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MULTIPLE PRODUCT OF TRIGONOMETRICAL FUNCTIONS

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ABSTRACT. Sin nx and Cos nx can be expanded in terms of sums of sine and cosine multiple angles; also using same hypothesis, one can expand the product $Sin^m x\, Cos^n x$; here an attempt has been made to express

$$\prod_{i=1}^{k} Sin^{m_i} x_i \prod_{j=1}^{l} Cos^{n_j} y_j$$

as a sum of multiple angles of sine and cosine in linear form.

1. Introduction

If n is an integer then $(Cos\ x + i\ Sin\ x)^n = Cos\ nx + i\ Sin\ nx$ using binomial series in left side and equating real and imaginary part we can find series for Cos nx and Sin nx in terms of powers of Sin x and Cos x. Similarly Sin x and Cos x can be expressed in terms of power of exponential form

$$Cos\ x = (e^{ix} + e^{-ix})/2 \text{ and } Sin\ x = (e^{ix} - e^{-ix})/2i$$

and using this $Sin^n x$ and $Cos^n x$ can be expanded in multiple angle of sine and cosine.

Same hypothesis can be generalized in terms of product of $Sin^m x\, Cos^n x$. Here our work is based on the same direction but the way of expression of the series is quite different and simple.

2. Notations & Formulations

(i) Sine = \$ and Cosine = ⊄

(ii) $f^+ (ax \pm by) = f(ax + by) + f(ax - by)$

(iii) $f^- (ax \pm by) = f(ax + by) - f(ax - by)$
We shall assume $f^+(\pm by) = f^-(\pm by) = f(by)$

(iv) $f^+ (\overline{ax + by}) = f(ax + by) + f(ax - by)$

(v) $f^- (\overline{ax + by}) = f(ax + by) - f(ax - by)$

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(vi) $f^{-}(\overline{ax + by} \pm cz) = f^{+}(ax + by \pm cz) - f^{-}(ax - by \pm cz)$

(vii) $f(m_i, x_i) = f^{m_i}(x_i)$ (m_i as power of functions)

(viii) $\tau_i = m_i - 2\alpha_i$, and $\mu_j = n_j - 2\beta_j$

(ix) $$\delta = \sum_{(\alpha_1 \ldots \alpha_k;\, \beta_1 \ldots \beta_l) = (0 \ldots 0;0 \ldots 0)}^{([\frac{m_1}{2}] \ldots [\frac{m_k}{2}];\, [\frac{n_1}{2}] \ldots [\frac{n_l}{2}])} A_{\alpha_1 \ldots \alpha_k;\, \beta_1 \ldots \beta_l}$$

(x) $$\delta \begin{cases} A & \text{for Condition x} \\ B & \text{for Condition y} \end{cases} = \begin{cases} \delta A & \text{if Condition x holds} \\ \delta B & \text{if Condition y holds} \end{cases}$$

(xi) $\sum_{i=1}^{n} {}^{\bullet}\, \alpha_i = \alpha_1 \pm \alpha_2 \pm \ldots\ldots \pm \alpha_n$

(xii) $\eta_{1n} = \sum_{i=1}^{n} {}^{\bullet}$

(xiii) $\Pi(1,K) = \prod_{i=1}^{n}$

(xiv) $\alpha_i + \beta_j = \lambda$, $\lambda = 0, 1, \ldots [\frac{m_k + n_l}{2}]$

$\alpha_i = 0, 1, \ldots [\frac{m_i}{2}]$; $\beta_j = 0, 1, \ldots [\frac{n_j}{2}]$; $i = 1, 2, \ldots k$; $j = 1, 2, \ldots l$

Theorem-1: If m_1 is an even positive integer, $n_1 \in N$ and $x_1, y_1 \in R$ then

$$\$ (m_1, x_1) \not\subset (n_1, y_1) = \begin{cases} \delta \not\subset^{+} [\tau_1 x_1 \pm \mu_1 y_1] & \text{if } n_1 = \text{odd} \\ \delta \not\subset^{+} [\tau_1 x_1 \pm \mu_1 y_1] + K & \text{if } n_1 = \text{even} \end{cases} \quad \ldots\ldots (1.1)$$

where $A_{\alpha_1\beta_1} = \frac{(-1)^{\alpha_1 + m_1/2}}{2^{m_1+n_1-1}}\, {}^{m_1}C_{\alpha_1}\, {}^{n_1}C_{\beta_1}$ and $K = (-1/2)\, A_{[\frac{m_1}{2}][\frac{n_1}{2}]}$

$\alpha_1 = 0, 1, \ldots [\frac{m_1}{2}]$; $\beta_1 = 0, 1, \ldots [\frac{n_1}{2}]$

Corollary-1: If $y_1 = x_1$, then (1.1) will reduce to the form

$$\$\,(m_1, x_1) \not\subset (n_1, x_1) = \begin{cases} \sum_{\lambda=0}^{[\frac{m_1+n_1}{2}]} A_\lambda \not\subset [m_1+n_1-2\lambda]\, x_1 & \text{if } m_1 = \text{even},\ n_1 = \text{odd} \\ \sum_{\lambda=0}^{[\frac{m_1+n_1}{2}]} A_\lambda \not\subset [m_1+n_1-2\lambda]\, x_1 + K' & \text{if } m_1 = \text{even},\ n_1 = \text{even} \end{cases} \quad \text{...... (1.2)}$$

where $A_\lambda = \sum_{\alpha_1=0}^{[\frac{m_1+n_1}{2}]} \frac{(-1)^{\alpha_1+m_1/2}}{2^{m_1+n_1-1}}\ {}^{m_1}C_{\alpha_1}\,{}^{n_1}C_{\lambda-\alpha_1}$ and $K' = (-1/2)\, A_{[\frac{m_1+n_1}{2}]}$; $\lambda = 0, 1, \dots [\frac{m_1+n_1}{2}]$

Theorem-2: If m_1 is an odd positive integer, $n_1 \in N$ and $x_1, y_1 \in R$ then

$$\$\,(m_1, x_1) \not\subset (n_1, y_1) = \delta\, \$^{+} [\tau_1 x_1 \pm \mu_1 y_1] \quad \text{...... (1.3)}$$

where $A_{\alpha_1\beta_1} = \frac{(-1)^{\alpha_1+(m_1-1)/2}}{2^{m_1+n_1-1}}\ {}^{m_1}C_{\alpha_1}\,{}^{n_1}C_{\beta_1}$; $\alpha_1 = 0, 1, \dots [\frac{m_1}{2}]$; $\beta_1 = 0, 1, \dots [\frac{n_1}{2}]$

Corollary-2: If $y_1 = x_1$, then (1.3) will reduce to the form

$$\$\,(m_1, x_1) \not\subset (n_1, x_1) = \sum_{\lambda=0}^{[\frac{m_1+n_1}{2}]} A_\lambda\, \$\, [m_1+n_1-2\lambda]\, x_1 \quad \text{...... (1.4)}$$

where $\mathbf{A}_\lambda = \sum_{\alpha_1=0}^{[\frac{m_1+n_1}{2}]} \frac{(-1)^{\alpha_1+(m_1-1)/2}}{2^{m_1+n_1-1}}\ {}^{m_1}C_{\alpha_1}\,{}^{n_1}C_{\lambda-\alpha_1}$; $\lambda = 0, 1, \dots [\frac{m_1+n_1}{2}]$

Theorem-3: If $m_1, n_1 \in N$ and $x_1, y_1 \in R$ then

$$\not\subset (m_1, x_1) \not\subset (n_1, y_1) = \begin{cases} \delta \not\subset^{+} [\tau_1 x_1 \pm \mu_1 y_1] + K & \text{if } m_1, n_1 \text{ both are even} \\ \delta \not\subset^{+} [\tau_1 x_1 \pm \mu_1 y_1] & \text{otherwise} \end{cases} \quad \text{...... (1.5)}$$

where $A_{\alpha_1\beta_1} = \frac{1}{2^{m_1+n_1-1}}\ {}^{m_1}C_{\alpha_1}\,{}^{n_1}C_{\beta_1}$ and $K = (-1/2)\, A_{[\frac{m_1}{2}][\frac{n_1}{2}]}$

$\alpha_1 = 0, 1, \dots [\frac{m_1}{2}]$; $\beta_1 = 0, 1, \dots [\frac{n_1}{2}]$

Corollary-3: If $y_1 = x_1$, then (1.5) will reduce to the form

$$\not\subset (m_1, x_1) \not\subset (n_1, x_1) = \not\subset (m_1+n_1, x_1) = \begin{cases} \sum_{\lambda=0}^{[\frac{m_1+n_1}{2}]} A_\lambda \not\subset [m_1+n_1-2\lambda]\, x_1 + K' & \text{if } m_1+n_1 = \text{even} \\ \sum_{\lambda=0}^{[\frac{m_1+n_1}{2}]} A_\lambda \not\subset [m_1+n_1-2\lambda]\, x_1 & \text{if } m_1+n_1 = \text{odd} \end{cases} \quad \text{...... (1.6)}$$

where $\mathbf{A}_\lambda = \frac{1}{2^{m_1+n_1-1}} \; {}^{m_1+n_1}C_\lambda$ and $K' = (-1/2)\, A_{[\frac{m_1+n_1}{2}]}$; $\lambda = 0, 1, \ldots [\frac{m_1+n_1}{2}]$

Theorem-4: If $m_1, n_1 \in N$ and $x_1, y_1 \in R$ then

$$\$ (m_1, x_1)\, \$ (n_1, y_1) = \delta\, \$^+ [\tau_1 x_1 \pm \mu_1 y_1] \qquad \text{if } m_1+n_1 = \text{odd} \qquad \text{...... (1.7)}$$

Keeping m_1 odd, similarly for n_1 = odd

where $A_{\alpha_1\beta_1} = \frac{(-1)^{\alpha_1+\beta_1+(m_1+n_1-1)/2}}{2^{m_1+n_1-1}} \; {}^{m_1}C_{\alpha_1}\, {}^{n_1}C_{\beta_1}$; $\alpha_1 = 0, 1, \ldots [\frac{m_1}{2}]$; $\beta_1 = 0, 1, \ldots [\frac{n_1}{2}]$

Corollary-4: If $y_1 = x_1$, then (1.7) will reduce to the form

$$\$ (m_1, x_1)\, \$ (n_1, x_1) = \$ (m_1+n_1, x_1) = \sum_{\lambda=0}^{[\frac{m_1+n_1}{2}]} A_\lambda\, \$ [m_1+n_1-2\lambda]\, x_1 \qquad \text{...... (1.8)}$$

where $\mathbf{A}_\lambda = \frac{(-1)^{\lambda+(m_1+n_1-1)/2}}{2^{m_1+n_1-1}} \; {}^{m_1+n_1}C_\lambda$; $\lambda = 0, 1, \ldots [\frac{m_1+n_1}{2}]$

Theorem-5: If m_1, n_1 are even positive integers and $x_1, y_1 \in R$ then

$$\$ (m_1, x_1)\, \$ (n_1, y_1) = \delta \not\subset^+ [\tau_1 x_1 \pm \mu_1 y_1] + K \qquad \text{...... (1.9)}$$

where $A_{\alpha_1\beta_1} = \frac{(-1)^{\alpha_1+\beta_1+(m_1+n_1)/2}}{2^{m_1+n_1-1}} \; {}^{m_1}C_{\alpha_1}\, {}^{n_1}C_{\beta_1}$ and $K = (-1/2)\, A_{[\frac{m_1}{2}][\frac{n_1}{2}]}$

$\alpha_1 = 0, 1, \ldots [\frac{m_1}{2}]$; $\beta_1 = 0, 1, \ldots [\frac{n_1}{2}]$

Corollary-5: If $y_1 = x_1$, then (1.9) will reduce to the form

$$\$ (m_1, x_1) \$ (n_1, x_1) = \$ (m_1+n_1, x_1) = \sum_{\lambda=0}^{[\frac{m_1+n_1}{2}]} A_\lambda \not\subset [m_1+n_1-2\lambda]\, x_1 + K' \qquad ...... (2.0)$$

where $\mathbf{A}_\lambda = \frac{(-1)^{\lambda+(m_1+n_1)/2}}{2^{m_1+n_1-1}}\ {}^{m_1+n_1}C_\lambda$ and $K' = (-1/2)\, A_{[\frac{m_1+n_1}{2}]}$; $\lambda = 0, 1, \ldots [\frac{m_1+n_1}{2}]$

Theorem-6: If m_1, n_1 are both odd positive integers and $x_1, y_1 \in R$ then

$$\$ (m_1, x_1) \$ (n_1, y_1) = \delta \not\subset^{-} \overline{[\tau_1 x_1 + \mu_1 y_1]} \qquad ...... (2.1)$$

where $A_{\alpha_1\beta_1} = \frac{(-1)^{\alpha_1+\beta_1+(m_1+n_1)/2}}{2^{m_1+n_1-1}}\ {}^{m_1}C_{\alpha_1}\, {}^{n_1}C_{\beta_1}$; $\alpha_1 = 0, 1, \ldots [\frac{m_1}{2}]$; $\beta_1 = 0, 1, \ldots [\frac{n_1}{2}]$

Corollary-6: If $y_1 = x_1$, then (2.1) will reduce to the form

$$\$ (m_1, x_1) \$ (n_1, x_1) = \$ (m_1+n_1, x_1) = \sum_{\lambda=0}^{[\frac{m_1+n_1}{2}]} A_\lambda \not\subset [m_1+n_1-2\lambda]\, x_1 + K' \qquad ...... (2.2)$$

where $\mathbf{A}_\lambda = \frac{(-1)^{\lambda+(m_1+n_1)/2}}{2^{m_1+n_1-1}}\ {}^{m_1+n_1}C_\lambda$ and $K' = (-1/2)\, A_{[\frac{m_1+n_1}{2}]}$; $\lambda = 0, 1, \ldots [\frac{m_1+n_1}{2}]$

Theorem-7: If m_i's are even, $m_i, n_j \in N$ and $x_i, y_j \in R$ then

$$\Pi (1,K) \$ (m_i, x_i)\, \Pi (1,l) \not\subset (n_j, y_j) = \begin{cases} \delta \not\subset^{+} [\eta_{1k}\, \tau_i x_i \pm \eta_{1l}\, \mu_j y_j] & \text{if } \sum_{j=1}^{l} n_j = \text{odd} \\ \delta \not\subset^{+} [\eta_{1k}\, \tau_i x_i \pm \eta_{1l}\, \mu_j y_j] + K & \text{if } \sum_{j=1}^{l} n_j = \text{even} \end{cases} \qquad ...... (2.3)$$

where $A_{\alpha_1\ldots\alpha_k;\, \beta_1\ldots\beta_l} = \frac{(-1)^{\sum_{i=1}^{k} \alpha_i + (\sum_{i=1}^{k} m_i)/2}}{2^{\sum_{i=1}^{k} m_i + \sum_{j=1}^{l} n_j - 1}}\ \Pi (1,K)\, {}^{m_i}C_{\alpha_i}\, \Pi (1,l)\, {}^{n_j}C_{\beta_j}$

and $K = (-1/2)\, A_{[\frac{m_1}{2}]\ldots[\frac{m_k}{2}];\, [\frac{n_1}{2}]\ldots[\frac{n_l}{2}]}$; $\alpha_i = 0, 1, \ldots [\frac{m_i}{2}]$; $\beta_j = 0, 1, \ldots [\frac{n_j}{2}]$;

$i = 0, 1, \ldots k$; $j = 0, 1, \ldots l$

Corollary-7: If $x_i = x$ and $y_j = y$ then (2.3) will reduce to the simple form

$$\$\,(\Sigma m_i, x) \not\subset (\Sigma n_j, y) = \$\,(M, x) \not\subset (N, y) = \$\,(2k, x) \not\subset (N, y), \quad k \in N \qquad ..... (2.4)$$

Then the result immediately can be verified as a sum of multiple angles of Cosine series by the known theorem (1) and hence follows the respective corollary.

Theorem-8: If $\sum_{i=1}^{k} m_i$ is odd, $m_i, n_j \in N$ and $x_i, y_j \in R$ then

$$\Pi\,(1,K)\ \$\,(m_i, x_i)\ \Pi\,(1,l) \not\subset (n_j, y_j) = \delta \not\subset^{+} [\eta_{1k}\ \tau_i\ x_i \pm \eta_{1l}\ \mu_j\ y_j] \qquad ...... (2.5)$$

where $A_{\alpha_1 \ldots \alpha_k;\, \beta_1 \ldots \beta_l} = \dfrac{(-1)^{\sum_{i=1}^{k} \alpha_i + (\sum_{i=1}^{k} m_i - 1)/2}}{2^{\sum_{i=1}^{k} m_i + \sum_{j=1}^{l} n_j - 1}}\ \Pi\,(1,K)\ {}^{m_i}C_{\alpha_i}\ \Pi\,(1,l)\ {}^{n_j}C_{\beta_j}$

$\alpha_i = 0, 1, \ldots. [\frac{m_i}{2}]$; $\beta_j = 0, 1, \ldots. [\frac{n_j}{2}]$; $i = 0, 1, \ldots. k$; $j = 0, 1, \ldots l$

Corollary-8: If $x_i = x$ and $y_j = y$ then (2.5) will reduce to the simple form

$$\$\,(\Sigma m_i, x) \not\subset (\Sigma n_j, y) = \$\,(M, x) \not\subset (N, y) = \$\,(2k+1, x) \not\subset (N, y), \quad k \in N \qquad ..... (2.6)$$

Then the result immediately can be verified as a sum of multiple angles of Sine series by the known theorem (2) and hence follows the respective corollary.

Theorem-9: If $\sum_{i=1}^{k} m_i$ is even with at least a pair of m_i is odd,

then for $m_i, n_j \in N$ and $x_i, y_j \in R$

[Assuming (m_P, m_{P+1}) pair is odd in power of Sine for $1 < P < k$ and $1 < P+1 < k$, implies rest sum Σm_i is even]

$$\Pi\,(1,K)\ \$\,(m_i, x_i)\ \Pi\,(1,l) \not\subset (n_j, y_j) = \delta \not\subset^{-} [\overline{\tau_P\ x_P + \tau_{P+1}\ x_{P+1}} \pm \eta_{1k}\ \tau_i\ x_i \pm \eta_{1l}\ \mu_j\ y_j] \qquad ...... (2.7)$$

(η_{1k} except $_{P}$, $_{P+1}$ term) where $A_{\alpha_1\ldots\alpha_k;\,\beta_1\ldots\beta_l} = \dfrac{(-1)^{\sum_{i=1}^{k}\alpha_i + (\sum_{i=1}^{k} m_i)/2}}{2^{\sum_{i=1}^{k} m_i + \sum_{j=1}^{l} n_j - 1}} \;\Pi\,(1,K)\; {}^{m_i}C_{\alpha_i}\; \Pi\,(1,l)\; {}^{n_j}C_{\beta_j}$

$\alpha_i = 0, 1, \ldots \left[\frac{m_i}{2}\right]; \beta_j = 0, 1, \ldots \left[\frac{n_j}{2}\right]; i = 0, 1, \ldots k; j = 0, 1, \ldots l$

Corollary-9: If $x_i = x$ and $y_j = y$ then (2.7) will reduce to the simple form

$\$\,(\Sigma m_i, x) \not\subset (\Sigma n_j, y) = \$\,(M, x) \not\subset (N, y) = \$\,(2k, x) \not\subset (N, y), \quad k \in N$ ..... (2.8)

Then the result immediately can be verified as a sum of multiple angles of Cosine series by the known theorem (1) and hence follows the respective corollary.

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