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CERTAIN CLASSES OF BI-UNIVALENT FUNCTIONS WITH RESPECT TO SYMMETRIC POINTS

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Abstract. In this investigation, we propose to make use of the Horadam polynomials, we consider two subclasses of bi-univalent functions with respect to symmetric points. For functions belonging to these classes, we find the second Hankel determinant inequality for defined class of bi-univalent functions. Some interesting remarks of the results presented here are also investigated.
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1. Introduction and definitions

Let A denote the class of functions of the form

$$f(z) = z + a_2 z^2 + \cdots \tag{1.1}$$

which are analytic in

$$\Delta = \{z : z \in \mathbb{C} \text{ and } |z| < 1\}.$$

We denote S by the class of univalent functions in Δ. Further, we know that every univalent function has an inverse f^{-1}, defined by

$$f^{-1}(f(z)) = z, \qquad z \in \Delta$$

and

where

$$f^{-1}(w) = w - a_2 w^2 + (2a_2^2 - a_3)w^3 - (5a_2^3 - 5a_2 a_3 + a_4)w^4 + \cdots .$$

A function f A is said to be bi-univalent in Δ if both a function f and it's inverse f^{-1} are univalent in Δ. Let σ denote the class of bi-univalent functions in Δ given by (1.1). After Brannan and Taha [13], Srivastava et al. [43] only revived the study of bi-univalent functions and its related work. After this paper we could see huge number of papers [1, 2, 5, 6, 7, 8, 9, 10, 12, 14, 16, 17, 19, 21, 26, 27, 30, 32, 35, 36, 37, 39, 40, 41, 46, 47] (also, the references therein) in this line by defining various subclasses of bi-univalent functions to discuss the initial TaylorMaclaurin coefficient estimates $|a_2|$, $|a_3|$ and $|a_4|$.

For analytic functions f and g in Δ, f is said to be subordinate to g if there exists an analytic function w such that

$$w(0) = 0, \qquad |w(z)| < 1 \quad \text{and} \quad f(z) = g(w(z)), \qquad z \in \Delta.$$

It is denoted by

$$f \prec g \qquad (z \in \Delta) \qquad \text{that is} \qquad f(z) \prec g(z), \qquad z \in \Delta.$$

In particular, when g is univalent in Δ,

$$f \prec g \qquad (z \in \Delta) \;\Leftrightarrow\; f(0) = g(0) \quad \text{and} \quad f(\Delta) \subset g(\Delta).$$

For $a,\ b,\ p,\ q \in \mathbb{R}$, the Horadam polynomials $\hbar_n(x,\ a,\ b;\ p,\ q) := \hbar_n(x)$ are given by the recurrence relation (see [24, 25]):

$$\hbar_n(x) \quad = \quad px\hbar_{n-1}(x) + q\hbar_{n-2}(x), \qquad n \in \mathbb{N} \tag{1.2}$$

here

$$\hbar_1(x) = a; \qquad \hbar_2(x) = bx. \tag{1.3}$$

The generating function of the Horadam polynomials $\hbar_n(x)$ (see [25]) is given by

$$\Pi(x,\ z) := \sum_{n=1}^{\infty} \hbar_n(x) z^{n-1} = \frac{a + (b - ap)xz}{1 - pxz - qz^2}\ . \tag{1.4}$$

Here, and in what follows, the argument $x \in \mathbb{R}$ is independent of the argument $z \in \mathbb{C}$; that is, $x \neq \Re(z)$.

It is observed that for special values of the parameters involved in this polynomial leads to different some other polynomials like the Lucas polynomials, the Fibonacci polynomials, the Pell polynomials, the Pell-Lucas polynomials and the Chebyshev polynomials for more details (see, [1, 2, 25, 40]).

A function f A is said to be starlike with respect to symmetric points, if it satisfies

$$\Re\left(\frac{zf'(z)}{f(z) - f(-z)}\right) > 0, \qquad (z \in \Delta).$$

The class of starlike functions with respect to symmetric points is denoted by S_s^* and was introduced by Sakaguchi [38].

A function f A is said to be convex with respect to symmetric points, if it satisfies

$$\Re\left(\frac{(zf'(z))'}{(f(z) - f(-z))'}\right) > 0, \qquad (z \in \Delta).$$

The class of convex functions with respect to symmetric points is denoted by K_s and was introduced by Das and Singh [18]. Latter, the aforementioned classes are discussed for functions in the class of bi-univalent functions by many renowned researchers [6, 8, 7, 10, 17, 21, 30, 39, 44] also, the works done by the references therein.

Definition 1.1. A function $f \in \sigma$ is said to be in the class $P_s^\sigma(\alpha,\ x)$, if

$$\frac{2\alpha z^2 f''(z) + 2zf'(z)}{\alpha z\left(f(z) - f(-z)\right)' + (1-\alpha)\left(f(z) - f(-z)\right)} \prec \Pi(x,\ z) + 1 - a, \qquad z \in \Delta$$

and

$$\frac{2\alpha w^2 g''(w) + 2wg'(w)}{\alpha w\left(g(w) - g(-w)\right)' + (1-\alpha)\left(g(w) - g(-w)\right)} \prec \Pi(x,\ w) + 1 - a, \qquad w \in \Delta$$

hold.

Various results for the special values of the parameters involved in the class are given as follws:

(1) In particular, when $\alpha = 0$, we have $P_s^\sigma(0,\ x) := S_s^\sigma(x)$, if

$$\frac{2zf'(z)}{f(z) - f(-z)} \prec \Pi(x,\ z) + 1 - a, \qquad z \in \Delta$$

and

$$\frac{2wg'(w)}{g(w) - g(-w)} \prec \Pi(x,\ w) + 1 - a, \qquad w \in \Delta$$

hold.

(2) If $a = p = x = 1$, $b = 2$ and $q = 0$, then we have

$$\frac{2zf'(z)}{f(z) - f(-z)} \prec \frac{1+z}{1-z} \qquad z \in \Delta$$

and

$$\frac{2wg'(w)}{g(w) - g(-w)} \prec \frac{1-w}{1+w}, \qquad w \in \Delta$$

Definition 1.2. A function $f \in \sigma$ is said to be in the class $M_s^\sigma(\alpha,\ x)$, if

$$(1-\alpha)\frac{2zf'(z)}{f(z) - f(-z)} + \alpha\frac{2\left(zf'(z)\right)'}{\left(f(z) - f(-z)\right)'} \prec \Pi(x,\ z) + 1 - a, \qquad z \in \Delta$$

and

$$(1-\alpha)\frac{2wg'(w)}{g(w) - g(-w)} + \alpha\frac{2\left(wg'(w)\right)'}{\left(g(w) - g(-w)\right)'} \prec \Pi(x,\ w) + 1 - a, \qquad w \in \Delta$$

Variety of results for the special values of the parameters involved in the class are given as follws:

(1) In particular, when $\alpha = 0$, we have $M_s^\sigma(0,\ x) := S_s^\sigma(x)$, if

$$\frac{2zf'(z)}{f(z) - f(-z)} \prec \Pi(x,\ z) + 1 - a, \qquad z \in \Delta$$

and

$$\frac{2wg'(w)}{g(w) - g(-w)} \prec \Pi(x,\ w) + 1 - a, \qquad w \in \Delta$$

hold.

(2) When $\alpha = 1$, $M_s^{\sigma}(1,\ x) := K_s^{\sigma}(x)$ if

$$\frac{2\left(zf'(z)\right)'}{\left(f(z)-f(-z)\right)'} \prec \Pi(x,\ z)+1-a, \qquad z \in \Delta$$

and

$$\frac{2\left(wg'(w)\right)'}{\left(g(w)-g(-w)\right)'} \prec \Pi(x,\ w)+1-a, \qquad w \in \Delta$$

hold.

(3) If $a = p = x = 1$, $b = 2$ and $q = 0$, then we have

$$\frac{2\left(zf'(z)\right)'}{\left(f(z)-f(-z)\right)'} \prec \frac{1+z}{1-z} \qquad z \in \Delta$$

and

$$\frac{2\left(wg'(w)\right)'}{\left(g(w)-g(-w)\right)'} \prec \frac{1-w}{1+w}, \qquad w \in \Delta$$

defined as

$$H_q(n) = \begin{vmatrix} a_n & a_{n+1} & \cdots & a_{n+q-1} \\ a_{n+1} & a_{n+2} & \cdots & a_{n+q-2} \\ \vdots & \vdots & \vdots & \vdots \\ a_{n+q-1} & a_{n+q-2} & \cdots & a_{n+2q-2} \end{vmatrix} \qquad (a_1 = 1).$$

The Hankel determinants $H_2(1) = a_3 - a_2^2$ and $H_2(2) = a_2a_4 - a_2^3$ are well-known as Fekete-Szegö and second Hankel determinant functionals respectively. Further Fekete and Szegö [23] introduced the generalized functional $a_3 - \delta a_2^2$, where δ is some real number. In 1969, Keogh and Merkes [28] studied the Fekete-Szegö problem for the classes S^* and K. In 2001, Srivastava et al. [42] solved completely the Fekete-Szegö problem for the family $C_1 := \{f \in A\ :\ \Re\left(e^{i\eta}f'(z)\right) > 0,\ -\frac{\pi}{2} < \eta < \frac{\pi}{2},\ z \in \mathbb{D}\}$ and obtained improvement of $|a_3 - a_2^2|$ for the smaller set C_1. Recently, Kowalczyk et al. [29] discussed the developments involving the Fekete-Szegö functional $|a_3 - \delta a_2^2|$, where $0 \leq \delta \leq 1$ as well as the corresponding Hankel determinant for the Taylor-Maclaurin coefficients $\{a_n\}_{n\in\mathbb{N}\setminus\{1\}}$ of normalized univalent functions of the form (1.1). Similarly, several authors have investigated upper bounds for the Hankel determinant of functions belonging to various subclasses of univalent functions [3, 4, 11, 12, 15, 31, 33, 34, 45] and the references therein. On the other hand, Zaprawa [46, 47] extended the study on Fekete-Szegö problem to some specific classes of bi-univalent functions. Very recently, the upper bounds of $H_2(2)$ for the classes $S^*_\sigma(\beta)$ and $K_\sigma(\beta)$ were discussed by Deniz et al. [19]. Later, the upper bounds of $H_2(2)$ for various subclasses of σ were obtained by Altınkaya and Yalçın [9], Çağlar et al. [16], Kanas et al. [26], Karthiyayini and Sivasankari [27], Motamednezhad et al. [32], Orhan et al. [35, 36, 37] and Srivastava et al. [41].

2. Main Result

In the following theorem, we estimate the second Hankel determinant inequality for functions functions in $P_s^{\sigma}(\alpha,\ x)$. To prove our result we need the following lemmas.

Lemma 2.1. [20] *Let u be analytic function in the unit disk Δ, with $u(0) = 0$, and $|u(z)| < 1$ for all $z \in \mathbb{D}$, with the power series expansion $u(z) = u_1 z + u_2 z^2 + \cdots$. Then, $|u_n| \leq 1$ for all $n \in \mathbb{N}$. Furthermore, $|u_n| = 1$ for some $n \in \mathbb{N}$ if and only if $u(z) = e^{i\theta} z^n$, $\theta \in \mathbb{R}$.*

Lemma 2.2. [26] *If $\psi(z) = \psi z + \psi_2 z^2 + \cdots$, $z \in \Delta$, is a Schwarz function, then*

$$\psi_2 = x(1 - \psi_1^2)\ ,$$
$$\psi_3 = (1 - \psi_1^2)(1 - |x|^2)s - \psi_1(1 - \psi_1^2)x^2,$$

for some x, s, with $|x| \leq 1$ and $|s| \leq 1$.

Theorem 2.3. *For $0 < \alpha \leq 1$ and let $f(z) = z + a_2 z^2 + \cdots$ be in the class $P_s^\sigma(\alpha,\ x)$. Then*

$$|a_2 a_4 - a_3^2| \leq \begin{cases} R, & \text{if } Q \leq 0, \quad P \leq -Q \\ P + Q + R, & \text{if } \left(Q \geq 0, P \geq -\dfrac{Q}{2}\right) \text{ or } (Q \leq 0, P \geq -Q) \\ \dfrac{4PR - Q^2}{4P}, & \text{if } Q > 0, P \leq \dfrac{-Q}{2}, \end{cases}$$

where,

$$\begin{aligned} P &= \frac{4bx(bp^2x^3 + (ap + b)qx)(1 + \alpha)^3 + b^4x^4(2\alpha^2 - 3\alpha - 1)}{32(1 + 3\alpha)(1 + \alpha)^4} + \frac{b^2x^2}{4(1 + 2\alpha)^2} \\ &\quad - \frac{b^3x^3}{16(1 + \alpha)^2(1 + 2\alpha)} - \frac{2bx(aq + bpx^2) + b^2x^2}{8(1 + \alpha)(1 + 3\alpha)} \\ Q &= \frac{b^3x^3}{16(1 + \alpha)^2(1 + 2\alpha)} + \frac{2bx(aq + bpx^2) + b^2x^2}{8(1 + \alpha)(1 + 3\alpha)} - \frac{b^2x^2}{2(1 + 2\alpha)^2} \\ R &= \frac{b^2x^2}{4(1 + 2\alpha)^2}. \end{aligned}$$

Proof. For $0 < \alpha \leq 1$ let f of the form (1.1) be in the class $P_s^\sigma(\alpha,\ x)$, then there exists Schwarz functions $u(z)$ and $v(w)$ given by

$$u(z) = u_1 z + u_2 z^2 + u_3 z^3 + \cdots \qquad \forall z \in \Delta$$

and

$$v(w) = v_1 w + v_2 w^2 + v_3 w^3 + \cdots \qquad \forall w \in \Delta$$

such that

$$\frac{2\alpha z^2 f''(z) + 2zf'(z)}{\alpha z\left(f(z) - f(-z)\right)' + (1 - \alpha)\left(f(z) - f(-z)\right)} = \Pi(x, u(z)) + 1 - a$$
$$\frac{2\alpha w^2 g''(w) + 2wg'(w)}{\alpha w\left(g(w) - g(-w)\right)' + (1 - \alpha)\left(g(w) - g(-w)\right)} \prec \Pi(x,\ v(w)) + 1 - a.$$

Or, equivalently,

$$\frac{2\alpha z^2 f''(z) + 2zf'(z)}{\alpha z\left(f(z) - f(-z)\right)' + (1-\alpha)\left(f(z) - f(-z)\right)} = 1 + h_2(x)u_1 z + [h_2(x)u_2 + h_3(x)u_1^2]z^2 + \cdots \tag{2.1}$$

$$\frac{2\alpha w^2 g''(w) + 2wg'(w)}{\alpha w\left(g(w) - g(-w)\right)' + (1-\alpha)\left(g(w) - g(-w)\right)} = 1 + h_2(x)v_1 w + [h_2(x)v_2 + h_3(x)v_1^2]w^2 + \cdots . \tag{2.2}$$

It is fairly we known that,

$$|u(z)| = |u_1 z + u_2 z^2 + u_3 z^3 + \cdots| < 1$$

and

$$|v(w)| = |v_1 w + v_2 w^2 + v_3 w^3 + \cdots| < 1$$

then

$$|u_k| \leq 1 \qquad \text{and} \qquad |v_k| \leq 1, \qquad k \in \mathbb{N}.$$

Also, from the Definition 1.1, we have

$$\frac{2\alpha z^2 f''(z) + 2zf'(z)}{\alpha z\left(f(z) - f(-z)\right)' + (1-\alpha)\left(f(z) - f(-z)\right)} \tag{2.3}$$
$$= 1 + 2(1+\alpha)a_2 z + 2(1+2\alpha)a_3 z^2 + [4(1+3\alpha)a_4 - 2a_2a_3(1+\alpha)(1+2\alpha)]z^3 + \cdots$$

$$\frac{2\alpha w^2 g''(w) + 2wg'(w)}{\alpha w\left(g(w) - g(-w)\right)' + (1-\alpha)\left(g(w) - g(-w)\right)} \tag{2.4}$$
$$= 1 - 2(1+\alpha)a_2 w + 2(1+2\alpha)(2a_2^2 - a_3)w^2 - [4(1+3\alpha)(5a_2^3 - 5a_2a_3 + a_4) - 2(1+\alpha)(1+2\alpha)(2a_2^3 - a_2a_3)]w^3 + \cdots .$$

Comparing the coefficients of like terms in equations (2.1),(2.3) and (2.2), (2.4), we have

$$2(1+\alpha)a_2 = h_2(x)u_1 \tag{2.5}$$

$$2(1+2\alpha)a_3 = h_2(x)u_2 + h_3(x)u_1^2 \tag{2.6}$$

$$4(1+3\alpha)a_4 - 2a_2a_3(1+\alpha)(1+2\alpha) = h_2(x)u_3 + 2h_3(x)u_1u_2 + h_4(x)u_1^3 \tag{2.7}$$

$$-2(1+\alpha)a_2 = h_2(x)v_1 \tag{2.8}$$

$$2(1+2\alpha)(2a_2^2 - a_3) = h_2(x)v_2 + h_3(x)v_1^2 \tag{2.9}$$

and

$$2(1+\alpha)(1+2\alpha)(2a_2^3 - a_2a_3) - 4(1+3\alpha)(5a_2^3 - 5a_2a_3 + a_4) = h_2(x)v_3 + 2h_3(x)v_1v_2 + h_4(x)v_1^3. \tag{2.10}$$

From equations (2.5) and (2.8), we get

$$u_1 = -v_1. \tag{2.11}$$

It follows from (2.5) that

$$a_2 = \frac{h_2(x)u_1}{2(1+\alpha)}. \tag{2.12}$$

Subtracting (2.9) from (2.6), we have

$$a_3 = \frac{h_2(x)(u_2 - v_2)}{4(1+2\alpha)} + a_2^2 = \frac{h_2(x)(u_2 - v_2)}{4(1+2\alpha)} + \frac{h_2^2(x)u_1^2}{4(1+\alpha)^2} \tag{2.13}$$

Subtracting (2.10) from (2.7), we have

$$a_4 = \frac{(1+\alpha)(1+2\alpha)}{2(1+3\alpha)}a_2^3 - \frac{1}{2}(5a_2^3 - 5a_2a_3) + \frac{1}{8(1+3\alpha)}[h_2(x)(u_3 - v_3) + 2h_3(x)u_1(u_2+v_2) + 2h_4(x)u_1^3]. \tag{2.14}$$

Now, from the equations (2.12), (2.13) and (2.14), we have

$$\begin{aligned} a_2a_4 - a_3^2 = {} & h_2^4(x)u_1^4\left(\frac{(1+\alpha)(1+2\alpha)}{32(1+3\alpha)(1+\alpha)^4} - \frac{1}{16(1+\alpha)^4}\right) + \frac{h_2(x)h_4(x)u_1^4}{8(1+\alpha)(1+3\alpha)} \\ & + \frac{h_2^3(x)(x-y)(1-u_1^2)u_1^2}{32(1+\alpha)^2(1+2\alpha)} \\ & + \frac{h_2^2(x)u_1}{16(1+\alpha)(1+3\alpha)}\Big[(1-u_1^2)[(1-|x|^2)s - (1-|y|^2)t] - u_1(1-u_1^2)(x^2+y^2)\Big] \\ & + \frac{h_2(x)h_3(x)(x+y)(1-u_1^2)u_1^2}{8(1+\alpha)(1+3\alpha)} - \frac{h_2^2(x)(x-y)^2(1-u_1^2)^2}{16(1+2\alpha)^2}. \end{aligned} \tag{2.15}$$

By the Lemma 2.2, we find that

$$\begin{aligned} u_2 - v_2 &= (x-y)(1-u_1^2) \quad \text{since} \quad u_1 = -v_1 \\ u_2 + v_2 &= (x+y)(1-u_1^2) \\ u_3 &= (1-u_1^2)(1-|x|^2)s - u_1(1-u_1^2)x^2 \\ v_3 &= (1-v_1^2)(1-|y|^2)t - v_1(1-v_1^2)y^2 \\ u_3 - v_3 &= (1-u_1^2)[(1-|x|^2)s - (1-|y|^2)t] - u_1(1-u_1^2)(x^2+y^2) \end{aligned}$$

for some x, y, s, t with $|x| \le 1, |y| \le 1, |s| \le 1, |t| \le 1$. Taking modulus on both sides, then equation (2.15) becomes

$$\begin{aligned} |a_2a_4 - a_3^2| \le {} & \frac{\Big|[(1+\alpha)(1+2\alpha) - 2(1+3\alpha)]h_2^4(x)u_1^4 + 4h_2(x)h_4(x)u_1^4(1+\alpha)^3\Big|}{32(1+3\alpha)(1+\alpha)^3} \\ & + \frac{h_2^3(x)u_1^2(1-u_1^2)(|x|+|y|)}{32(1+\alpha)2(1+2\alpha)} + \frac{h_2^2(x)u_1(1-u_1^2)}{8(1+\alpha)(1+3\alpha)} \\ & - \frac{h_2^2(x)u_1(1-u_1^2)(|x|^2+|y|^2)}{16(1+\alpha)(1+3\alpha)} + \frac{h_2^2(x)u_1^2(1-u_1^2)(|x|^2+|y|^2)}{16(1+\alpha)(1+3\alpha)} \\ & + \frac{h_2(x)h_3(x)u_1^2(1-u_1^2)(|x|+|y|)}{8(1+\alpha)(1+3\alpha)} - \frac{h_2^2(x)}{16(1+2\alpha)^2}(1+u_1^4-2u_1^2)(|x|+|y|)^2 \end{aligned}$$

Since, $|u_1| \le 1$, we may assume that $u_1 = u \in [0,1]$, and for $\gamma_1 = |x| \le 1$ and $\gamma_2 = |y| \le 1$, we obtain

$$|a_2a_4 - a_3^2| \le F(\gamma_1,\ \gamma_2) = S_1 + S_2(\gamma_1+\gamma_2) + S_3(\gamma_1^2+\gamma_2^2) + S_4(\gamma_1+\gamma_2)^2$$

where,

$$S_1 = \frac{|[(1+\alpha)(1+2\alpha) - 2(1+3\alpha)]h_2^4(x)u^4 + 4h_2(x)h_4(x)u^4(1+\alpha)^3|}{32(1+3\alpha)(1+\alpha)^3} + \frac{h_2^2(x)u(1-u^2)}{8(1+\alpha)(1+3\alpha)} \geq 0$$

$$S_2 = \frac{h_2^3(x)u^2(1-u^2)(|x|+|y|)}{32(1+\alpha)^2(1+2\alpha)} + \frac{h_2(x)h_3(x)u^2(1-u^2)(|x|+|y|)}{8(1+\alpha)(1+3\alpha)} \geq 0$$

$$S_3 = \frac{h_2^2(x)u^2(1-u^2)}{16(1+\alpha)(1+3\alpha)} - \frac{h_2^2(x)u(1-u^2)}{16(1+\alpha)(1+3\alpha)} \leq 0$$

$$S_4 = \frac{h_2^2(x)}{16(1+2\alpha)^2}(1+u^4-2u^2) \geq 0.$$

Now we need to maximize the function $F(\gamma_1, \gamma_2)$ on the closed square $S\ :\ [0,1]\times[0,1]$ for $u \in [0,1]$ with regards to $F(\gamma_1,\gamma_2) = F(\gamma_2,\gamma_1)$, it is sufficient that, we investigate maximum of

$$G(\gamma_2) = F(\gamma_2,\gamma_2) = S_1 + 2\gamma_2 S_2 + 2\gamma_2^2(S_3 + 2S_4) \quad on \quad \gamma_2 \in [0,1] \tag{2.16}$$

according to $u \in (0,\ 1)$, $u = 0$ and $u = 1$. Firstly, if we let $u = 1$, then we obtain

$$max\Big\{G(\gamma_2); \gamma_2 \in [0,1]\Big\} = \frac{|[(1+\alpha)(1+2\alpha) - 2(1+3\alpha)]h_2^4(x) + 4h_2(x)h_4(x)(1+\alpha)^3|}{32(1+3\alpha)(1+\alpha)^4}.$$

Secondly, letting $u = 0$, we get

$$G(\gamma_2) = \frac{4h_2^2(x)\gamma_2^2}{16(1+2\alpha)^2} = \frac{h_2^2(x)\gamma_2^2}{4(1+2\alpha)^2}.$$

Hence we can see that

$$max\Big\{G(\gamma_2); \gamma_2 \in [0,1]\Big\} = G(1) = \frac{h_2^2(x)}{4(1+2\alpha)^2}.$$

Finally, we let $u \in (0,1)$. Considering the equation (2.16) for $0 \leq \gamma_2 \leq 1$, we get

(1) If $S_3 + 2S_4 \geq 0$, it is clear that

$$G'(\gamma_2) = 4(S_3 + 2S_4)\gamma_2 + 2S_2 > 0$$

for $0 < \gamma_2 < 1$ and any fixed $u \in (0,1)$ ie., $G(\gamma_2)$ is an increasing function. Hence $max\Big\{G(\gamma_2); \gamma_2 \in [0,1]\Big\} = G(1) = S_1 + 2S_2 + 2S_3 + 4S_4$

(2) If $S_3 + 2S_4 < 0$, then we consider for critical point

$$\gamma_{2_0} = -\frac{S_2}{2(S_3 + 2S_4)} = \frac{S_2}{2K}$$

for any fixed $u \in (0,1)$, where $K = -(S_3 + 2S_4) > 0$ the following two cases

Case 1: For $\gamma_{2_0} = \frac{S_2}{2K} > 1$, it follows that $K < \frac{S_2}{2} \leq S_2$ and so $S_2 + S_3 + 2S_4 \geq 0$, therefore $G(0) = S_1 \leq S_1 + 2S_2 + 2S_3 + 4S_4 = G(1)$.

Case 2: For $\gamma_{2_0} = \frac{S_2}{2K} \leq 1$, since $S_2 \geq 0$, we get $\frac{S_2^2}{2K} \leq S_2$. Therefore, $G(0) = S_1 \leq S_1 + \frac{S_2^2}{2K} = G(\gamma_{2_0}) < S_1 + S_2$. Considering the above cases for point of u, it follows that the function $G(\gamma_2)$ get, its maximum when $S_3 + 2S_4 \geq 0$, it means

$$max\Big\{G(\gamma_2); \gamma_2 \in [0,1]\Big\} = G(1) = S_1 + 2S_2 + 2S_3 + 4S_4$$

Therefore, $maxF(\gamma_1, \gamma_2) = F(1,1)$ on the boundary of the square. Let $K : [0,1] \to \mathbb{R}$

$$K(u) = maxF(\gamma_1, \gamma_2) = F(1,1) = S_1 + 2S_2 + 2S_3 + 4S_4.$$

By replacing the values of S_1, S_2, S_3, S_4 in the above function K, we have

$$K(u) = \left\{ \frac{|4h_2(x)h_4(x)(1+\alpha)^3 + h_2^4(x)(2\alpha^2 - 3\alpha - 1)|}{32(1+3\alpha)(1+\alpha)^4} + \frac{h_2^2(x)}{4(1+2\alpha)^2} - \frac{h_2^3(x)}{16(1+\alpha)^2(1+2\alpha)} - \frac{2h_2(x)h_3(x) + h_2^2(x)}{8(1+\alpha)(1+3\alpha)} \right\} u^4$$
$$+ \left\{ \frac{h_2^3(x)}{16(1+\alpha)^2(1+2\alpha)} + \frac{2h_2(x)h_3(x) + h_2^2(x)}{8(1+\alpha)(1+3\alpha)} - \frac{h_2^2(x)}{2(1+2\alpha)^2} \right\} u^2 + \frac{h_2^2(x)}{4(1+2\alpha)^2}.$$

Letting $u^2 = c$, we get

$$S_1 + 2S_2 + 2S_3 + S_4 = Pc^2 + Qc + R, \qquad (2.17)$$

where,

$$P = \frac{|4h_2(x)h_4(x)(1+\alpha)^3 + h_2^4(x)(2\alpha^2 - 3\alpha - 1)|}{32(1+3\alpha)(1+\alpha)^4} + \frac{h_2^2(x)}{4(1+2\alpha)^2} - \frac{h_2^3(x)}{16(1+\alpha)^2(1+2\alpha)} - \frac{2h_2(x)h_3(x) + h_2^2(x)}{8(1+\alpha)(1+3\alpha)}$$

$$Q = \frac{h_2^3(x)}{16(1+\alpha)^2(1+2\alpha)} + \frac{2h_2(x)h_3(x) + h_2^2(x)}{8(1+\alpha)(1+3\alpha)} - \frac{h_2^2(x)}{2(1+2\alpha)^2}$$

$$R = \frac{h_2^2(x)}{4(1+2\alpha)^2}.$$

Then with the help of optimal value of quadratic expression, further, using (1.2) and (1.3), we get the required result. This completes the proof of the theorem. □

Corollary 2.4. *Let $f(z) = z + a_2z^2 + \cdots$ be in the class $S_s^\sigma(x)$. Then*

$$|a_2a_4 - a_3^2| \leq \begin{cases} R_1, & \text{if } Q_1 \leq 0, \quad P_1 \leq -Q_1 \\ P_1 + Q_1 + R_1, & \text{if } \left(Q_1 \geq 0, P_1 \geq -\frac{Q_1}{2}\right) \text{ or } (Q_1 \leq 0,\ P_1 \geq -Q_1) \\ \frac{4P_1R_1 - Q_1^2}{4P_1}, & \text{if } Q_1 > 0, P_1 \leq \frac{-Q_1}{2}, \end{cases}$$

where,

$$P_1 = \frac{4bx(bp^2x^3 + (ap+b)qx) - b^4x^4}{32} + \frac{b^2x^2}{4} - \frac{b^3x^3}{16} - \frac{2bx(aq + bpx^2) + b^2x^2}{8}$$

$$Q_1 = \frac{b^3x^3}{16} + \frac{2bx(aq + bpx^2) + b^2x^2}{8} - \frac{b^2x^2}{2}$$

$$R_1 = \frac{b^2x^2}{4}.$$

Now, we state the following theorem for functions in the class $M_s^\sigma(\alpha,\ x)$ without proof. Since the proof is line similar to the proof of Theorem 2.3, so we omit the details.

Theorem 2.5. *For $0 < \alpha \leq 1$ and let $f(z) = z + a_2 z^2 + \cdots$ be in the class $M_s^\sigma(\alpha,\, x)$. Then*

$$|a_2a_4 - a_3^2| \leq \begin{cases} W, & \textit{if } V \leq 0, \quad U \leq -V \\ U + V + W, & \textit{if } \left(V \geq 0, U \geq -\dfrac{V}{2}\right) \textit{or } (V \leq 0, U \geq -V) \\ \dfrac{4UW - V^2}{4U}, & \textit{if } V > 0, U \leq \dfrac{-V}{2}, \end{cases}$$

where,

$$\begin{aligned} U &= \frac{4bx(bp^2x^3 + (ap+b)qx)(1+\alpha)^2 - b^4x^4}{32(1+3\alpha)(1+\alpha)^3} + \frac{b^2x^2}{4(1+2\alpha)^2} \\ &\quad - \frac{b^3x^3}{16(1+\alpha)^2(1+2\alpha)} - \frac{2bx(aq+bpx^2) + b^2x^2}{8(1+\alpha)(1+3\alpha)} \\ V &= \frac{b^3x^3}{16(1+\alpha)^2(1+2\alpha)} + \frac{2bx(aq+bpx^2) + b^2x^2}{8(1+\alpha)(1+3\alpha)} - \frac{b^2x^2}{2(1+2\alpha)^2} \\ W &= \frac{b^2x^2}{4(1+2\alpha)^2}. \end{aligned}$$

Corollary 2.6. *For $0 < \alpha \leq 1$ and let $f(z) = z + a_2z^2 + \cdots$ be in the class $K_s^\sigma(x)$. Then*

$$|a_2a_4 - a_3^2| \leq \begin{cases} W_1, & \textit{if } V_1 \leq 0, \quad U_1 \leq -V_1 \\ U_1 + V_1 + W_1, & \textit{if } \left(V_1 \geq 0, U_1 \geq -\dfrac{V_1}{2}\right) \textit{or } (V_1 \leq 0,\, U_1 \geq -V_1) \\ \dfrac{4U_1W_1 - V_1^2}{4U_1}, & \textit{if } V_1 > 0, U_1 \leq \dfrac{-V_1}{2}, \end{cases}$$

where,

$$\begin{aligned} U_1 &= \frac{16bx(bp^2x^3 + (ap+b)qx) - b^4x^4}{1024} + \frac{b^2x^2}{36} - \frac{b^3x^3}{192} - \frac{2bx(aq+bpx^2) + b^2x^2}{64} \\ V_1 &= \frac{1}{192}\left[b^3x^3 + 6bx(aq+bpx^2) + 3b^2x^2\right] - \frac{b^2x^2}{18} \\ W_1 &= \frac{b^2x^2}{36}. \end{aligned}$$

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