Stochastic Modeling and Applications
Vol. 22 No.2 (December, 2018)

INTERPOLATION PROBLEM FOR MULTIDIMENSIONAL STATIONARY SEQUENCES WITH MISSING OBSERVATIONS

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Abstract. The problem of the mean-square optimal estimation of the linear functionals which depend on the unknown values of a stochastic stationary sequence from observations of the sequence with missings is considered. Formulas for calculating the mean-square error and the spectral characteristic of the optimal linear estimate of the functionals are derived under the condition of spectral certainty, where the spectral density of the sequence is exactly known. The minimax (robust) method of estimation is applied in the case where the spectral density of the sequence is not known exactly while some sets of admissible spectral densities are given. Formulas that determine the least favourable spectral densities and the minimax spectral characteristics are derived for some special sets of admissible densities.

1. Introduction

The problem of estimation of the unknown values of stochastic processes is of constant interest in the theory and applications of stochastic processes. The formulation of the interpolation, extrapolation and filtering problems for stationary stochastic sequences with known spectral densities and reducing them to the corresponding problems of the theory of functions belongs to A. N. Kolmogorov [17]. Effective methods of solution of the estimation problems for stationary stochastic sequences and processes were developed by N. Wiener [42] and A. M. Yaglom [43, 44]. Further results are presented in the books by Yu. A. Rozanov [39] and E. J. Hannan [12]. The crucial assumption of most of the methods developed for estimating the unobserved values of stochastic processes is that the spectral densities of the involved stochastic processes are exactly known. However, in practice complete information on the spectral densities is impossible in most cases. In this situation one finds parametric or nonparametric estimate of the unknown spectral density and then apply one of the traditional estimation methods provided that the selected density is the true one. This procedure can result in significant increasing of the value of error of estimate as K. S. Vastola and H. V. Poor [41] have demonstrated with the help of some examples. To avoid this effect one can search estimates which are optimal for all densities from a certain class of admissible spectral densities. These estimates are called minimax since they minimize the maximum value of the error. The paper by Ulf Grenander [11] was the first one where this

2000 *Mathematics Subject Classification.* Primary 60G10, 60G25, 60G35; Secondary 62M20, 93E10, 93E11.

Key words and phrases. Stationary sequence, mean square error, minimax-robust estimate, least favourable spectral density, minimax spectral characteristic.

approach to extrapolation problem for stationary processes was proposed. Several models of spectral uncertainty and minimax-robust methods of data processing can be found in the survey paper by S. A. Kassam and H. V. Poor [16]. J. Franke [7], [8], J. Franke and H. V. Poor [9] investigated the minimax extrapolation and filtering problems for stationary sequences with the help of convex optimization methods. This approach makes it possible to find equations that determine the least favorable spectral densities for different classes of densities. In the papers by M. P. Moklyachuk [24] - [27] the problems of extrapolation, interpolation and filtering for functionals which depend on the unknown values of stationary processes and sequences are investigated. The estimation problems for functionals which depend on the unknown values of multivariate stationary stochastic processes is the aim of the book by M. Moklyachuk and O. Masytka [29]. Results of investigations of the interpolation, extrapolation and filtering problems for periodically correlated stochastic sequences presented in the book by M. P. Moklyachuk and I. I. Golichenko [28]. In their papers M. M. Luz and M. P. Moklyachuk [19] - [23] deal with the estimation problems for functionals which depend on the unknown values of stochastic sequences with stationary increments. Prediction problem for stationary sequences with missing observations is investigated in papers by Bondon [1, 2], Cheng, Miamee and Pourahmadi [5], Cheng and Pourahmadi [6], Kasahara, Pourahmadi and Inoue [15], Pourahmadi, Inoue and Kasahara [36], Pelagatti [35]. In papers by Moklyachuk and Sidei [30] - [34] an approach is developed to investigation of the interpolation, extrapolation and filtering problems for stationary stochastic processes with missing observations.

In this article we consider the problem of the mean-square optimal estimation of the linear functional $A_s\vec{\xi} = \sum\limits_{l=0}^{s-1} \sum\limits_{j=M_l}^{M_l+N_{l+1}} \vec{a}(j)^\top \vec{\xi}(j)$, $M_l = \sum\limits_{k=0}^{l} (N_k + K_k)$, $N_0 = K_0 = 0$, which depends on the unknown values of a stochastic stationary sequence $\vec{\xi}(j) = \{\xi_k(j)\}_{k=1}^{T}$, $j \in \mathbb{Z}$ from observations of the sequence at points $j \in \mathbb{Z}\backslash S$, where $S = \bigcup\limits_{l=0}^{s-1} \{M_l, M_l + 1, \dots, M_l + N_{l+1}\}$. The problem is investigated in the case of spectral certainty, where the spectral density of the sequence is exactly known, and in the case of spectral uncertainty, where the spectral density of the sequence is unknown while a class of admissible spectral densities is given.

2. The Hilbert space projection method of linear interpolation

Let $\vec{\xi}(j) = \{\xi_k(j)\}_{k=1}^{T}$, $j \in \mathbb{Z}$, be a (wide sense) multidimensional stationary stochastic sequence with zero mean values, $E\vec{\xi}(j) = \vec{0}$, and with the correlation function $R(n) = E\vec{\xi}(j+n)(\vec{\xi}(j))^* = \left\{E\xi_k(j+n)\overline{\xi_l(j)}\right\}_{k,l=1}^{T}$ which admit the spectral decomposition (see Gikhman and Skorokhod [10])

$$R(n) = \frac{1}{2\pi} \int\limits_{-\pi}^{\pi} e^{ik\lambda} F(\lambda) d\lambda,$$

where $F(\lambda)=\{f_{kl}(\lambda)\}_{k,l=1}^{T}$ is the spectral density function of the sequence $\vec{\xi}(j)$ that satisfies the minimality condition

$$\int_{-\pi}^{\pi} \mathrm{Tr}\,\left[(F(\lambda))^{-1}\right]\,d\lambda<\infty. \tag{2.1}$$

This condition is necessary and sufficient in order that the error-free interpolation of unknown values of the sequence is impossible [39].

The stationary stochastic sequence $\vec{\xi}(j)$, $j\in\mathbb{Z}$, admits the spectral decomposition [10, 14]

$$\vec{\xi}(j)=\int\limits_{-\pi}^{\pi} e^{ij\lambda}Z(d\lambda), \tag{2.2}$$

where $Z(\Delta)$ is the vector- valued orthogonal stochastic measure of the sequence such that

$$EZ(\Delta_1)(Z(\Delta_2))^{*}=\frac{1}{2\pi}\int_{\Delta_1\cap\Delta_2}F(\lambda)d\lambda.$$

Consider the problem of the mean-square optimal estimation of the linear functional

$$A_s\vec{\xi}=\sum_{l=0}^{s-1}\sum_{j=M_l}^{M_l+N_{l+1}}\vec{a}(j)^{\top}\vec{\xi}(j),\ M_l=\sum_{k=0}^{l}(N_k+K_k),\ N_0=K_0=0,$$

which depends on the unknown values of a stochastic stationary sequence $\vec{\xi}(j), j\in\mathbb{Z}$, from observations of the sequence at points $j\in\mathbb{Z}\backslash S$, where $S=\bigcup\limits_{l=0}^{s-1}\{M_l,M_l+1,\dots,M_l+N_{l+1}\}$.

It follows from the spectral decomposition (2.2) of the sequence $\vec{\xi}(j)$ that we can represent the functional $A_s\vec{\xi}$ in the following form

$$A_s\vec{\xi}=\int\limits_{-\pi}^{\pi}(A_s(e^{i\lambda}))^{\top}Z(d\lambda), \tag{2.3}$$

where

$$A_s(e^{i\lambda})=\sum_{l=0}^{s-1}\sum_{j=M_l}^{M_l+N_{l+1}}\vec{a}(j)e^{ij\lambda}.$$

We will consider values $\xi_k(j), k=1,\dots,T, j\in\mathbb{Z}$, of the sequence $\vec{\xi}(j)$ as elements of the Hilbert space $H=L_2(\Omega,\mathcal{F},P)$ generated by random variables ξ with zero mathematical expectations, $E\xi=0$, finite variations, $E|\xi|^2<\infty$, and the inner product $(\xi,\eta)=E(\xi\overline{\eta})$. Denote by $H^s(\xi)$ the subspace of the Hilbert space $H=L_2(\Omega,\mathcal{F},P)$ generated by elements $\{\xi_k(j): j\in\mathbb{Z}\backslash S, k=\overline{1,T}\}$.

Denote by $L_2(F)$ the Hilbert space of vector-valued functions $\vec{a}(\lambda)=\{a_k(\lambda)\}_{k=1}^{T}$ such that

$$\int_{-\pi}^{\pi}\vec{a}(\lambda)^{\top}F(\lambda)\overline{\vec{a}(\lambda)}d\lambda=\int_{-\pi}^{\pi}\sum_{k,l=1}^{T}a_k(\lambda)\overline{a_l(\lambda)}\,f_{kl}(\lambda)\,d\lambda<\infty.$$

Denote by $L_2^s(F)$ the subspace of $L_2(F)$ generated by functions of the form

$$e^{in\lambda}\delta_k,\ \delta_k=\{\delta_{kl}\}_{l=1}^{T},\ k=1,\dots,T,\ n\in Z\backslash S.$$

The mean square optimal linear estimate $\hat{A}_s\vec{\xi}$ of the functional $A_s\vec{\xi}$ from observations of the sequence $\vec{\xi}(j)$ at points $j\in\mathbb{Z}\backslash S$ is an element of the $H^s(\xi)$. It can be represented in the form

$$\hat{A}_s\vec{\xi}=\int\limits_{-\pi}^{\pi}(h(e^{i\lambda}))^{\top}Z(d\lambda), \tag{2.4}$$

where $h(e^{i\lambda})=\left\{h_k(e^{i\lambda})\right\}_{k=1}^{T}\in L_2^s(F)$ is the spectral characteristic of the estimate $\hat{A}_s\vec{\xi}$.

The mean square error $\Delta(h;F)$ of the estimate $\hat{A}_s\vec{\xi}$ is given by the formula

$$\Delta(h;F)=E\left|A_s\vec{\xi}-\hat{A}_s\vec{\xi}\right|^2=\frac{1}{2\pi}\int\limits_{-\pi}^{\pi}(A_s(e^{i\lambda})-h(e^{i\lambda}))^{\top}F(\lambda)\overline{(A_s(e^{i\lambda})-h(e^{i\lambda}))}d\lambda.$$

The Hilbert space projection method proposed by A. N. Kolmogorov [17] makes it possible to find the spectral characteristic $h(e^{i\lambda})$ and the mean square error $\Delta(h;F)$ of the optimal linear estimate of the functional $A_s\vec{\xi}$ in the case where the spectral density $F(\lambda)$ of the sequence is exactly known and the minimality condition (2.1) is satisfied. The spectral characteristic can be found from the following conditions:

$$1) h(e^{i\lambda})\in L_2^s(F),$$
$$2) A_s(e^{i\lambda})-h(e^{i\lambda})\perp L_2^s(F).$$

It follows from the second condition that for any $\eta\in H^s(\xi)$ the following equations should be satisfied

$$\left(A_s\vec{\xi}-\hat{A}_s\vec{\xi},\eta\right)=E\left[(A_s\vec{\xi}-\hat{A}_s\vec{\xi})\overline{\eta}\right]=0$$

The last relation is equivalent to equations

$$E\left[(A_s\vec{\xi}-\hat{A}_s\vec{\xi})\overline{\xi_k(j)}\right]=0,\ \ j\in\mathbb{Z}\backslash S, k=\overline{1,T}.$$

By using representations (2.3), (2.4) and definition of the inner product in the space H we get

$$\int\limits_{-\pi}^{\pi}\left(A_s(e^{i\lambda})-h(e^{i\lambda})\right)^{\top}F(\lambda)e^{-ik\lambda}d\lambda=0,\ k\in\mathbb{Z}\backslash S.$$

It follows from this condition that the function $(A_s(e^{i\lambda})-h(e^{i\lambda}))^{\top}F(\lambda)$ is of the form

$$(A_s(e^{i\lambda})-h(e^{i\lambda}))^{\top}F(\lambda)=(C_s(e^{i\lambda}))^{\top},$$
$$C_s(e^{i\lambda})=\sum_{l=0}^{s-1}\sum_{j=M_l}^{M_l+N_{l+1}}\vec{c}(j)e^{ij\lambda},$$

where $\vec{c}(j), j\in S$ are unknown coefficients that we have to find.

From the last relation we deduce that the spectral characteristic $h(e^{i\lambda})$ of the optimal linear estimate of the functional $A_s\vec{\xi}$ is of the form

$$h(e^{i\lambda}) = A_s(e^{i\lambda}) - (F^{-1}(\lambda))^{\top} C_s(e^{i\lambda}). \tag{2.5}$$

To find equations for calculation of the unknown coefficients $\vec{c}(j), j \in S$, we use the decomposition of the function $(F^{-1}(\lambda))^{\top}$ into the Fourier series

$$(F^{-1}(\lambda))^{\top} = \sum_{m=-\infty}^{\infty} B(m)e^{im\lambda}, \tag{2.6}$$

where $B(m)$ are the Fourier coefficients of the function $(F^{-1}(\lambda))^{\top}$.

Inserting (2.6) into (2.5) we obtain the following representation of the spectral characteristic

$$h(e^{i\lambda}) = \left(\sum_{l=0}^{s-1}\sum_{j=M_l}^{M_l+N_{l+1}} \vec{a}(j)e^{ij\lambda}\right) - \\ - \left(\sum_{m=-\infty}^{\infty} B(m)e^{im\lambda}\right)\left(\sum_{l=0}^{s-1}\sum_{j=M_l}^{M_l+N_{l+1}} \vec{c}(j)e^{ij\lambda}\right). \tag{2.7}$$

It follows from the first condition, $h(e^{i\lambda}) \in L_2^s(F)$, that the Fourier coefficients of the function $h(e^{i\lambda}) = \left\{h_k(e^{i\lambda})\right\}_{k=1}^{T}$ are equal to zero for $j \in S$, namely

$$\int_{-\pi}^{\pi} h_k(e^{i\lambda})e^{-ij\lambda}d\lambda = 0,\ j \in S.$$

Using the last relations and (2.7) we get the following system of equations that determine the unknown coefficients $\vec{c}(j), j \in S$,

$$\begin{aligned} \vec{a}(M_{k-1}) - &\sum_{l=0}^{s-1}\sum_{j=M_l}^{M_l+N_{l+1}} B(M_{k-1}-j)\vec{c}(j) = 0; \\ \vec{a}(M_{k-1}+1) - &\sum_{l=0}^{s-1}\sum_{j=M_l}^{M_l+N_{l+1}} B(M_{k-1}+1-j)\vec{c}(j) = 0; \\ &\dots \\ \vec{a}(M_{k-1}+N_k) - &\sum_{l=0}^{s-1}\sum_{j=M_l}^{M_l+N_{l+1}} B(M_{k-1}+N_k-j)\vec{c}(j) = 0, \end{aligned} \tag{2.8}$$

where $k = 1, \dots, s$.

Denote by $\mathbf{\vec{a}}_s^{\top} = (\vec{a}_1^{\top}, \vec{a}_2^{\top}, \dots, \vec{a}_s^{\top})$, $\vec{a}_k^{\top} = (\vec{a}(M_{k-1})^{\top}, \dots, \vec{a}(M_{k-1}+N_k)^{\top})$, $k = 1, \dots, s$, $q = N_1 + N_2 + \dots + N_s + s$. Let B_s be a $q \times q$ matrix

$$B_s = \begin{pmatrix} B_{11} & B_{12} & \dots & B_{1s} \\ B_{21} & B_{22} & \dots & B_{2s} \\ \vdots & \vdots & \ddots & \vdots \\ B_{s1} & B_{s2} & \dots & B_{ss} \end{pmatrix},$$

where B_{mn} are $(N_m + 1) \times (N_n + 1)$ compound matrices constructed of the block-matrices of dimension $T \times T$ that are Fourier coefficients of the function $(F^{-1}(\lambda))^{\top}$:

$$B_{mn}(k,j) = \frac{1}{2\pi}\int_{-\pi}^{\pi}(F^{-1}(\lambda))^{\top}e^{-i(k-j)\lambda}d\lambda = B(k-j),$$

$$k = M_{m-1},\dots,\, M_{m-1}+N_m,$$
$$j = M_{n-1},\dots,\, M_{n-1}+N_n,$$
$$m,n = 1,\dots,s.$$

Making use the introduced notations we can write formulas (2.8) in the form of equation

$$\vec{\mathbf{a}}_s = B_s\vec{\mathbf{c}}_s, \tag{2.9}$$

where $\vec{\mathbf{c}}_s^{\top} = (\vec{c}_1^{\top}, \vec{c}_2^{\top},\, \dots,\, \vec{c}_s^{\top})$, $\vec{c}_k^{\top} = (\vec{c}(M_{k-1})^{\top},\dots,\vec{c}(M_{k-1}+N_k)^{\top})$, $k = 1,\dots,s$, is a vector constructed from the unknown coefficients $\vec{c}(j), j \in S$. Since the matrix B_s is reversible [36], we get the formula

$$\vec{\mathbf{c}}_s = B_s^{-1}\vec{\mathbf{a}}_s. \tag{2.10}$$

Hence, the unknown coefficients $\vec{c}(j), j \in S$, are calculated by the formula

$$\vec{c}(j) = \left(B_s^{-1}\vec{\mathbf{a}}_s\right)(j),$$

where $\left(B_s^{-1}\vec{\mathbf{a}}_s\right)(j)$ is the j-th component of the vector $B_s^{-1}\vec{\mathbf{a}}_s$, and the formula for calculating the spectral characteristic of the estimate $\hat{A}_s\xi$ is of the form

$$h(e^{i\lambda}) = \left(\sum_{l=0}^{s-1}\sum_{j=M_l}^{M_l+N_{l+1}}\vec{a}(j)e^{ij\lambda}\right) - \\ - \left(\sum_{m=-\infty}^{\infty}B(m)e^{im\lambda}\right)\left(\sum_{l=0}^{s-1}\sum_{j=M_l}^{M_l+N_{l+1}}\left(B_s^{-1}\vec{\mathbf{a}}_s\right)(j)e^{ij\lambda}\right). \tag{2.11}$$

The mean square error of the estimate of the function can be calculated by the formula

$$\Delta(h;F) = \frac{1}{2\pi}\int_{-\pi}^{\pi}(C_s(e^{i\lambda}))^{\top}F^{-1}(\lambda)\overline{C_s(e^{i\lambda})}d\lambda = \\ = \int_{-\pi}^{\pi}\left(\sum_{l=0}^{s-1}\sum_{k=M_l}^{M_l+N_{l+1}}\vec{c}(k)e^{ik\lambda}\right)^{\top}\left(\sum_{m=-\infty}^{\infty}B(m)e^{im\lambda}\right)\left(\sum_{l=0}^{s-1}\sum_{j=M_l}^{M_l+N_{l+1}}\overline{\vec{c}(j)}e^{-ij\lambda}\right)d\lambda \\ = \langle\vec{\mathbf{c}}_s, B_s\vec{\mathbf{c}}_s\rangle = \left\langle B_s^{-1}\vec{\mathbf{a}}_s, \vec{\mathbf{a}}_s\right\rangle, \tag{2.12}$$

where $\langle\cdot,\cdot\rangle$ is the inner product.
Let us summarize our results and present them in the form of a theorem.

Theorem 2.1. Let $\vec{\xi}(j) = \{\xi_k(j)\}_{k=1}^{T}$ be a multidimensional stationary stochastic sequence with the spectral density $F(\lambda)$ that satisfies the minimality condition (2.1). The spectral characteristic $h(e^{i\lambda})$ and the mean square error $\Delta(h,F)$ of the optimal linear estimate $\hat{A}_s\vec{\xi}$ of the functional $A_s\vec{\xi}$ from observations of the sequence $\vec{\xi}(j)$ at points $j \in \mathbb{Z}\backslash S$, where $S = \bigcup_{l=0}^{s-1}\{M_l, \dots, M_l + N_{l+1}\}$, can be calculated by formulas (2.11), (2.12).

Example 1. Consider the problem of linear interpolation of the functional $A_2\vec{\xi} = \vec{a}(0)^\top\vec{\xi}(0) + \vec{a}(n)^\top\vec{\xi}(n)$, $n > 1$, which depends on the unknown values $\vec{\xi}(0)$, $\vec{\xi}(n)$ of the multidimensional stochastic sequence $\vec{\xi}(j)$ from observations at points $j \in \mathbb{Z}\backslash S$, where $S = \{0\} \cup \{n\}$. In this case the spectral characteristic (2.7) of the estimate $\hat{A}_2\vec{\xi}$ can be calculated by the formula

$$h(e^{i\lambda}) = (\vec{a}(0) + \vec{a}(n)e^{in\lambda}) - (F^{-1}(\lambda))^\top \cdot (\vec{c}(0) + \vec{c}(n)e^{in\lambda}), \tag{2.13}$$

where $F(\lambda)$ is a known spectral density, the function $(F^{-1}(\lambda))^\top$ admits the decomposition $(F^{-1}(\lambda))^\top = \sum_{m=-\infty}^{\infty} B(m)e^{i\lambda m}$, and coefficients $\vec{c}(0), \vec{c}(n)$ are determined by the system of equations

$$\vec{a}(0) = B(0)\vec{c}(0) + B(-n)\vec{c}(n),$$
$$\vec{a}(n) = B(n)\vec{c}(0) + B(0)\vec{c}(n).$$

The matrix B_2 is of the form

$$B_2 = \begin{pmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{pmatrix},$$

$$B_{11} = B(0),\ B_{12} = B(-n),\ B_{21} = B(n),\ B_{22} = B(0).$$

Let $\vec{\mathbf{a}}_2^\top = (\vec{a}_1^\top, \vec{a}_2^\top)$, where $\vec{a}_1 = \vec{a}(0)$, $\vec{a}_2 = \vec{a}(n)$ and let $\vec{\mathbf{c}}_2^\top = (\vec{c}_1^\top, \vec{c}_2^\top)$, where $\vec{c}_1 = \vec{c}(0)$, $\vec{c}_2 = \vec{c}(n)$. In this case equations (2.9) and (2.10) can be rewritten as

$$\vec{\mathbf{a}}_2 = B_2\vec{\mathbf{c}}_2,$$
$$\vec{\mathbf{c}}_2 = B_2^{-1}\vec{\mathbf{a}}_2.$$

Consider this problem for the two-dimensional stationary sequence $\vec{\xi}(n) = \{\xi_k(n)\}_{k=1}^2$ where $\xi_1(n) = \xi(n)$ is a stationary stochastic sequence with the spectral density $f(\lambda)$, and $\xi_2(n) = \xi(n) + \eta(n)$, where $\eta(n)$ is an uncorrelated with $\xi(n)$ stationary stochastic sequence with the spectral density $g(\lambda)$. The matrix of spectral densities is of the form

$$F(\lambda) = \begin{pmatrix} f(\lambda) & f(\lambda) \\ f(\lambda) & f(\lambda) + g(\lambda) \end{pmatrix}.$$

Its determinant equals

$$D = |F(\lambda)| = f(\lambda)\, g(\lambda),$$

and the inverse matrix is as follows

$$F(\lambda)^{-1} = \begin{pmatrix} \frac{1}{f(\lambda)} + \frac{1}{g(\lambda)} & \frac{-1}{g(\lambda)} \\ \frac{-1}{g(\lambda)} & \frac{1}{g(\lambda)} \end{pmatrix}.$$

Let

$$f(\lambda) = \frac{1}{2\pi\ |1 - b_1 e^{i\lambda}|^2}, \quad g(\lambda) = \frac{1}{2\pi\ |1 - b_2 e^{i\lambda}|^2},\ b_1, b_2 \in R.$$

We have the matrix B_2:

$$B_2 = 2\pi \begin{pmatrix} 2+b_1^2+b_2^2 & -1-b_2^2 & 0 & 0 \\ -1-b_2^2 & 1+b_2^2 & 0 & 0 \\ 0 & 0 & 2+b_1^2+b_2^2 & -1-b_2^2 \\ 0 & 0 & -1-b_2^2 & 1+b_2^2 \end{pmatrix}.$$

Its determinant equals

$$D = (2\pi)^4\,(1+b_1^2)^2(1+b_2^2)^2.$$

The inverse matrix B_2^{-1} is equal to

$$\frac{1}{2\pi}\begin{pmatrix} \frac{1}{1+b_1^2} & \frac{1}{1+b_1^2} & 0 & 0 \\ \frac{1}{1+b_1^2} & \frac{2+b_1^2+b_2^2}{(1+b_1^2)(1+b_2^2)} & 0 & 0 \\ 0 & 0 & \frac{1}{1+b_1^2} & \frac{1}{1+b_1^2} \\ 0 & 0 & \frac{1}{1+b_1^2} & \frac{2+b_1^2+b_2^2}{(1+b_1^2)(1+b_2^2)} \end{pmatrix}.$$

The vector $\vec{\mathbf{c}}_2$ is as follows

$$\vec{\mathbf{c}}_2 = \frac{1}{2\pi}\left(\frac{\alpha+\beta}{1+b_1^2}, \frac{\alpha}{1+b_1^2}+\frac{(2+b_1^2+b_2^2)\beta}{(1+b_1^2)(1+b_2^2)}, \frac{\gamma+\delta}{1+b_1^2}, \frac{\gamma}{1+b_1^2}+\frac{(2+b_1^2+b_2^2)\delta}{(1+b_1^2)(1+b_2^2)}\right)^{\top}$$

where $\vec{a}(0) = (\alpha,\,\beta)^{\top}$, $\vec{a}(n) = (\gamma,\,\delta)^{\top}$. Thus, the spectral characteristic of the optimal estimate of the random variable $A_2\vec{\xi}$ is calculated by the formula

$$(h(e^{i\lambda}))^{\top} = (h_1(e^{i\lambda}),\, h_2(e^{i\lambda}))^{\top},$$

where

$$h_1(e^{i\lambda}) = \left(\frac{(\alpha+\beta)b_1}{1+b_1^2} - \frac{\beta b_2}{1+b_2^2}\right)(e^{i\lambda}+e^{-i\lambda})+$$
$$+\left(\frac{(\gamma+\delta)b_1}{1+b_1^2} - \frac{\delta b_2}{1+b_2^2}\right)(e^{i(n+1)\lambda}+e^{i(n-1)\lambda}),$$

$$h_2(e^{i\lambda}) = \frac{\beta b_2}{1+b_2^2}(e^{i\lambda}+e^{-i\lambda}) + \frac{\delta b_2}{1+b_2^2}(e^{i(n+1)\lambda}+e^{i(n-1)\lambda}).$$

The mean square error is of the form

$$\Delta(F) = \frac{(\alpha+\beta)^2+(\gamma+\delta)^2}{2\pi(1+b_1^2)} + \frac{\beta^2+\delta^2}{2\pi(1+b_2^2)}.$$

Example 2. Let all conditions of the previous example are satisfied. Consider the problem of linear interpolation of the functional

$$A_2\vec{\xi} = \vec{a}(0)^{\top}\vec{\xi}(0) + \vec{a}(1)^{\top}\vec{\xi}(1) + \vec{a}(n)^{\top}\vec{\xi}(n),$$

where $\vec{a}(0) = (\alpha_1,\beta_1)^{\top}$, $\vec{a}(1) = (\alpha_2,\beta_2)^{\top}$, $\vec{a}(n) = (\gamma,\delta)^{\top}$, $n>2$. In this case the matrix B_2 is of the form

$$B_2 = \begin{pmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{pmatrix},$$

where

$$B_{11} = \begin{pmatrix} B(0) & B(-1) \\ B(1) & B(0) \end{pmatrix},\; B_{12} = \begin{pmatrix} B(-n) \\ B(-n+1) \end{pmatrix},$$

$$B_{12} = \begin{pmatrix} B(n) & B(n-1) \end{pmatrix}, \, B_{22} = \begin{pmatrix} B(0) \end{pmatrix},$$

$$B(0) = 2\pi \begin{pmatrix} 2+b_1^2+b_2^2 & -1-b_2^2 \\ -1-b_2^2 & 1+b_2^2 \end{pmatrix}, \, B(1) = B(-1) = 2\pi \begin{pmatrix} -b_1-b_2 & b_2 \\ b_2 & -b_2 \end{pmatrix},$$

$$B(-n) = B(-n+1) = B(n) = B(n-1) = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}.$$

The inverse matrix B_2^{-1} is equal to

$$\frac{1}{2\pi}\begin{pmatrix} \frac{1+b_1^2}{A} & \frac{1+b_1^2}{A} & \frac{b_1}{A} & \frac{b_1}{A} & 0 & 0 \\ \frac{1+b_1^2}{A} & \frac{1+b_1^2}{A}+\frac{1+b_2^2}{B} & \frac{b_1}{A} & \frac{b_1}{A}+\frac{b_2}{B} & 0 & 0 \\ \frac{b_1}{A} & \frac{b_1}{A} & \frac{1+b_1^2}{A} & \frac{1+b_1^2}{A} & 0 & 0 \\ \frac{b_1}{A} & \frac{b_1}{A}+\frac{b_2}{B} & \frac{1+b_1^2}{A} & \frac{1+b_1^2}{A}+\frac{1+b_2^2}{B} & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{1}{1+b_1^2} & \frac{1}{1+b_1^2} \\ 0 & 0 & 0 & 0 & \frac{1}{1+b_1^2} & \frac{1}{1+b_1^2}+\frac{1}{1+b_2^2} \end{pmatrix},$$

where $A = 1+b_1^2+b_1^4$, $B = 1+b_2^2+b_2^4$. Then we will have $\vec{\mathbf{c}}_2^\top = (\vec{c}(0)^\top, \vec{c}(1)^\top, \vec{c}(n)^\top)$, where $\vec{c}(0) = (c_1(0), c_2(0))^\top, \vec{c}(1) = (c_1(1), c_2(1))^\top$,

$$c_1(0) = \frac{(1+b_1^2)(\alpha_1+\beta_1)+b_1(\alpha_2+\beta_2)}{A},$$

$$c_2(0) = \frac{(1+b_1^2)(\alpha_1+\beta_1)+b_1(\alpha_2+\beta_2)}{A} + \frac{(1+b_2^2)\beta_1+b_2\beta_2}{B},$$

$$c_1(1) = \frac{b_1(\alpha_1+\beta_1)+(1+b_1^2)(\alpha_2+\beta_2)}{A},$$

$$c_2(1) = \frac{b_1(\alpha_1+\beta_1)+(1+b_1^2)(\alpha_2+\beta_2)}{A} + \frac{b_2\beta_1+(1+b_2^2)\beta_2}{B},$$

$$\vec{c}(n) = \left(\frac{\gamma+\delta}{1+b_1^2}, \frac{\gamma}{1+b_1^2}+\frac{\delta}{(1+b_1^2)}+\frac{\delta}{(1+b_2^2)}\right)^\top.$$

Thus, the spectral characteristic of the optimal estimate of the random variable $A_2\vec{\xi}$ is calculated by the formula

$$(h(e^{i\lambda}))^\top = (h_1(e^{i\lambda}),\, h_2(e^{i\lambda}))^\top,$$

where

$$h_1(e^{i\lambda}) = \left(b_1\frac{(1+b_1^2)(\alpha_1+\beta_1)+b_1(\alpha_2+\beta_2)}{1+b_1^2+b_1^4} - b_2\frac{(1+b_2^2)\beta_1+b_2\beta_2}{1+b_2^2+b_2^4}\right)e^{-i\lambda}+$$
$$+\left(b_1\frac{b_1(\alpha_1+\beta_1)+(1+b_1^2)(\alpha_2+\beta_2)}{1+b_1^2+b_1^4} - b_2\frac{b_2\beta_1+(1+b_2^2)\beta_2}{1+b_2^2+b_2^4}\right)e^{2i\lambda}+$$
$$+\left(\frac{(\gamma+\delta)b_1}{1+b_1^2} - \frac{\delta b_2}{1+b_2^2}\right)(e^{i(n+1)\lambda}+e^{i(n-1)\lambda}),$$

$$h_2(e^{i\lambda}) = b_2\frac{(1+b_2^2)\beta_1+b_2\beta_2}{1+b_2^2+b_2^4}e^{-i\lambda}+$$
$$+b_2\frac{b_2\beta_1+(1+b_2^2)\beta_2}{1+b_2^2+b_2^4}e^{2i\lambda}+\frac{\delta b_2}{1+b_2^2}(e^{i(n+1)\lambda}+e^{i(n-1)\lambda}).$$

The mean square error is of the form

$$\Delta(F) = \frac{1}{2\pi}\left(\frac{((\alpha_1+\beta_1)^2+(\alpha_2+\beta_2)^2)(1+b_1^2)+2(\alpha_1+\beta_1)(\alpha_2+\beta_2)b_1}{1+b_1^2+b_1^4}\right)+ \\ +\frac{1}{2\pi}\left(\frac{(1+b_2^2)(\beta_1^2+\beta_2^2)+2\beta_1\beta_2 b_2}{1+b_2^2+b_2^4}\right)+ \\ +\frac{1}{2\pi}\left(\frac{(\gamma+\delta)^2}{(1+b_1^2)}+\frac{\delta^2}{(1+b_2^2)}\right).$$

3. Minimax-robust method of interpolation

The traditional methods of estimation of the functional $A_s\vec{\xi}$ which depends on unknown values of a stationary stochastic sequence $\vec{\xi}(j)$ can be applied in the case where the spectral density $F(\lambda)$ of the considered stochastic sequence $\vec{\xi}(j)$ is exactly known. In practise, however, we do not have complete information on spectral density of the sequence. For this reason we apply the minimax(robust) method of estimation of the functional $A_s\vec{\xi}$, that is we find an estimate that minimizes the maximum of the mean square errors for all spectral densities from the given class of admissible spectral densities D. For description of minimax method we propose the following definitions (see Moklyachuk and Masytka [29]).

Definition 3.1. For a given class of spectral densities D a spectral density $F_0(\lambda) \in D$ is called the least favourable in D for the optimal linear estimation of the functional $A_s\vec{\xi}$ if the following relation holds true

$$\Delta(F_0) = \Delta(h(F_0); F_0) = \max_{F\in D} \Delta(h(F); F).$$

Definition 3.2. For a given class of spectral densities D the spectral characteristic $h^0(e^{i\lambda})$ of the optimal linear estimate of the functional $A_s\vec{\xi}$ is called minimax-robust if

$$h^0(e^{i\lambda}) \in H_D = \bigcap_{F\in D} L_2^s(F),$$

$$\min_{h\in H_D}\max_{F\in D}\Delta(h;F) = \sup_{F\in D}\Delta\left(h^0;F\right).$$

It follows from the introduced definitions and the obtained formulas that the following statement holds true.

Lemma 3.1. The spectral density $F_0(\lambda) \in D$ is the least favourable in the class of admissible spectral densities D for the optimal linear estimate of the functional $A_s\vec{\xi}$ if the Fourier coefficients of the function $F_0^{-1}(\lambda)$ define a matrix B_s^0 that is a solution to the optimization problem

$$\max_{F\in D}\left\langle B_s^{-1}\vec{\mathbf{a}}_s,\, \vec{\mathbf{a}}_s\right\rangle = \left\langle (B_s^0)^{-1}\vec{\mathbf{a}}_s,\, \vec{\mathbf{a}}_s\right\rangle. \tag{3.1}$$

The minimax spectral characteristic $h^0 = h(F_0)$ can be calculated by the formula (2.11) if $h(F_0) \in H_D$.

The least favourable spectral density F_0 and the minimax spectral characteristic h^0 form a saddle point of the function $\Delta(h;F)$ on the set $H_D \times D$. The saddle point inequalities

$$\Delta(h;F_0) \geq \Delta\left(h^0;F_0\right) \geq \Delta\left(h^0;F\right) \qquad \forall F \in D, \forall h \in H_D$$

hold true if $h^0 = h(F_0)$ and $h(F_0) \in H_D$, where F_0 is a solution to the constrained optimization problem

$$\tilde{\Delta}(F) = -\Delta\left(h^0;F\right) \to \inf, \ F(\lambda) \in D, \tag{3.2}$$

where

$$\Delta\left(h^0;F\right) = \frac{1}{2\pi}\int_{-\pi}^{\pi}(C_s^0(e^{i\lambda}))^{\top}[F^0(\lambda)]^{-1}F(\lambda)[F^0(\lambda)]^{-1}\overline{C_s^0(e^{i\lambda})}d\lambda,$$

$$C_s^0(e^{i\lambda}) = \sum_{l=0}^{s-1}\sum_{j=M_l}^{M_l+N_{l+1}}\left((B_s^0)^{-1}\vec{\mathbf{a}}_s\right)(j)e^{ij\lambda}.$$

The constrained optimization problem (3.2) is equivalent to the unconstrained optimization problem

$$\Delta_D(F) = \tilde{\Delta}(F) + \delta(F\,|D\,) \to \inf,$$

where $\delta(F\,|D\,)$ is the indicator function of the set D. Solution F_0 to this problem is characterized by the condition $0 \in \partial\Delta_D(F_0)$, where $\partial\Delta_D(F_0)$ is the subdifferential of the convex functional $\Delta_D(F)$ at point F_0. This condition makes it possible to find the least favourable spectral densities in some special classes of spectral densities D [13], [37], [38].

Note, that the form of the functional $\Delta\left(h^0;F\right)$ is convenient for application the Lagrange method of indefinite multipliers for finding solution to the problem (3.2). Making use the method of Lagrange multipliers and the form of subdifferentials of the indicator functions we describe relations that determine least favourable spectral densities in some special classes of spectral densities (see books [28, 26, 29] for additional details).

4. Least favourable spectral densities in the class D_0^-

Consider the problem of the optimal estimation of the functional $A_s\vec{\xi}$ which depends on the unknown values of a stationary stochastic sequence $\vec{\xi}(j)$ in the case where the spectral density is from the class

$$D_0^- = \left\{ F(\lambda) \left| \frac{1}{2\pi}\int_{-\pi}^{\pi}(F^{-1}(\lambda))^{\top}d\lambda = P \right. \right\},$$

where $P = \{p_{ij}\}_{i,j=1}^{T}$ is a given matrix. To find solutions to the constrained optimization problem (3.2) we use the Lagrange multipliers method. With the help of this method we get the equation

$$(F^0(\lambda)^{\top})^{-1}C_s^0(e^{i\lambda})(C_s^0(e^{i\lambda}))^*(F^0(\lambda)^{\top})^{-1} = (F^0(\lambda)^{\top})^{-1}\vec{\alpha}\cdot\vec{\alpha}^*(F^0(\lambda)^{\top})^{-1},$$

where $\vec{\alpha} = \{\alpha_k\}_{k=1}^{T}$ is a vector of the Lagrange multipliers. From this relation we find that the Fourier coefficients of the matrix function $(F^0(\lambda)^\top)^{-1}$ satisfy the following equation

$$\left(\sum_{l=0}^{s-1}\sum_{k=M_l}^{M_l+N_{l+1}} \vec{c}(k)e^{ik\lambda}\right)\cdot\left(\sum_{l=0}^{s-1}\sum_{k=M_l}^{M_l+N_{l+1}} \vec{c}(k)e^{ik\lambda}\right)^{*} = \vec{\alpha}\cdot\vec{\alpha}^{*}, \tag{4.1}$$

where $\vec{c}(k), k \in S$, are components of the vector $\vec{\mathbf{c}}_s$ that satisfies the equation $B_s^0\vec{\mathbf{c}}_s = \vec{\mathbf{a}}_s$, the matrix B_s^0 consists from matrices $B_{mn}^0(k,j)$, each of which is determined by the Fourier coefficients of the function $(F^0(\lambda)^\top)^{-1}$

$$B_{mn}^0(k,j) = \frac{1}{2\pi}\int_{-\pi}^{\pi}(F^0(\lambda)^\top)^{-1}e^{-i(k-j)\lambda}d\lambda = B^0(k-j),$$

$$k = M_{m-1},\dots, M_{m-1}+N_m,$$
$$j = M_{n-1},\dots, M_{n-1}+N_n,$$
$$m,n = 1,\dots,s.$$

The Fourier coefficients $B(k) = B(-k), k \in S$, satisfy both equation (4.1) and equation $B_s^0\vec{\mathbf{c}}_s = \vec{\mathbf{a}}_s$. These coefficients can be found from the equation $B_s^0\vec{\alpha}_s = \vec{\mathbf{a}}_s$, where $\vec{\alpha}_s = (\vec{\alpha}, 0, \dots, 0)$. The last relation can be presented in the form of the system of equations

$$B(k)\vec{\alpha} = \vec{a}(k),\ k \in S.$$

From the first equation of the system (for $k = 0$) we find the unknown value $\vec{\alpha} = B^{-1}(0)\vec{a}(0)$. It follows from the extremum condition (3.1) and the restriction on the spectral densities from the class D_0^- that the Fourier coefficient

$$B(0) = \frac{1}{2\pi}\int_{-\pi}^{\pi}(F^0(\lambda)^\top)^{-1}d\lambda = P.$$

Thus,

$$B(k) = \vec{a}(k)\vec{a}(0)^{-1}P,$$

where $\vec{a}(0)^{-1}$ is a vector that satisfies the equation

$$\vec{a}(0)^{-1}\cdot\vec{a}(0) = 1.$$

Let

$$B(k) = B(-k) = \begin{cases} \vec{a}(k)\vec{a}(0)^{-1}P & \text{if } k \in S; \\ 0 & \text{if } k \in \{0,\dots,M_{s-1}+N_s\}\setminus S. \end{cases}$$

Let the vector sequences $\vec{a}(k), k \in S$ be such that the matrix function

$$(F^0(\lambda)^\top)^{-1} = \sum_{k=-(M_{s-1}+N_s)}^{M_{s-1}+N_s} B(k)e^{ik\lambda}.$$

is positive definite and has the determinant which does not equal zero identically. In this case the matrix function can be represented in the form [18]

$$(F^0(\lambda))^{-1} = \left(\sum_{k=0}^{M_{s-1}+N_s} A_k e^{-ik\lambda}\right) \cdot \left(\sum_{k=0}^{M_{s-1}+N_s} A_k e^{-ik\lambda}\right)^*, \ \lambda \in [-\pi, \pi],$$

where $A_k = 0, k \in \{0, \dots, M_{s-1} + N_s\} \setminus S$. Hence, $F_0(\lambda)$ is the spectral density of the autoregressive stochastic sequence of order $M_{s-1} + N_s$ generated by the equation

$$\sum_{k=0}^{M_{s-1}+N_s} A_k \vec{\xi}(n-k) = \sum_{l=0}^{s-1} \sum_{k=M_l}^{M_l+N_{l+1}} A_k \vec{\xi}(n-k) = \vec{\epsilon}_n, \tag{4.2}$$

where $\vec{\varepsilon}(n)$ is a vector-valued stochastic white noise sequence.

The minimax spectral characteristic $h(F_0)$ of the optimal linear estimate of the functional $A_s\vec{\xi}$ can be calculated by the formula (2.5), where

$$C_s(e^{i\lambda}) = \sum_{l=0}^{s-1} \sum_{k=M_l}^{M_l+N_{l+1}} \vec{c}(k) e^{ik\lambda} = \vec{\alpha} = P^{-1}\vec{a}(0),$$

namely

$$\begin{aligned} h(F_0) &= \sum_{l=0}^{s-1} \sum_{k=M_l}^{M_l+N_{l+1}} \vec{a}(k) e^{ik\lambda} - \left(\sum_{k=-(M_{s-1}+N_s)}^{M_{s-1}+N_s} B(k) e^{ik\lambda}\right) P^{-1}\vec{a}(0) \\ &= \sum_{k=1}^{N_1} \vec{a}(k) e^{-ik\lambda} + \sum_{l=1}^{s-1} \sum_{k=M_l}^{M_l+N_{l+1}} \vec{a}(k) e^{-ik\lambda}. \end{aligned} \tag{4.3}$$

Summing up our reasoning we come to conclusion that the following theorem holds true.

Theorem 4.1. The least favourable in the class D_0^- spectral density for the optimal linear estimation of the functional $A_s\vec{\xi}$ determined by a sequence $\vec{a}(k), k \in S$, such that the matrix function $\sum_{k=-(M_{s-1}+N_s)}^{M_{s-1}+N_s} B(k) e^{ik\lambda}$, where

$$B(k) = B(-k) = \vec{a}(k)\vec{a}(0)^{-1}P, \ k \in S,$$

is positive definite and has the determinant which does not equal to zero identically, is the spectral density of the autoregressive sequence (4.2) with the Fourier coefficients $B(k)$. The minimax spectral characteristics $h(F_0)$ is given by formula (4.3).

5. Least favourable spectral densities in the class D_W

Consider the problem of the optimal estimation of the functional $A_s\vec{\xi}$ which depends on the unknown values of a stationary stochastic sequence $\vec{\xi}(j)$ in the case where the spectral density of the sequence is from the set of spectral densities with restrictions on the moments of the function $(F^{-1}(\lambda))^\top$. Let

$$D_W = \left\{ F(\lambda) \left| \frac{1}{2\pi} \int_{-\pi}^{\pi} (F^{-1}(\lambda))^\top \cos(w\lambda) d\lambda = B(w)\ , w = 0, 1, \dots, W \right. \right\},$$

where the sequence of matrices $B(w) = B(-w),\ w = 0, \dots, W$, are such that the matrix function $\sum_{w=-W}^{W} B(w)e^{iw\lambda}$ is positive definite and has the determinant which does not equal zero identically (see the book by M. G. Krein and A. A. Nudelman [18] for more details). To find solutions to the constrained optimization problem (3.2) for the set D_W of admissible spectral densities we use the Lagrange multipliers method and the equation

$$\left(\sum_{l=0}^{s-1} \sum_{k=M_l}^{M_l+N_{l+1}} \vec{c}(k)e^{ik\lambda}\right) \cdot \left(\sum_{l=0}^{s-1} \sum_{k=M_l}^{M_l+N_{l+1}} \vec{c}(k)e^{ik\lambda}\right)^* = \\ = \left(\sum_{w=0}^{W} \vec{\alpha}(w)e^{iw\lambda}\right) \cdot \left(\sum_{w=0}^{W} \vec{\alpha}(w)e^{iw\lambda}\right)^*, \quad (5.1)$$

where $\vec{\alpha}(w), w = 0, 1, \dots, W$ are the Lagrange multipliers and $\vec{c}(k), k = 0, \dots, W$ are solutions to the equation

$$B_s^0 \vec{\mathbf{c}}_s = \vec{\mathbf{a}}_s.$$

Consider two cases: $W \geq M_{s-1}+N_s$ and $W < M_{s-1}+N_s$. Let $W \geq M_{s-1}+N_s$. In this case the given Fourier coefficients $B(w)$ define the matrix B_s^0 and the optimization problem (3.1) is degenerate. Let $\vec{\alpha}(M_{s-1}+N_s+1) = \ldots = \vec{\alpha}(W) = \vec{0}$ and $\vec{\alpha}(j) = \vec{0}, j \notin S$. Components $\vec{\alpha}(j), j \in S$, of the vector $\vec{\alpha}_s$ can be found from the equation $B_s^0\vec{\alpha}_s = \vec{\mathbf{a}}_s$. Hence, the relation (5.1) holds true. Thus the least favorable is every density $F(\lambda) \in D_W$ and the density of the autoregression stochastic sequence

$$F^0(\lambda)) = \left(\sum_{w=-W}^{W} B(|w|)^\top e^{iw\lambda}\right)^{-1} = \left(\sum_{k=0}^{W} A_k e^{ik\lambda}\right)\left(\sum_{k=0}^{W} A_k e^{ik\lambda}\right)^* \quad (5.2)$$

is least favorable, too.

Let $W < M_{s-1}+N_s$. Then the matrix B_s is determined by the known $B(w), w \in S \cap \{0, \dots, W\}$, and the unknown $B(w), w \in S\backslash\{0, \dots, W\}$, Fourier coefficients of the function $(F^{-1}(\lambda))^\top$. The unknown coefficients $\vec{\alpha}(k)$, $k \in S \cap \{0, \dots, W\}$, and $B(w), w \in S\backslash\{0, \dots, W\}$, can be found from the equation $B_s\vec{\alpha}_s^0 = \vec{\mathbf{a}}_s$, with $\vec{\alpha}_s^0 = (\vec{\alpha}(0), \dots, \vec{\alpha}(W_1), 0, \dots, 0)$, where W_1 is determined from the relation $\{0, \dots, W_1\} = \{0, \dots, W\} \cap S$.

The last equation can be presented as the system of equations

$$B(0)\vec{\alpha}(0)+B(1)\vec{\alpha}(1)+\ldots+B(-W_1)\vec{\alpha}(W_1)=\vec{a}(0);$$
$$B(1)\vec{\alpha}(0)+B(0)\vec{\alpha}(0)+\ldots+B(-W_1+1)\vec{\alpha}(W_1)=\vec{a}(1);$$
$$\ldots$$
$$B(W_1)\vec{\alpha}(0)+B(W_1-1)\vec{\alpha}(0)+\ldots+B(0)\vec{\alpha}(W_1)=\vec{a}(W_1);$$
$$\ldots$$
$$B(M_{s-1}+N_s)\vec{\alpha}(0)+B(1)\vec{\alpha}(1)+\ldots+B(0)\vec{\alpha}(W_1)=\vec{a}(M_{s-1}+N_s).$$

From the first W_1 equations we can find the unknown coefficients $\vec{\alpha}(k)$ and from the next equations we find the Fourier coefficients $B(w), w\in S\setminus\{0,\ldots,W\}$.

If the sequence $B(w)=B(-w), w\in S$, that is constructed from the sequence $B(w), w\in S\cap\{0,\ldots,W\}$ and the calculated coefficients $B(w), w\in S\setminus\{0,\ldots,W\}$, determines positive definite function with the determinant which does not equal zero identically , then the least favourable spectral density $F^0(\lambda)$ is determined by the Fourier coefficients $B(w), w\in S$ of the function $(F^0(\lambda)^\top)^{-1}$

$$F^0(\lambda)=\left(\sum_{k=0}^{M_{s-1}+N_s}(B(k)^\top e^{ik\lambda}+B(-k)^\top e^{-ik\lambda})\right)^{-1}=$$
$$=\left(\sum_{k=0}^{M_{s-1}+N_s}A_k e^{ik\lambda}\right)\left(\sum_{k=0}^{M_{s-1}+N_s}A_k e^{ik\lambda}\right)^*. \quad (5.3)$$

Let us summarize our results and present them in the form of a theorem.

Theorem 5.1. The least favourable spectral density in the class D_W for the optimal linear estimate of the functional $A_s\vec{\xi}$ in the case where $W\geq M_{s-1}+N_s$ is the spectral density (5.2) of the autoregression stochastic sequence of order W determined by coefficients $B(w), w=0,1,\ldots,W$. In the case where $W<M_{s-1}+N_s$ and solutions $B(w)=B(-w), w\in S\setminus\{0,\ldots,W\}$, of equation $B_s\vec{\alpha}_s^0=\vec{\mathbf{a}}_s$ together with coefficients $B(w)=B(-w), w\in S\cap\{0,\ldots,W\}$, form a positive definite function with the determinant which does not equal zero identically, the least favourable spectral density in D_W is the density (5.3) of the autoregression stochastic sequence of the order $M_{s-1}+N_s$. The minimax characteristic of the estimate is calculated by formula (2.11).

6. Least favourable spectral densities in the class D_V^U

Consider the problem of the optimal estimation of the functional $A_s\vec{\xi}$ which depends on the unknown values of a stationary stochastic sequence $\vec{\xi}(j)$ in the case where the spectral density of the sequence is from the set of spectral densities

$$D_V^U=\left\{F(\lambda)\left|0\leq V(\lambda)\leq F(\lambda)\leq U(\lambda),\frac{1}{2\pi}\int_{-\pi}^{\pi}(F^{-1}(\lambda))^\top d\lambda=P\right.\right\},$$

where $V(\lambda),U(\lambda)$ are given spectral densities. To find solutions to the constrained optimization problem (3.2) for the set D_V^U of admissible spectral densities we use the condition $0 \in \partial\Delta_D(F^0)$. It follows from the condition $0 \in \partial\Delta_D(F^0)$ for $D = D_V^U$ that the Fourier coefficients of the function $(F^0(\lambda)^\top)^{-1}$ satisfy both equation

$$B_s^0 \vec{\mathbf{c}}_s = \vec{\mathbf{a}}_s$$

and the equation

$$\left(\sum_{l=0}^{s-1}\sum_{k=M_l}^{M_l+N_{l+1}} \left(\left(B_s^0\right)^{-1}\vec{\mathbf{a}}_s\right)(k)e^{ik\lambda}\right)\left(\sum_{l=0}^{s-1}\sum_{k=M_l}^{M_l+N_{l+1}} \left(\left(B_s^0\right)^{-1}\vec{\mathbf{a}}_s\right)(k)e^{ik\lambda}\right)^* = \\ = \Gamma_1(\lambda)+\Gamma_2(\lambda)+\vec{\alpha}\cdot\vec{\alpha}^*, \quad (6.1)$$

where $\Gamma_1(\lambda) \geq 0$ and $\Gamma_1(\lambda) = 0$ if $F^0(\lambda) \geq V(\lambda)$; $\Gamma_2(\lambda) \leq 0$ and $\Gamma_2(\lambda) = 0$ if $F_0(\lambda) \leq U(\lambda)$. Therefore, in the case where $V(\lambda) \leq F^0(\lambda) \leq U(\lambda)$, the function $(F^0(\lambda))^{-1}$ is of the form

$$(F^0(\lambda))^{-1} = \sum_{k=0}^{M_{s-1}+N_s} (B(k)^\top e^{ik\lambda} + B(-k)^\top e^{-ik\lambda}) = \\ = \left(\sum_{k=0}^{M_{s-1}+N_s} A_k e^{ik\lambda}\right)\left(\sum_{k=0}^{M_{s-1}+N_s} A_k e^{ik\lambda}\right)^*,$$

where $B(k) = B(-k) = \vec{a}(k)\vec{a}(0)^{-1}P$. The least favourable in the class D_V^U is the density of the autoregression stochastic sequence of the order $M_{s-1}+N_s$ if the following inequality holds true

$$V(\lambda) \leq \left(\sum_{k=0}^{M_{s-1}+N_s} (B(k)^\top e^{ik\lambda} + B(-k)^\top e^{-ik\lambda})\right)^{-1} \leq U(\lambda),\ \lambda \in [-\pi,\pi]. \quad (6.2)$$

The following theorem holds true.

Theorem 6.1. If the coefficients $B(k) = B(-k) = \vec{a}(k)\vec{a}(0)^{-1}P$, $k \in S$, satisfy the inequality (6.2) and form a positive definite function with the determinant which does not equal zero identically, then the least favourable in the class D_V^U spectral density for the optimal linear estimate of the functional $A_s\vec{\xi}$ is density (4.2) of the autoregression stochastic sequence of order $M_{s-1}+N_s$. The minimax characteristic $h(F^0)$ of the estimate can be calculated by the formula (4.3). If the inequality (6.2) is not satisfied, then the least favourable spectral density in D_V^U is determined by relation (6.1) and the constrained optimization problem (3.1). The minimax characteristic of the estimate is calculated by formula (2.11).

7. Conclusions

In this article we describe methods of solution of the problem of the mean-square optimal linear estimation of the functional $A_s\vec{\xi} = \sum_{l=0}^{s-1}\sum_{j=M_l}^{M_l+N_{l+1}} \vec{a}(j)^\top\vec{\xi}(j)$, $M_l =$

$\sum\limits_{k=0}^{l}(N_k+K_k)$, $N_0=K_0=0$, which depends on the unknown values of the stationary stochastic sequence $\vec{\xi}(j)=\{\xi_k(j)\}_{k=1}^{T}$. Estimates are based on observations of the sequence $\vec{\xi}(j)$ at points $j\in\mathbb{Z}\backslash S$, where $S=\bigcup\limits_{l=0}^{s-1}\{M_l,M_l+1,\dots,M_l+N_{l+1}\}$. We provide formulas for calculating the values of the mean square error and the spectral characteristic of the optimal linear estimate of the functional in the case where the spectral density of the sequence $\vec{\xi}(j)$ is exactly known. In the case where the spectral density is unknown while a set of admissible spectral densities is given, the minimax approach is applied. We obtain formulas that determine the least favourable spectral densities and the minimax spectral characteristics of the optimal linear estimates of the functional $A_s\vec{\xi}$ for concrete classes of admissible spectral densities. It is shown that spectral densities the autoregressive stochastic sequences are the least favourable in some classes of spectral densities.

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