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ON RECURRENT RIEMANNIAN MANIFOLDS

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ABSTRACT. The object of the present paper is to study the nature of 1-form for a non-zero and non-constant scalar curvature in a Ricci recurrent Riemannian manifold and obtained some results for conharmonically recurrent Riemannian manifold.

1. Introduction and outline of paper

The notion of Recurrent space in Riemannian geometry relies to a broad interest and were studied by many authors like De,U.C.([2],[3]), khan,Q.([8],[9]), Shaikh,A.A.([10],[11]), and Walker([13]) etc. The concept of connection comes the idea of transporting data along a curve. An affine connection gives the directional derivatives of vector fields. In this paper we use Riemannian connection the idea of Riemannian connection come from the Affine connection and we denotes 1-forms by η and ξ. We denote the connection by nabla(∇).

Let M be the C^{∞} Riemannian manifolds of dimension n and $\chi(M)$ be the set of differentiable vector fields on manifold M. Let P, Q $\in \chi(M)$ then $\nabla_P Q \in \chi(M)$ is said to be the covariant differentiation of P over Q and K(P,Q,S) be the Riemannian curvatur tensor of order (1, 3).
A Riemannian manifolds M is called recurrent manifolds ([7]) if

$$(\nabla_A K)(P, Q, S) = \eta(A)K(P, Q, S) \tag{1.1}$$

Where η is 1-form known as recurrence parameter and is non zero. If the 1-form $\eta = 0$ in the above equation , then the recurrent manifold becomes symmetric manifolds ([6]).
Contracting (1.1) over P, we obtain

$$(\nabla_A Ric)(Q, S) = \eta(A)Ric(Q, S) \tag{1.2}$$

Where a is 1-form and Ric = $(C_1^1 K) = \Sigma_{i=1}^{n}{}' K(e_i, V, W, e_i)$; e_i are orthonormal basis vectors of the tangent space T_A at A$\in M_n$ from equation (1.2), we have

$$(\nabla_A R)(S) = \eta(A)R(S) \tag{1.3}$$

Where R is known as Ricci tensor of order (1 1), defined as

$$Ric(P, S) = g(R(P), S) \tag{1.4}$$

Contracting (1.3) over S, we obtain

$$Ar = \eta(A)r \tag{1.5}$$

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Where r is known as scalar curvature tensor. In this paper, I have assume a non-flat C^∞ Riemannian manifolds in which recurrent conharmonic curvature tensor N satisfies the following condition-

$$(\nabla_A N)(P,Q,S) = \eta(A)N(P,Q,S) \tag{1.6}$$

Where a is 1-form known as recurrence parameter which is non zero and N(P,Q,S) is defined by ([4]).

$$\begin{aligned} N(P,Q,S) &= K(P,Q,S) - \frac{1}{(n-2)}[Ric(Q,S)P - Ric(P,S)Q \\ &\quad + g(Q,S)R(P) - g(P,S)R(Q)]. \end{aligned} \tag{1.7}$$

Such n-dimensional Riemannian manifolds has been called a conharmonically recurrent Riemannian manifold ([5]). If the 1-form $\eta = 0$, then the manifold becomes to conharmonically symmetric. Weyl conformal curvature tensor C(P,Q,S), Projective curvature tensor T(P,Q,S) are given by ([4]).

$$\begin{aligned} C(P,Q,S) &= K(P,Q,S) - \frac{1}{(n-2)}[Ric(Q,S)P - Ric(P,S)Q \\ &\quad + g(Q,S)R(P) - g(P,S)R(Q)] \\ &\quad + \frac{r}{(n-1)(n-2)}[g(Q,S)P - g(P,S)Q] \end{aligned} \tag{1.8}$$

$$T(P,Q,S) = K(P,Q,S) - \frac{1}{n-1}[Ric(Q,S)P - Ric(P,S)Q] \tag{1.9}$$

and

$$W(P,Q,S) = K(P,Q,S) - \frac{r}{n(n-1)}[g(Q,S)P - g(P,S)Q]. \tag{1.10}$$

A Riemannian manifold is called projectively, conformally and concircularly recurrent Riemannian manifolds ([12]) if

$$\begin{aligned} (\nabla_A T)(P,Q,S) &= \eta(A)T(P,Q,S) \\ (\nabla_A C)(P,Q,S) &= \eta(A)C(P,Q,S) \\ (\nabla_A W)(P,Q,S) &= \eta(A)W(P,Q,S) \end{aligned} \tag{1.11}$$

Where η is 1-form known as recurrence parameter.
A manifold is called Einstein manifold if ([1])

$$Ric(Q,S) = \lambda g(Q,S) \tag{1.12}$$

where λ is constant. From (1.12), we have

$$R(Q) = \lambda Q \tag{1.13}$$

Contracting over Q, we obtain

$$r = n\lambda \tag{1.14}$$

2. Nature of the Recurrence parameter on a Ricci Recurrent Riemannian manifold

First assume the case when r is constant and is not equal to zero.
Then from (1.5), we have

$$\eta(A)r = 0 \tag{2.1}$$

covariant differentiation of (2.1) over B, we obtain

$$(\nabla_B \eta)(A)r = 0 \tag{2.2}$$

Interchanging A , B in the above equation and then subtracting them, we get

$$[(\nabla_B \eta)(A) - (\nabla_A \eta)(B)]r = 0$$

since r≠0, then

$$(\nabla_B \eta)(A) - (\nabla_A \eta)(B) = 0$$

This shows that the recurrence parameter is closed. Now, we consider the case when the scalar curvarure r is not constant.
Then from (1.5), we have

$$BAr = (\nabla_B \eta)(A)r - \eta(A)(Br) \tag{2.3}$$

Interchanging A & B in (2.3) and then subtracting them, we obtain

$$[(\nabla_B \eta)(A) - (\nabla_A \eta)(B)]r + \eta(A)(Br) - \eta(B)(Ar) = 0 \tag{2.4}$$

Using(1.5) in (2.4), we obtain

$$[(\nabla_B \eta)(A) - (\nabla_A \eta)(B)]r = 0$$

Since r≠ 0, then

$$(\nabla_B \eta)(A) - (\nabla_A \eta)(B) = 0.$$

Thus we can state the following:

Theorem 2.1. *In the Ricci recurrent Riemannian manifold the recurrence parameter is closed for non-zero and non constant scalar curvature r in both the cases.*

3. Conharmonically recurrent Riemannian manifold

Let M be the n-dimensional conharmonically recurent Riemannian manifold.
Hence from eqn. (1.6) & (1.7), we obtain that

$$\begin{aligned}
&(\nabla_A K)(P,Q,S) - \eta(A)K(P,Q,S)\\
&= \frac{1}{n-2}[(\nabla_A Ric)(Q,S)P - (\nabla_A Ric)(P,S)Q + g(Q,S)(\nabla_A R)(P)\\
&-g(P,S)Q(\nabla_A R)(Y) - \eta(A)(Ric(Q,S)P - Ric(P,S)Q\\
&+g(Q,S)R(P) - g(P,S)R(Q))]
\end{aligned} \tag{3.1}$$

Permuting (3.1) twice over A,P and Q; adding these three equations and using Bianchi's second identity, we obtain

$$\begin{aligned}
&\eta(A)K(P,Q,S)+\eta(P)K(Q,A,S)+\eta(Q)K(A,P,S)\\
&+\frac{1}{(n-2)}[(\nabla_A Ric)(Q,S)P-(\nabla_A Ric)(P,S)Q+g(Q,S)(\nabla_A R)(P)\\
&-g(P,S)(\nabla_A R)(Q)+(\nabla_P Ric)(A,S)Q-(\nabla_P Ric)(Q,S)A\\
&+g(A,S)(\nabla_P R)(Q)-g(Q,S)(\nabla_P R)(A)+(\nabla_Q Ric)(P,S)A\\
&-(\nabla_Q Ric)(A,S)P+g(P,S)(\nabla_Q R)(A)-g(A,S)(\nabla_Q R)(P)\\
&-\eta(A)(Ric(Q,S)P-Ric(P,S)Q+g(Q,S)R(P)-g(P,S)R(Q))\\
&-\eta(P)(Ric(A,S)Q-Ric(Q,S)A+g(A,S)R(Q)-g(Q,S)R(A))\\
&-\eta(Q)(Ric(P,S)A-Ric(A,S)P+g(P,S)R(A)-g(A,S)R(P))]\\
&=0 \qquad (3.2)
\end{aligned}$$

Contracting (3.2) over P, we obtain

$$\begin{aligned}
&\eta(A)Ric(Q,S)-\eta(Q)Ric(A,S)+'K(Q,A,S,\rho)\\
&+\frac{1}{n-2}[(n-1)(\nabla_A Ric)(Q,S)+g(Q,S)(Ar)-g((\nabla_A R)(Q),S)\\
&+(1-n)(\nabla_Q Ric)(A,S)+g((\nabla_Q R)(P),S)-g(A,S)(Qr)\\
&+(1-n)\eta(X)Ric(Q,S)-\eta(X)g(Q,S)r+\eta(X)g(R(Q),S)\\
&-\eta(Q)Ric(A,S)+\eta(A)Ric(Q,S)-\eta(R(Q))g(A,S)+\eta(R(A))g(Q,S)\\
&+(n-1)\eta(Q)Ric(A,S)-\eta(Q)g(R(A),S)+\eta(Q)g(A,S)r]\\
&=0 \qquad (3.3)
\end{aligned}$$

where the metric tensor g and ρ is defined by

$$g(A,\rho)=\eta(A),\forall A \qquad (3.4)$$

Factoring off S in (3.3), we obtain

$$\begin{aligned}
\eta(A)R(Q) \quad - \quad &\eta(Q)R(A)-K(Q,A,\rho)\\
&+\frac{1}{(n-2)}[(n-1)(\nabla_A R)(Q)+Q(Ar)-(\nabla_A R)(Q)-(\nabla_Q R)(A)\\
&-(\nabla_A R)(Q)+\frac{1}{2}A(Qr)-\frac{1}{2}Q(Ar)+(1-n)(\nabla_Q R)(A)\\
&+(\nabla_Q R)(A)-A(Qr)-(n-1)\eta(A)R(Q)-\eta(A)Qr\\
&+\eta(A)R(Q)-\eta(Q)R(A)+\eta(A)R(Q)-\eta(R(Q))A\\
&+\eta(R(A)Q+(n-1)\eta(Q)R(A)-\eta(P)R(A)+\eta(P)Ar]\\
&=0.
\end{aligned}$$

or,

$$\begin{aligned} K(Q,A,\rho) &= \frac{1}{n-2}[(n-3)(\nabla_A R)(Q) \\ &\quad -(n-3)(\nabla_Q R)(A)+\eta(A)R(Q)-\eta(Q)R(A) \\ &\quad +\eta(R(A))Q-\eta(R(Q))A-\eta(A)Qr+\eta(Q)Ar] \end{aligned} \tag{3.5}$$

Contracting equation (3.5) with respect to Q, we obtain

$$\begin{aligned} Ric(A,\rho) &= \frac{1}{n-2}[(n-3)(Ar)-\frac{1}{2}(n-3)(Ar)+\eta(A)r \\ &\quad -\eta(R(A))+(n-1)\eta(R(A))-(n-1)\eta(A)r]. \end{aligned}$$

or,

$$Ric(A,p)=\frac{n-3}{2(n-2)}(Ar)-\frac{1}{n-2}\eta(A)r+\eta(R(A)) \tag{3.6}$$

Using (1.4) & (3.4) in (3.6), we obtain

$$(n-3)Ar-2(n-2)\eta(A)r=0.$$

we can state the following:

Theorem 3.1. *If the scalar curvature r is constant in an n-dimensional ($n\geq 4$) conharmonically recurrent Riemannian manifold, then it must be vanish*

Taking covariant differentiation on (1.7) over A, we obtain

$$\begin{aligned} (\nabla_A N)(P,Q,S) &= (\nabla_A K)(P,Q,S) \\ &\quad -\frac{1}{n-2}[(\nabla_A Ric)(Q,S)P-(\nabla_A Ric)(P,S)Q \\ &\quad +g(Q,S)(\nabla_A R)(P)-g(P,S)(\nabla_A R)(Q)]. \end{aligned} \tag{3.7}$$

Let M_n be the Ricci-recurent Riemannian manif0ld, Then using (1.2) & (1.3) in (3.7), we obtain

$$\begin{aligned} (\nabla_A N)(P,Q,S) &= (\nabla_A K)(P,Q,S)-\frac{\eta(A)}{n-2}[Ric(Q,S)P-Ric(P,S)Q \\ &\quad +g(Q,S)R(P)-g(P,S)R(Q)]. \end{aligned} \tag{3.8}$$

From above equation (3.8) we conclude that if any one of the equation (1.1) or (1.6) hold then the second also holds.

Thus we can state the following:

Theorem 3.2. *The necessary and sufficient condition for an n ($n\geq 3$)-dimensional and the same recurrence parameter Ricci-Recurrent Riemanian manifold to be the recurrent manifold is that it is conharmonically recurrent manifold.*

Now we prove the following:

Theorem 3.3. *In the n-dimensional Einstein manifold M_n the conharmonic curvature tensor satisfies the identity:*

$$(\nabla_A N)(P,Q,S)+(\nabla_P N)(Q,A,S)+(\nabla_Q N)(A,P,S)=0$$

Proof. Using (1.12) and (1.13) in (1.7), it follows that

$$N(P,Q,S) = K(P,Q,S) - \frac{2\lambda}{n-2}[g(Q,S)P - g(P,S)Q] \tag{3.9}$$

Taking covariant differentiation on (3.9) over A, we obtain

$$(\nabla_A N)(P,Q,S) = (\nabla_A K)(P,Q,S) \tag{3.10}$$

permuting (3.10) twice over A, P & Q then adding these equations and using Bianchi's second identity, we get the appropriate result. □

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