a_3;b;y_1/t,y_2/t,y_3/t;n_1,n_2)\right)(x):s,p\right\}$$

$$=x^{\rho-\alpha_1-\alpha_2+\eta-1}\Gamma(p)\frac{\Gamma(s+\theta k+\sum_{j=1}^{l}\theta_jk_j)}{\Gamma(s+p+\theta k+\sum_{j=1}^{l}\theta_jk_j)}\frac{\Gamma(1-\rho-dk-\sum_{j=1}^{l}d_jk_j-\beta_1)\Gamma(1-\rho-dk-\sum_{j=1}^{l}d_jk_j-\eta+\alpha_1+\alpha_2)}{\Gamma(1-\rho-dk-\sum_{j=1}^{l}d_jk_j)\Gamma(1-\rho-dk-\sum_{j=1}^{l}d_jk_j-\eta+\alpha_1+\alpha_2+\beta_2)}$$

$$\times\frac{\Gamma(1-\rho-dk-\sum_{j=1}^{l}d_jk_j-\eta+\alpha_1+\beta_2)}{\Gamma(1-\rho-dk-\sum_{j=1}^{l}d_jk_j+\alpha_1-\beta_1)}\sum_{k=0}^{[N/M]}\sum_{k_1,\cdots,k_l=0}^{\infty}\frac{(-N)_{Mk}}{k!}A_{N,k}F_{\mu_j,c}^{\lambda_j}(\sigma x^d)^k\prod_{j=1}^{l}\frac{(\sigma_j x^{d_j})^{k_j}}{k_j!}$$

$$\times{}_4F_{D,4}^{3(\gamma,\delta,\tau,v,p_1,p_2)}\left[\begin{matrix}a,1-\rho-dk-\sum_{j=1}^{l}d_jk_j-\beta_1,1-\rho-dk-\sum_{j=1}^{l}d_jk_j-\eta+\alpha_1+\alpha_2,\\ b,1-\rho-dk-\sum_{j=1}^{l}d_jk_j,1-\rho-dk-\sum_{j=1}^{l}d_jk_j-\eta+\alpha_1+\alpha_2+\beta_2,\end{matrix}\right.$$

$$\left.\begin{matrix}1-\rho-dk-\sum_{j=1}^{l}d_jk_j-\eta+\alpha_1+\beta_2,s+\theta k+\sum_{j=1}^{l}\theta_jk_j,a_1,a_2,a_3;\\ 1-\rho-dk-\sum_{j=1}^{l}d_jk_j+\alpha_1-\beta_1,s+p+\theta k+\sum_{j=1}^{l}\theta_jk_j;\end{matrix}\ \frac{y_1}{x},\frac{y_2}{x},\frac{y_3}{x}\right] \tag{41}$$

5.2 Laplace Transform

Definition: The Laplace transform of a function $f(z)$ is defined as (see [16])

$$L\{f(z)\}=\int_0^\infty e^{-qz}f(z)dz \tag{42}$$

Theorem 5: Let $x>0,\ \alpha_1,\ \alpha_2,\ \beta_1,\ \beta_2,\ \eta,\ \rho,\ \lambda_j,\ \mu_j,\ c\in\mathbb{C},\ d,\ d_j\in\mathbb{R}^+$ where $j=1,2,\cdots l;$ $|t|\leq 1$ be such that $\Re(\eta)>0, \Re(\mu_j)>0.$ Further, let
$\Re(\rho+dk+\sum_{j=1}^{l}d_jk_j+m_1+m_2+m_3)>\max\{0,\Re(\alpha_1+\alpha_2+\beta_1-\eta),\Re(\alpha_2-\beta_2)\},$
then the following fractional integral formula holds true:

$$L\left\{I_{0,x}^{\alpha_1,\alpha_2,\beta_1,\beta_2,\eta}\left(t^{\rho-1}S_N^M(\sigma z^\theta t^d)E_{\mu_j,c}^{\lambda_j}(\sigma_1 z^{\theta_1}t^{d_1},\ldots,\sigma_l z^{\theta_l}t^{d_l})F_D^{3(\gamma,\delta,\tau,v,p_1,p_2)}(a,a_1,a_2,a_3;b;y_1zt,y_2zt,y_3zt;n_1,n_2)\right)(x)\right\}$$

$$=\frac{x^{\rho-\alpha_1-\alpha_2+\eta-1}}{q}\frac{\Gamma(1+\theta k+\sum_{j=1}^{l}\theta_jk_j)\Gamma(\rho+dk+\sum_{j=1}^{l}d_jk_j)\Gamma(\rho+dk+\sum_{j=1}^{l}d_jk_j+\eta-\alpha_1-\alpha_2-\beta_1)}{\Gamma(\rho+dk+\sum_{j=1}^{l}d_jk_j+\beta_2)\Gamma(\rho+dk+\sum_{j=1}^{l}d_jk_j+\eta-\alpha_1-\alpha_2)}$$

$$\times\frac{\Gamma(\rho+dk+\sum_{j=1}^{l}d_jk_j+\beta_2-\alpha_2)}{\Gamma(\rho+dk+\sum_{j=1}^{l}d_jk_j+\eta-\alpha_2-\beta_1)}\sum_{k=0}^{[N/M]}\sum_{k_1,\cdots,k_l=0}^{\infty}\frac{(-N)_{Mk}}{k!}A_{N,k}F_{\mu_j,c}^{\lambda_j}\left(\frac{\sigma x^d}{q^\theta}\right)^k\prod_{j=1}^{l}\frac{1}{k_j!}\left(\frac{\sigma_j x^{d_j}}{q^{\theta_j}}\right)^{k_j}$$

$$\times{}_4F_{D,3}^{3(\gamma,\delta,\tau,v,p_1,p_2)}\left[\begin{matrix}a,\rho+dk+\sum_{j=1}^{l}d_jk_j,\rho+dk+\sum_{j=1}^{l}d_jk_j+\eta-\alpha_1-\alpha_2-\beta_1,\\ b,\rho+dk+\sum_{j=1}^{l}d_jk_j+\beta_2,\rho+dk+\sum_{j=1}^{l}d_jk_j+\eta-\alpha_1-\alpha_2,\end{matrix}\right.$$

$$\left.\begin{matrix}\rho+dk+\sum_{j=1}^{l}d_jk_j+\beta_2-\alpha_2,1+\theta k+\sum_{j=1}^{l}\theta_jk_j,a_1,a_2,a_3;\\ \rho+dk+\sum_{j=1}^{l}d_jk_j+\eta-\alpha_2-\beta_1;\end{matrix}\ \frac{xy_1}{q},\frac{xy_2}{q},\frac{xy_3}{q}\right] \tag{43}$$

Proof: To prove the equation (43), using the definition (42) of Laplace transform, we get

$$L\left\{I_{0,x}^{\alpha_1,\alpha_2,\beta_1,\beta_2,\eta}\left(t^{\rho-1}S_N^M(\sigma z^\theta t^d)E_{\mu_j,c}^{\lambda_j}(\sigma_1 z^{\theta_1}t^{d_1},\ldots,\sigma_l z^{\theta_l}t^{d_l})F_D^{3(\gamma,\delta,\tau,v,p_1,p_2)}(a,a_1,a_2,a_3;b;y_1zt,y_2zt,y_3zt;n_1,n_2)\right)(x)\right\}$$

$$=\int_0^\infty e^{-qz}\left\{I_{0,x}^{\alpha_1,\alpha_2,\beta_1,\beta_2,\eta}\left(t^{\rho-1}S_N^M(\sigma z^\theta t^\lambda)E_{\mu_j,c}^{\lambda_j}(\sigma_1 z^{\theta_1}t^{d_1},\ldots,\sigma_l z^{\theta_l}t^{d_l})\right.\right.$$

$$\left.\left.\times F_D^{3(\gamma,\delta,\tau,v,p_1,p_2)}(a,a_1,a_2,a_3;b;y_1zt,y_2zt,y_3zt;n_1,n_2)\right)(x)\right\}dz \tag{44}$$

Now using the equation (22), then the above equation (44) reduces to

$$= \int_0^\infty e^{-qz} z^{\theta k+\sum_{j=1}^{l}\theta_j k_j+m_1+m_2+m_3} \Bigg\{ x^{\rho-\alpha_1-\alpha_2+\eta-1} \sum_{k=0}^{[N/M]} \sum_{k_1,\cdots,k_l=0}^{\infty} \frac{(-N)_{Mk}}{k!} A_{N,k} F_{\mu_j,c}^{\lambda_j}(\sigma x^d)^k$$

$$\times \prod_{j=1}^{l} \frac{(\sigma_j x^{d_j})^{k_j}}{k_j!} \sum_{m_1,m_2,m_3=0}^{\infty} \frac{(a)_{m_1+m_2+m_3}(a_1)_{m_1}(a_2)_{m_2}(a_3)_{m_3}}{(b)_{m_1+m_2+m_3}} \frac{B_{p_1,p_2}^{(\gamma,\delta,\tau,v)}(a-n_1+m_1+m_2+m_3, b-a+n_2)}{B(a-n_1+m_1+m_2+m_3, b-a+n_2)}$$

$$\times \frac{(y_1 x)^{m_1}}{m_1!} \frac{(y_2 x)^{m_2}}{m_2!} \frac{(y_3 x)^{m_3}}{m_3!} \frac{\Gamma(\rho+dk+\sum_{j=1}^{l} d_j k_j+\eta-\alpha_1-\alpha_2-\beta_1+m_1+m_2+m_3)}{\Gamma(\rho+dk+\sum_{j=1}^{l} d_j k_j+\eta-\alpha_1-\alpha_2+m_1+m_2+m_3)}$$

$$\times \frac{\Gamma(\rho+dk+\sum_{j=1}^{l} d_j k_j+m_1+m_2+m_3)}{\Gamma(\rho+dk+\sum_{j=1}^{l} d_j k_j+\beta_2+m_1+m_2+m_3)} \frac{\Gamma(\rho+dk+\sum_{j=1}^{l} d_j k_j+\beta_2-\alpha_2+m_1+m_2+m_3)}{\Gamma(\rho+dk+\sum_{j=1}^{l} d_j k_j+\eta-\alpha_2-\beta_2+m_1+m_2+m_3)} \Bigg\} dz \quad (45)$$

After interchanging the order of integration and summation and using $\Gamma(\lambda+n) = (\lambda)_n \Gamma(\lambda)$, we get

$$= x^{\rho-\alpha_1-\alpha_2+\eta-1} \frac{\Gamma(\rho+dk+\sum_{j=1}^{l} d_j k_j)\Gamma(\rho+dk+\sum_{j=1}^{l} d_j k_j+\eta-\alpha_1-\alpha_2-\beta_1)}{\Gamma(\rho+dk+\sum_{j=1}^{l} d_j k_j+\beta_2)\Gamma(\rho+dk+\sum_{j=1}^{l} d_j k_j+\eta-\alpha_1-\alpha_2)}$$

$$\times \frac{\Gamma(\rho+dk+\sum_{j=1}^{l} d_j k_j+\beta_2-\alpha_2)}{\Gamma(\rho+dk+\sum_{j=1}^{l} d_j k_j+\eta-\alpha_2-\beta_1)} \sum_{k=0}^{[N/M]} \sum_{k_1,\cdots,k_l=0}^{\infty} \frac{(-N)_{Mk}}{k!} A_{N,k} F_{\mu_j,c}^{\lambda_j}(\sigma x^d)^k \prod_{j=1}^{l} \frac{(\sigma_j x^{d_j})^{k_j}}{k_j!}$$

$$\times \sum_{m_1,m_2,m_3=0}^{\infty} \frac{(a)_{m_1+m_2+m_3}(\rho+dk+\sum_{j=1}^{l} d_j k_j)_{m_1+m_2+m_3}(\rho+dk+\sum_{j=1}^{l} d_j k_j+\eta-\alpha_1-\alpha_2-\beta_1)_{m_1+m_2+m_3}}{(b)_{m_1+m_2+m_3}(\rho+dk+\sum_{j=1}^{l} d_j k_j+\beta_2)_{m_1+m_2+m_3}\Gamma(\rho+dk+\sum_{j=1}^{l} d_j k_j+\eta-\alpha_1-\alpha_2)_{m_1+m_2+m_3}}$$

$$\times \frac{\Gamma(\rho+dk+\sum_{j=1}^{l} d_j k_j+\beta_2-\alpha_2)_{m_1+m_2+m_3}(a_1)_{m_1}(a_2)_{m_2}(a_3)_{m_3}}{\Gamma(\rho+dk+\sum_{j=1}^{l} d_j k_j+\eta-\alpha_2-\beta_1)_{m_1+m_2+m_3}} \frac{B_{p_1,p_2}^{(\gamma,\delta,\tau,v)}(a-n_1+m_1+m_2+m_3, b-a+n_2)}{B(a-n_1+m_1+m_2+m_3, b-a+n_2)}$$

$$\times \frac{(xy_1)^{m_1}}{m_1!} \frac{(xy_2)^{m_2}}{m_2!} \frac{(xy_3)^{m_3}}{m_3!} \int_0^\infty e^{-qz} z^{\theta k+\sum_{j=1}^{l}\theta_j k_j+m_1+m_2+m_3} dz \quad (46)$$

Now evaluating the z-integral and after a little simplification, we get

$$=\frac{x^{\rho-\alpha_1-\alpha_2+\eta-1}}{q}\frac{\Gamma(1+\theta k+\sum_{j=1}^{l}\theta_j k_j)\Gamma(\rho+dk+\sum_{j=1}^{l}d_j k_j)\Gamma(\rho+dk+\sum_{j=1}^{l}d_j k_j+\eta-\alpha_1-\alpha_2-\beta_1)}{\Gamma(\rho+dk+\sum_{j=1}^{l}d_j k_j+\beta_2)\Gamma(\rho+dk+\sum_{j=1}^{l}d_j k_j+\eta-\alpha_1-\alpha_2)}$$

$$\times\frac{\Gamma(\rho+dk+\sum_{j=1}^{l}d_j k_j+\beta_2-\alpha_2)}{\Gamma(\rho+dk+\sum_{j=1}^{l}d_j k_j+\eta-\alpha_2-\beta_1)}\sum_{k=0}^{[N/M]}\sum_{k_1,\cdots,k_l=0}^{\infty}\frac{(-N)_{Mk}}{k!}A_{N,k}F_{\mu_j,c}^{\lambda_j}\left(\frac{\sigma x^d}{q^\theta}\right)^k\prod_{j=1}^{l}\frac{1}{k_j!}\left(\frac{\sigma_j x^{d_j}}{q^{\theta_j}}\right)^{k_j}$$

$$\times\sum_{m_1,m_2,m_3=0}^{\infty}\frac{(a)_{m_1+m_2+m_3}(\rho+dk+\sum_{j=1}^{l}d_j k_j)_{m_1+m_2+m_3}(\rho+dk+\sum_{j=1}^{l}d_j k_j+\eta-\alpha_1-\alpha_2-\beta_1)_{m_1+m_2+m_3}}{(b)_{m_1+m_2+m_3}(\rho+dk+\sum_{j=1}^{l}d_j k_j+\beta_2)_{m_1+m_2+m_3}\Gamma(\rho+dk+\sum_{j=1}^{l}d_j k_j+\eta-\alpha_1-\alpha_2)_{m_1+m_2+m_3}}$$

$$\times\frac{\Gamma(\rho+dk+\sum_{j=1}^{l}d_j k_j+\beta_2-\alpha_2)_{m_1+m_2+m_3}(1+\theta k+\sum_{j=1}^{l}\theta_j k_j)_{m_1+m_2+m_3}(a_1)_{m_1}(a_2)_{m_2}(a_3)_{m_3}}{\Gamma(\rho+dk+\sum_{j=1}^{l}d_j k_j+\eta-\alpha_2-\beta_1)_{m_1+m_2+m_3}}$$

$$\times\frac{B_{p_1,p_2}^{(\gamma,\delta,\tau,v)}(a-n_1+m_1+m_2+m_3,b-a+n_2)}{B(a-n_1+m_1+m_2+m_3,b-a+n_2)}\frac{1}{m_1!}\left(\frac{xy_1}{q}\right)^{m_1}\frac{1}{m_2!}\left(\frac{xy_2}{q}\right)^{m_2}\frac{1}{m_3!}\left(\frac{xy_3}{q}\right)^{m_3}\tag{47}$$

The above equation (47) in view of the equation (13) gives the required result (43).

Theorem 6: Let $x>0,\ \alpha_1,\ \alpha_2,\ \beta_1,\ \beta_2,\ \eta,\ \rho,\ \lambda_j,\ \mu_j,\ c\in\mathbb{C},\ d,\ d_j\in\mathbb{R}^+$ where $j=1,2,\cdots l;$ $|1/t|\le 1$ be such that $\Re(\eta)>0,\ \Re(\mu_j)>0.$ Further, let

$$\Re(\rho+dk+\sum_{j=1}^{l}d_j k_j-m_1-m_2-m_3)<1+\min\{\Re(-\beta_1),\Re(\alpha_1+\alpha_2-\eta),\Re(\alpha_1+\beta_2-\eta)\},$$

then the following fractional integral formula holds true:

$$L\left\{I_{x,\infty}^{\alpha_1,\alpha_2,\beta_1,\beta_2,\eta}\left(t^{\rho-1}S_N^M(\sigma z^\theta t^d)E_{\mu_j,c}^{\lambda_j}(\sigma_1 z^{\theta_1}t^{d_1},\dots,\sigma_l z^{\theta_l}t^{d_l})F_D^{3(\gamma,\delta,\tau,v,p_1,p_2)}(a,a_1,a_2,a_3;b;y_1/t,y_2/t,y_3/t;n_1,n_2)\right)(x)\right\}$$

$$=\frac{x^{\rho-\alpha_1-\alpha_2+\eta-1}}{q}\frac{\Gamma(1+\theta k+\sum_{j=1}^{l}\theta_j k_j)\Gamma(1-\rho-dk-\sum_{j=1}^{l}d_j k_j-\beta_1)\Gamma(1-\rho-dk-\sum_{j=1}^{l}d_j k_j-\eta+\alpha_1+\alpha_2)}{\Gamma(1-\rho-dk-\sum_{j=1}^{l}d_j k_j)\Gamma(1-\rho-dk-\sum_{j=1}^{l}d_j k_j-\eta+\alpha_1+\alpha_2+\beta_2)}$$

$$\times\frac{\Gamma(1-\rho-dk-\sum_{j=1}^{l}d_j k_j-\eta+\alpha_1+\beta_2)}{\Gamma(1-\rho-dk-\sum_{j=1}^{l}d_j k_j+\alpha_1-\beta_1)}\sum_{k=0}^{[N/M]}\sum_{k_1,\cdots,k_l=0}^{\infty}\frac{(-N)_{Mk}}{k!}A_{N,k}F_{\mu_j,c}^{\lambda_j}\left(\frac{\sigma x^d}{q^\theta}\right)^k\prod_{j=1}^{l}\frac{1}{k_j!}\left(\frac{\sigma_j x^{d_j}}{q^{\theta_j}}\right)^{k_j}$$

$$\times{}_4F_{D,3}^{3(\gamma,\delta,\tau,v,p_1,p_2)}\left[\begin{matrix}a,1-\rho-dk-\sum_{j=1}^{l}d_j k_j-\beta_1,1-\rho-dk-\sum_{j=1}^{l}d_j k_j-\eta+\alpha_1+\alpha_2,\\ b,1-\rho-dk-\sum_{j=1}^{l}d_j k_j,1-\rho-dk-\sum_{j=1}^{l}d_j k_j-\eta+\alpha_1+\alpha_2+\beta_2,\end{matrix}\right.$$

$$\left. \begin{matrix} 1-\rho-dk-\sum_{j=1}^{l} d_j k_j-\eta+\alpha_1+\beta_2, 1+\theta k+\sum_{j=1}^{l}\theta_j k_j, a_1, a_2, a_3; \\ 1-\rho-dk-\sum_{j=1}^{l} d_j k_j+\alpha_1-\beta_1; \end{matrix} \quad \frac{y_1}{qx}, \frac{y_2}{qx}, \frac{y_3}{qx} \right] \tag{48}$$

5.3 Verma Transform

Definition: The Verma transform of a function $f(z)$ is defined as (see [6])

$$V\{f(z)\}=\int_0^\infty (sz)^{r-1} e^{-\frac{1}{2}sz} W_{\kappa,\omega}(sz) f(z) dz \tag{49}$$

where $W_{\kappa,\omega}(z)$ represents Whittaker function.

Theorem 7: Let $x>0,\ \alpha_1,\ \alpha_2,\ \beta_1,\ \beta_2,\ \eta,\ \rho,\ \lambda_j,\ \mu_j,\ c \in \mathbb{C},\ d,\ d_j \in \mathbb{R}^+$ where $j=1,2,\cdots l;$ $|t|\le 1$ be such that $\Re(\eta)>0,\ \Re(\mu_j)>0.$ Further, let

$\Re(\rho+dk+\sum_{j=1}^{l} d_j k_j+m_1+m_2+m_3)>\max\{0, \Re(\alpha_1+\alpha_2+\beta_1-\eta), \Re(\alpha_2-\beta_2)\},$

then the following fractional integral formula holds true:

$$V\left\{I_{0,x}^{\alpha_1,\alpha_2,\beta_1,\beta_2,\eta}\left(t^{\rho-1} S_N^M(\sigma z^\theta t^d) E_{\mu_j,c}^{\lambda_j}(\sigma_1 z^{\theta_1} t^{d_1},\ldots,\sigma_l z^{\theta_l} t^{d_l}) F_D^{3(\gamma,\delta,\tau,v,p_1,p_2)}(a,a_1,a_2,a_3;b;y_1 zt,y_2 zt,y_3 zt;n_1,n_2)\right)(x)\right\}$$

$$=\frac{x^{\rho-\alpha_1-\alpha_2+\eta-1}}{s^r}\frac{\Gamma(\frac{1}{2}\pm\omega+\theta k+\sum_{j=1}^{l}\theta_j k_j)}{\Gamma(1-\kappa+\theta k+\sum_{j=1}^{l}\theta_j k_j)}\frac{\Gamma(\rho+dk+\sum_{j=1}^{l} d_j k_j)\Gamma(\rho+dk+\sum_{j=1}^{l} d_j k_j+\eta-\alpha_1-\alpha_2-\beta_1)}{\Gamma(\rho+dk+\sum_{j=1}^{l} d_j k_j+\beta_2)\Gamma(\rho+dk+\sum_{j=1}^{l} d_j k_j+\eta-\alpha_1-\alpha_2)}$$

$$\times\frac{\Gamma(\rho+dk+\sum_{j=1}^{l} d_j k_j+\beta_2-\alpha_2)}{\Gamma(\rho+dk+\sum_{j=1}^{l} d_j k_j+\eta-\alpha_2-\beta_1)}\sum_{k=0}^{[N/M]}\sum_{k_1,\cdots,k_l=0}^{\infty}\frac{(-N)_{Mk}}{k!}A_{N,k}\left(\frac{\sigma x^d}{s^\theta}\right)^k\prod_{j=1}^{l}\frac{1}{k_j!}\left(\frac{\sigma_j x^{d_j}}{s^{\theta_j}}\right)^{k_j}$$

$$\times {}_5F_{D,4}^{3(\gamma,\delta,\tau,v,p_1,p_2)}\left[\begin{matrix} a,\rho+dk+\sum_{j=1}^{l} d_j k_j, \rho+dk+\sum_{j=1}^{l} d_j k_j+\eta-\alpha_1-\alpha_2-\beta_1, \\ b,\rho+dk+\sum_{j=1}^{l} d_j k_j+\beta_2, \rho+dk+\sum_{j=1}^{l} d_j k_j+\eta-\alpha_1-\alpha_2, \end{matrix}\right.$$

$$\left. \begin{matrix} \rho+dk+\sum_{j=1}^{l} d_j k_j+\beta_2-\alpha_2, \frac{1}{2}\pm\omega+\theta k+\sum_{j=1}^{l}\theta_j k_j, a_1, a_2, a_3; \\ \rho+dk+\sum_{j=1}^{l} d_j k_j+\eta-\alpha_2-\beta_1, 1-\kappa+\theta k+\sum_{j=1}^{l}\theta_j k_j; \end{matrix} \quad \frac{xy_1}{s}, \frac{xy_2}{s}, \frac{xy_3}{s} \right] \tag{50}$$

Proof: To prove the equation (50) using the definition (49) of Verma transform, we get

$$V\left\{I_{0,x}^{\alpha_1,\alpha_2,\beta_1,\beta_2,\eta}\left(t^{\rho-1}S_N^M(\sigma z^\theta t^d)E_{\mu_j,c}^{\lambda_j}(\sigma_1 z^{\theta_1}t^{d_1},\ldots,\sigma_l z^{\theta_l}t^{d_l})F_D^{3(\gamma,\delta,\tau,v,p_1,p_2)}(a,a_1,a_2,a_3;b;y_1zt,y_2zt,y_3zt;n_1,n_2)\right)(x)\right\}$$

$$=\int_0^\infty z^{r-1}e^{-\frac{1}{2}sz}W_{\kappa,\omega}(sz)\left\{I_{0,x}^{\alpha_1,\alpha_2,\beta_1,\beta_2,\eta}\left(t^{\rho-1}S_N^M(\sigma z^\theta t^\lambda)E_{\mu_j,c}^{\lambda_j}(\sigma_1 z^{\theta_1}t^{d_1},\ldots,\sigma_l z^{\theta_l}t^{d_l})\right.\right.$$

$$\left.\left.\times F_D^{3(\gamma,\delta,\tau,v,p_1,p_2)}(a,a_1,a_2,a_3;b;y_1zt,y_2zt,y_3zt;n_1,n_2)\right)(x)\right\}dz \quad (51)$$

Now using the equation (22), then the above equation (51) reduces to

$$=\int_0^\infty z^{r+\theta k+\sum_{j=1}^l\theta_j k_j+m_1+m_2+m_3-1}e^{-\frac{1}{2}sz}W_{\kappa,\omega}(sz)\left\{x^{\rho-\alpha_1-\alpha_2+\eta-1}\sum_{k=0}^{[N/M]}\sum_{k_1,\cdots,k_l=0}^{\infty}\frac{(-N)_{Mk}}{k!}A_{N,k}F_{\mu_j,c}^{\lambda_j}(\sigma x^d)^k\right.$$

$$\times\prod_{j=1}^l\frac{(\sigma_j x^{d_j})^{k_j}}{k_j!}\sum_{m_1,m_2,m_3=0}^{\infty}\frac{(a)_{m_1+m_2+m_3}(a_1)_{m_1}(a_2)_{m_2}(a_3)_{m_3}}{(b)_{m_1+m_2+m_3}}\frac{B_{p_1,p_2}^{(\gamma,\delta,\tau,v)}(a-n_1+m_1+m_2+m_3,b-a+n_2)}{B(a-n_1+m_1+m_2+m_3,b-a+n_2)}$$

$$\times\frac{(y_1x)^{m_1}}{m_1!}\frac{(y_2x)^{m_2}}{m_2!}\frac{(y_3x)^{m_3}}{m_3!}\frac{\Gamma(\rho+dk+\sum_{j=1}^l d_jk_j+\eta-\alpha_1-\alpha_2-\beta_1+m_1+m_2+m_3)}{\Gamma(\rho+dk+\sum_{j=1}^l d_jk_j+\eta-\alpha_1-\alpha_2+m_1+m_2+m_3)}$$

$$\left.\times\frac{\Gamma(\rho+dk+\sum_{j=1}^l d_jk_j+m_1+m_2+m_3)}{\Gamma(\rho+dk+\sum_{j=1}^l d_jk_j+\beta_2+m_1+m_2+m_3)}\frac{\Gamma(\rho+dk+\sum_{j=1}^l d_jk_j+\beta_2-\alpha_2+m_1+m_2+m_3)}{\Gamma(\rho+dk+\sum_{j=1}^l d_jk_j+\eta-\alpha_2-\beta_2+m_1+m_2+m_3)}\right\}dz \quad (52)$$

After interchanging the order of integration and summation and using $\Gamma(\lambda+n)=(\lambda)_n\Gamma(\lambda)$, we get

$$=x^{\rho-\alpha_1-\alpha_2+\eta-1}\frac{\Gamma(\rho+dk+\sum_{j=1}^l d_jk_j)\Gamma(\rho+dk+\sum_{j=1}^l d_jk_j+\eta-\alpha_1-\alpha_2-\beta_1)}{\Gamma(\rho+dk+\sum_{j=1}^l d_jk_j+\beta_2)\Gamma(\rho+dk+\sum_{j=1}^l d_jk_j+\eta-\alpha_1-\alpha_2)}$$

$$\times\frac{\Gamma(\rho+dk+\sum_{j=1}^l d_jk_j+\beta_2-\alpha_2)}{\Gamma(\rho+dk+\sum_{j=1}^l d_jk_j+\eta-\alpha_2-\beta_1)}\sum_{k=0}^{[N/M]}\sum_{k_1,\cdots,k_l=0}^{\infty}\frac{(-N)_{Mk}}{k!}A_{N,k}F_{\mu_j,c}^{\lambda_j}(\sigma x^d)^k\prod_{j=1}^l\frac{(\sigma_j x^{d_j})^{k_j}}{k_j!}$$

$$\times\sum_{m_1,m_2,m_3=0}^{\infty}\frac{(a)_{m_1+m_2+m_3}(\rho+dk+\sum_{j=1}^l d_jk_j)_{m_1+m_2+m_3}(\rho+dk+\sum_{j=1}^l d_jk_j+\eta-\alpha_1-\alpha_2-\beta_1)_{m_1+m_2+m_3}}{(b)_{m_1+m_2+m_3}(\rho+dk+\sum_{j=1}^l d_jk_j+\beta_2)_{m_1+m_2+m_3}\Gamma(\rho+dk+\sum_{j=1}^l d_jk_j+\eta-\alpha_1-\alpha_2)_{m_1+m_2+m_3}}$$

$$\times\frac{\Gamma(\rho+dk+\sum_{j=1}^{l}d_jk_j+\beta_2-\alpha_2)_{m_1+m_2+m_3}(a_1)_{m_1}(a_2)_{m_2}(a_3)_{m_3}}{\Gamma(\rho+dk+\sum_{j=1}^{l}d_jk_j+\eta-\alpha_2-\beta_1)_{m_1+m_2+m_3}}\frac{B_{p_1,p_2}^{(\gamma,\delta,\tau,v)}(a-n_1+m_1+m_2+m_3,b-a+n_2)}{B(a-n_1+m_1+m_2+m_3,b-a+n_2)}$$

$$\times\frac{(xy_1)^{m_1}}{m_1!}\frac{(xy_2)^{m_2}}{m_2!}\frac{(xy_3)^{m_3}}{m_3!}\int_0^\infty z^{r+\theta k+\sum_{j=1}^{l}\theta_jk_j+m_1+m_2+m_3-1}e^{-\frac{1}{2}sz}W_{\kappa,\omega}(sz)dz \tag{53}$$

Now evaluating the z-integral and after a little simplification, we get

$$=\frac{x^{\rho-\alpha_1-\alpha_2+\eta-1}}{s^r}\frac{\Gamma(\frac{1}{2}\pm\omega+\theta k+\sum_{j=1}^{l}\theta_jk_j)}{\Gamma(1-\kappa+\theta k+\sum_{j=1}^{l}\theta_jk_j)}\frac{\Gamma(\rho+dk+\sum_{j=1}^{l}d_jk_j)\Gamma(\rho+dk+\sum_{j=1}^{l}d_jk_j+\eta-\alpha_1-\alpha_2-\beta_1)}{\Gamma(\rho+dk+\sum_{j=1}^{l}d_jk_j+\beta_2)\Gamma(\rho+dk+\sum_{j=1}^{l}d_jk_j+\eta-\alpha_1-\alpha_2)}$$

$$\times\frac{\Gamma(\rho+dk+\sum_{j=1}^{l}d_jk_j+\beta_2-\alpha_2)}{\Gamma(\rho+dk+\sum_{j=1}^{l}d_jk_j+\eta-\alpha_2-\beta_1)}\sum_{k=0}^{[N/M]}\sum_{k_1,\cdots,k_l=0}^{\infty}\frac{(-N)_{Mk}}{k!}A_{N,k}F_{\mu_j,c}^{\lambda_j}\left(\frac{\sigma x^d}{s^\theta}\right)^k\prod_{j=1}^{l}\frac{1}{k_j!}\left(\frac{\sigma_jx^{d_j}}{s^{\theta_j}}\right)^{k_j}$$

$$\times\sum_{m_1,m_2,m_3=0}^{\infty}\frac{(a)_{m_1+m_2+m_3}(\rho+dk+\sum_{j=1}^{l}d_jk_j)_{m_1+m_2+m_3}(\rho+dk+\sum_{j=1}^{l}d_jk_j+\eta-\alpha_1-\alpha_2-\beta_1)_{m_1+m_2+m_3}}{(b)_{m_1+m_2+m_3}(\rho+dk+\sum_{j=1}^{l}d_jk_j+\beta_2)_{m_1+m_2+m_3}\Gamma(\rho+dk+\sum_{j=1}^{l}d_jk_j+\eta-\alpha_1-\alpha_2)_{m_1+m_2+m_3}}$$

$$\times\frac{(\rho+dk+\sum_{j=1}^{l}d_jk_j+\beta_2-\alpha_2)_{m_1+m_2+m_3}(\frac{1}{2}\pm\omega+\theta k+\sum_{j=1}^{l}\theta_jk_j)_{m_1+m_2+m_3}(a_1)_{m_1}(a_2)_{m_2}(a_3)_{m_3}}{(\rho+dk+\sum_{j=1}^{l}d_jk_j+\eta-\alpha_2-\beta_1)_{m_1+m_2+m_3}(1-\kappa+\theta k+\sum_{j=1}^{l}\theta_jk_j)_{m_1+m_2+m_3}}$$

$$\times\frac{B_{p_1,p_2}^{(\gamma,\delta,\tau,v)}(a-n_1+m_1+m_2+m_3,b-a+n_2)}{B(a-n_1+m_1+m_2+m_3,b-a+n_2)}\frac{1}{m_1!}\left(\frac{xy_1}{s}\right)^{m_1}\frac{1}{m_2!}\left(\frac{xy_2}{s}\right)^{m_2}\frac{1}{m_3!}\left(\frac{xy_3}{s}\right)^{m_3} \tag{54}$$

The above equation (54) in view of the equation (13) gives the required result (50).

Theorem 8: Let $x>0,\ \alpha_1,\ \alpha_2,\ \beta_1,\ \beta_2,\ \eta,\ \rho,\ \lambda_j,\ \mu_j,\ c\in\mathbb{C},\ d,\ d_j\in\mathbb{R}^+$ where $j=1,2,\cdots l;$ $|1/t|\le 1$ be such that $\Re(\eta)>0,\ \Re(\mu_j)>0.$ Further, let

$$\Re(\rho+dk+\sum_{j=1}^{l}d_jk_j-m_1-m_2-m_3)<1+\min\{\Re(-\beta_1),\Re(\alpha_1+\alpha_2-\eta),\Re(\alpha_1+\beta_2-\eta)\},$$

then the following fractional integral formula holds true:

$$V\left\{I_{x,\infty}^{\alpha_1,\alpha_2,\beta_1,\beta_2,\eta}\left(t^{\rho-1}S_N^M(\sigma z^\theta t^d)E_{\mu_j,c}^{\lambda_j}(\sigma_1z^{\theta_1}t^{d_1},\ldots,\sigma_lz^{\theta_l}t^{d_l})F_D^{3(\gamma,\delta,\tau,v,p_1,p_2)}(a,a_1,a_2,a_3;b;y_1/t,y_2/t,y_3/t;n_1,n_2)\right)(x)\right\}$$

$$=\frac{x^{\rho-\alpha_1-\alpha_2+\eta-1}}{s^r}\frac{\Gamma(\frac{1}{2}\pm\omega+\theta k+\sum_{j=1}^{l}\theta_jk_j)}{\Gamma(1-\kappa+\theta k+\sum_{j=1}^{l}\theta_jk_j)}\frac{\Gamma(1-\rho-dk-\sum_{j=1}^{l}d_jk_j-\beta_1)\Gamma(1-\rho-dk-\sum_{j=1}^{l}d_jk_j-\eta+\alpha_1+\alpha_2)}{\Gamma(1-\rho-dk-\sum_{j=1}^{l}d_jk_j)\Gamma(1-\rho-dk-\sum_{j=1}^{l}d_jk_j-\eta+\alpha_1+\alpha_2+\beta_2)}$$

$$\times \frac{\Gamma(1-\rho-dk-\sum_{j=1}^{l} d_j k_j-\eta+\alpha_1+\beta_2)}{\Gamma(1-\rho-dk-\sum_{j=1}^{l} d_j k_j+\alpha_1-\beta_1)} \sum_{k=0}^{[N/M]} \sum_{k_1,\cdots,k_l=0}^{\infty} \frac{(-N)_{Mk}}{k!} A_{N,k} F_{\mu_j,c}^{\lambda_j}(\sigma x^d)^k \prod_{j=1}^{l} \frac{(\sigma_j x^{d_j})^{k_j}}{k_j!}$$

$$\times {}_5F_{D,4}^{3(\gamma,\delta,\tau,\nu,p_1,p_2)}\left[\begin{matrix} a, 1-\rho-dk-\sum_{j=1}^{l} d_j k_j-\beta_1, 1-\rho-dk-\sum_{j=1}^{l} d_j k_j-\eta+\alpha_1+\alpha_2, \\ b, 1-\rho-dk-\sum_{j=1}^{l} d_j k_j, 1-\rho-dk-\sum_{j=1}^{l} d_j k_j-\eta+\alpha_1+\alpha_2+\beta_2, \end{matrix}\right.$$

$$\left.\begin{matrix} 1-\rho-dk-\sum_{j=1}^{l} d_j k_j-\eta+\alpha_1+\beta_2, \frac{1}{2}\pm\omega+\theta k+\sum_{j=1}^{l}\theta_j k_j, a_1, a_2, a_3; \\ 1-\rho-dk-\sum_{j=1}^{l} d_j k_j+\alpha_1-\beta_1, 1-\kappa+\theta k+\sum_{j=1}^{l}\theta_j k_j; \end{matrix} \; \frac{y_1}{sx}, \frac{y_2}{sx}, \frac{y_3}{sx}\right] \quad (55)$$

6. Conclusion

From the last few decades, the concept of fractional calculus and integral transform are very unique in nature and play an important role in mathematics, physics and engineering. In this paper several fractional integral and integral transform formulas involving the product of general class of polynomials, multivariable Mittage-Leffler function and generalized Lauricella's hypergeometric function, are obtained and expressed in terms of generalized Lauricella's hypergeometric function. In a similar manner, by suitably specializing the parameters, several fractional integral and integral transform formulas involving the product of a number of simpler special functions, can be obtained.

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