

REDUCTION FORMULA FOR INTEGRAL FUNCTIONS

S. AGARWAL^{1#}, A.S. UNIYAL²

ABSTRACT. In our earlier paper [1] we have given trigonometrical formulae for multiple powers in terms of summation of multiple angles, here application based on these formulae have been made so that one can easily get the complicated integral in terms of simple integral and reduction form. In this paper we will express some integral results in simple form.

1. Introduction

We are well known about $\int \sin^m x \cos^n x dx$, for $m, n \in I^+$ and its reduction formulae. We are also well known about beta function. Here, in this paper we tried to solve the problems like reduction formulae of the type $\int \sin^m x dx$, $\int x^m \sin^n x dx$, $\int \sin^m x \cos^n x dx$, $\int e^{px} \sin^m x dx$, $\int e^{px} \sin x dx$ etc. are also well known integrals.

2. Trigonometrical Formulae Reference to Our Earlier Paper [1]

We can write our trigonometrical formulae [1] in general expression without any restriction on m and n .

$$\begin{aligned}
 [a] \sin^m x \cos^n y = & \sum_{(\alpha, \beta) = (0,0)}^{\binom{m}{2} \binom{n}{2}} [A_{\alpha\beta} \cos\{(m-2\alpha)x \pm (n-2\beta)y\} \\
 & + B_{\alpha\beta} \sin\{(m-2\alpha)x \pm (n-2\beta)y\}] + K \dots(1.1)
 \end{aligned}$$

On restriction

- (i) $m = \text{even}$ then (1.1) will reduce to Cosine series
- (ii) $m = \text{odd}$ then (1.1) will reduce to the Sine series

[b] If $y = x$ then (1.1) can be written as

$$\sin^m x \cos^n y = \sum_{\lambda=0}^{\binom{m+n}{2}} [A_{\lambda} \cos(m+n-2\lambda)x + B_{\lambda} \sin(m+n-2\lambda)x] + K' \dots(1.2)$$

[#]*Corresponding Author*

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$$[c] \cos^m x \cos^n y = \sum_{(\alpha, \beta) = (0,0)}^{\binom{m}{2} \binom{n}{2}} [A_{\alpha\beta} \cos\{(m-2\alpha)x \pm (n-2\beta)y\}] + K \quad \dots(1.3)$$

[d] If $y = x$ then (1.3) can be written as

$$\cos^{m+n} x = \sum_{\lambda=0}^{\binom{m+n}{2}} [A_{\lambda} \cos\{(m+n-2\lambda)x\}] + K' \quad \dots(1.4)$$

$$[e] \sin^m x \sin^n y = \sum_{(\alpha, \beta) = (0,0)}^{\binom{m}{2} \binom{n}{2}} [A_{\alpha\beta} \sin\{(m-2\alpha)x \pm (n-2\beta)y\}] \\ + B_{\alpha\beta} \cos\{(m-2\alpha)x \pm (n-2\beta)y\}] \\ + B_{\alpha\beta} \cos\{(m-2\alpha)x + (n-2\beta)y\}] + K \dots(1.5)$$

On restriction

- (i) If $m+n = \text{odd}$ then (1.5) will reduce to Sine series
- (ii) If $m+n = \text{even}$ then (1.5) will reduce to Cosine series

[f] If $y = x$ then (1.5) can be written as

$$\sin^{m+n} x = \sum_{\lambda=0}^{\binom{m+n}{2}} [A_{\lambda} \sin(m+n-2\lambda)x + B_{\lambda} \cos(m+n-2\lambda)x] + K' \quad \dots(1.6)$$

[g] Similarly the generalised theorems can be defined as above mathematical expressions.

3. Applications of Our Trigonometrical Formulae Reference to Earlier Paper [1]

Example-1: $\int \sin^m x \cos^n x \, dx$

$$= \left(\frac{1}{2^8}\right) \left[\frac{\sin 9x}{9} + \frac{\sin 7x}{7} - \frac{4\sin 5x}{5} - \frac{4\sin 3x}{3} + 6\sin x \right] + C$$

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Example-2: $\int \sin^4 x \cos^6 x \, dx$

$$= \int \left[\sum_{\lambda=0}^5 A_{\lambda} \cos(10-2\lambda)x + K' \right] dx$$

$$= \int \left[\sum_{\lambda=0}^4 A_{\lambda} \cos(10-2\lambda)x + (A_5 + K') \right] dx$$

$$= \sum_{\lambda=0}^4 A_{\lambda} \frac{\sin(10-2\lambda)x}{10-2\lambda} + \frac{1}{2} A_5 x$$

$$= \left(\frac{1}{2^9}\right) \left[\frac{\sin 10x}{10} + \frac{\sin 8x}{4} - \frac{\sin 6x}{2} - 2\sin 4x + \sin 2x + 6x \right] + C$$

Example-3: $\int \sin^3 x \cos^5 x \, dx$

$$= \left(\frac{1}{2^7}\right) \left[\frac{\cos 8x}{8} + \frac{\cos 6x}{3} - \frac{\cos 4x}{2} - 3 \cos 2x \right] + C$$

Example-4: $\int \sin^4 x \cos^5 7x \, dx$

$$= \left(\frac{1}{2^8}\right) \left[\frac{\sin 39x}{39} - \frac{4\sin 37x}{37} + \frac{6\sin 35x}{35} - \frac{4\sin 33x}{33} + \frac{\sin 31x}{31} \right.$$

$$+ \frac{\sin 25x}{5} - \frac{20\sin 23x}{23} + \frac{10\sin 21x}{7} - \frac{20\sin 19x}{19} + \frac{5\sin 17x}{17}$$

$$\left. + \frac{10\sin 11x}{11} - \frac{40\sin 9x}{9} + \frac{60\sin 7x}{7} - 8\sin 5x + \frac{10\sin 3x}{3} \right] + C$$

(The terms are arranged in descending order of Sine angles.)

Example-5: $\int e^{4x} \sin^8 x \cos^5 x \, dx$

$$= \left(\frac{1}{2^{12}}\right) e^{4x} \left[\frac{(13\sin 13x + 4\cos 13x)}{185} - \frac{3(11\sin 11x + 4\cos 11x)}{137} \right.$$

$$- \frac{2(9\sin 9x + 4\cos 9x)}{97} + \frac{14(7\sin 7x + 4\cos 7x)}{65}$$

$$- \frac{5(5\sin 5x + 4\cos 5x)}{41} - (3\sin 3x + 4\cos 3x)$$

$$\left. + \frac{20(\sin x + 4\cos x)}{17} \right] + C$$

Example-6: $\int \sin^5 40x \sin^2 10x \cos^6 30x \, dx$

$$= \sum_{(\alpha_1, \alpha_2, \beta_1) = (0,0,0)}^{(2,1,3)} A_{\alpha_1, \alpha_2, \beta_1} \sin[(5-2\alpha_1)40 \pm (2-2\alpha_2)10 \pm (6-2\beta_1)30]x$$

$$= \left(\frac{1}{2^{12}}\right) \left[\frac{\cos 400x}{400} - \frac{\cos 380x}{190} + \frac{\cos 360x}{360} + \frac{3\cos 340x}{170} \right.$$

$$- \frac{17\cos 320x}{320} + \frac{4\cos 300x}{75} + \frac{\cos 280x}{28} - \frac{3\cos 260x}{13}$$

$$+ \frac{17\cos 240x}{48} - \frac{3\cos 220x}{22} - \frac{21\cos 200x}{40} + \frac{23\cos 180x}{18}$$

$$- \frac{19\cos 160x}{16} - \frac{5\cos 140x}{14} + \frac{71\cos 120x}{24} - \frac{227\cos 100x}{50}$$

$$\left. + \frac{47\cos 80x}{20} + \frac{143\cos 60x}{30} - \frac{619\cos 40x}{40} + \frac{249\cos 20x}{10} \right] + C$$

(The terms are arranged in descending order of Cosine angles.)

Similarly, many integrals involving Sine and Cosine functions of multiple angles and multiple powers can be easily solved by using our theorems.

4. Conclusion

The theorems which we have given in our earlier paper [1] have wide applications in multiple integrals of multiple variables. The methods which we adopt are very systematic and follow sequential order to solve many more problems in integral calculus.

N.B.: In our next communication we will send the mathematical presentation about multiple integral of more than two or three variables.

References

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¹**S.AGARWAL:** ASSISTANT PROFESSOR, DEPARTMENT OF MATHEMATICS, KCMT, BAREILLY, INDIA
EMAIL: meshubhamagarwal@gmail.com

²**A.S. UNIYAL:** ASSOCIATE PROFESSOR, DEPARTMENT OF MATHEMATICS, M.B. (GOVT.) P.G. COLLEGE, HALDWANI, INDIA
(ON DEPUTATION, STATE LIAISON OFFICER NSS, UTTARAKHAND, INDIA)
EMAIL: asuniyal0111@gmail.com