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NUMERICAL RESOLUTION OF A DEGENERATE ELLIPTIC-PARABOLIC SEAWATER INTRUSION PROBLEM USING DISCRETE DUALITY FINITE VOLUME METHODS

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Abstract. In this paper, we propose an approximation for a seawater intrusion problem in a confined aquifer, this model consists in a coupled system of an elliptic and a de-generate parabolic equations. The anisotropic diffusion operators in both equations require special care while discretizing by a finite volume method DDFV (Discrete Duality Finite Volume). We first establish some a priori estimates satisfied by the sequences of approximate solutions. Then, it yields the compactness of these sequences. Passing to the limit in the numerical scheme, we finally obtain that the limit of the sequence of approximate solutions is a weak solution to the problem under study. The theoretical results are confirmed by some numerical experiments.

1. Introduction

Saltwater intrusion is the phenomenon where the movement of saltwater intrude into freshwater aquifers, it can lead to contamination of drinking water sources and other consequences. The hydraulic connection between groundwater and seawater leads naturally to the saltwater intrusion to some degree in most coastal aquifers. Because saline water has a higher mineral content than freshwater, it is denser and has a higher water pressure. As a result, saltwater can push inland beneath the freshwater. Certain human activities increased saltwater intrusion in many coastal areas, especially groundwater pumping from coastal freshwater wells. Water extraction drops the level of fresh groundwater, reducing its water pressure and allowing saltwater to flow further inland. Since freshwater and saltwater are miscible fluids, we have a transition zone separating them caused by hydrodynamic dispersion, in the literature, there exists several modelling approaches see [8, 9, 34, 32, 33, 5, 4, 3]. for more details about sea intrusion problem see [6, 7].
In this work we consider the approximation of the modeling of a degenerate seawater intrusion problem where the fluids are immiscible and the domains occupied by the fluids are separated by an interface called sharep interface (see [8, 9, 7, 6, 11, 12]). This approximation has been widely studied by Ahmed Ait Hammou Oulhaj in [13, 14] using the finite volume method 4. The problem has been treated by Mohamed El Alaoui Talibi and al. in [11] using the sensitive adjoint method. However, M. El Alaoui Talibi and M. H. Tber demonstrates the

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existence of solutions for degenerate sea water problem in [12]. Seawater intrusion problem has been studied by J. Bear and others in [7], M. Jazar and R. Monneau in [15], Ahmad Al Bitar in [10] and D. Ouazar and al. in [16]. In [1] the authors propose a comparison between the results of the methods a finite element method and a finite volume method.

In this paper, the discretization of the diffusion terms is based upon a discrete duality finite volume [18], which allows the tensor K to be anisotropic highly variable in space, and many be applied to almost general meshes.
The objective of this works is to prove the convergence of the family of discrete solution to the solution continuous problem, and of associate gradient to the gradient of solution, as the mesh tend to 0.
A huge literature exists in the engineering setting of discretization of the diffusion term, let us cite for instance the Discrete Duality Finite Volume schemes by Domelevo and Omnes [17, 18], the Scheme Using Stabilization and Hybrid Interfaces by Eymard, Gallouet and Herbin [19, 20], the Multi Points Flux Approximation schemes by Aavatsmark, Barkve, Boe and Mannseth [21, 22], the Mixed Finite Volume schemes by Droniou and Eymard [23, 24]. These methods and other methods (discontinuous Galerkin methods, finite element methods, mimetic finite difference methods ...) have been compared on a benchmark organized by Herbin and Hubert [25].

We denote by h the depth of the interface and f the freshwater hydraulic head. We consider the confined aquifer bounded by two horizontal and impermeable layers.

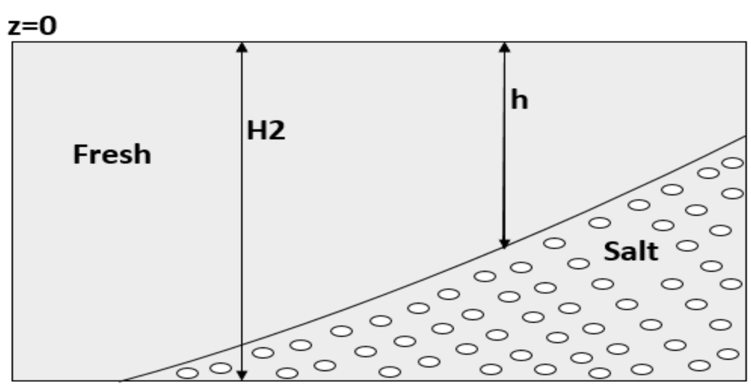

FIGURE 1. Saltwater intrusion phenomena.

The upper surface corresponds to $z = 0$ and the lower to $z = -H_2$, H_2 is the thickness of the aquifer assumed to be such that $H_2 > \delta > 0$. Then (h, f) satisfy the following system:

$$\begin{cases} \frac{\partial h}{\partial t} - div(K(x)T_s(h)\nabla h) + div(K(x)T_s(h)\nabla f) = -I_s & \text{in } \Omega \times [0,T], \\ -div(K(x)H_2\nabla f) + div(K(x)T_s(h)\nabla h) = I_f + I_s & \text{in } \Omega \times [0,T]. \end{cases} \tag{1.1}$$

$T_s(h) = H_2 - h$ is the thickness of saltwater zone and $K(.)$,the hydraulic conductivity, is uniformly positive definite matrix. With the initial condition

$$h(x,0) = h_0(x) \text{ on } \Omega. \tag{1.2}$$

such that

$$h_0 \in L^2(\Omega) \text{ satisfies } \delta \le h_0(x) \le H_2 \text{ for a.e. } x \in \Omega. \tag{1.3}$$

and the boundary conditions

$$\begin{cases} h = h_D & \text{in } \partial\Omega \times [0,T], \\ f = f_D & \text{in } \partial\Omega \times [0,T], \end{cases} \tag{1.4}$$

with

$$\begin{cases} h_D \in L^2(0,T;H^1(\Omega)) \text{ such that } \frac{dh_D}{dt} \in L^2(0,T;(H^1(\Omega)'), \\ \text{ and } \delta \le h_D(t,x) \le H_2 \text{ for a.e } (t,x) \in [0,T] \times L^2(0,T;H^1(\Omega)), \end{cases} \tag{1.5}$$

and

$$f_D \in L^2(0,T;H^1(\Omega)). \tag{1.6}$$

Where Ω is an open, bounded connected subset of IR^d ($d = 2$ or $d = 3$). Which supported tube polygonal ($d = 2$) or polyhedral ($d = 3$), and $\partial\Omega$ stands for its boundary.
K is a hydraulic conductivity matrix-valued function satisfying

$$\Lambda_A \le |\xi|^{-2} K(x)\xi.\xi \le \Lambda_B. \tag{1.7}$$

I_s and I_f are the supply functions represent the distributed supply surface of fresh and salt water into the aquifer such that

$$(I_s, I_f) \in L^2(0,T;L^2(\Omega)). \tag{1.8}$$

Now, we define the following weak formulation demonstrated in [26]

Theorem 1.1. *Assume that* (1.1)-(1.8)*. A weak solution to* (1.1) *is* $h \in W(0,T), f \in L^2(0,T;H^1(\Omega))$ *such that*

$$\begin{cases} \int_0^T \int_\Omega h\frac{\partial v}{\partial t} + \int_0^T \int_\Omega T_s(h)K(x)\nabla h \nabla v - \int_0^T \int_\Omega T_s(h)K(x)\nabla f \nabla v = -\int_0^T \int_\Omega I_s v \\ \int_0^T \int_\Omega H_2 \nabla f \nabla v - \int_0^T \int_\Omega T_s(h)K(x)\nabla h \nabla v = \int_0^T \int_\Omega (I_f + I_s)v \\ v \in L^2(0,T;H^1(\Omega)) \end{cases} \tag{1.9}$$

Later, we define the following Global existence demonstrated in [26]

Theorem 1.2. *Let $h_0 \in L^2(\Omega)$, h_D, f_D and I_s, I_f verified (1.3),(1.5),(1.6) and (1.8). Then for any $T > 0$, there exists a solution $h \in W(0,T) + h_D$, $f \in L^2(0,T,H^1(\Omega)) + f_D$ of the following variational equations.*

$$\left\{\begin{array}{l} \int_0^T \int_\Omega h\frac{\partial v}{\partial t} + \int_0^T \int_\Omega T_s(h)K(x)\nabla h\nabla v - \int_0^T \int_\Omega T_s(h)K(x)\nabla f\nabla v = -\int_0^T \int_\Omega I_s v \\ \int_0^T \int_\Omega H_2\nabla f\nabla v - \int_0^T \int_\Omega T_s(h)K(x)\nabla h\nabla v = \int_0^T \int_\Omega (I_f + I_s)v \\ v \in L^2(0,T,H^1(\Omega)). \end{array}\right. \tag{1.10}$$

With $W(0,T) = \{\omega \in L^2(0,T;H_0^1(\Omega)), \frac{d\omega}{dt} \in L^2(0,T;H_0^1(\Omega)')\}$

The organization of this paper is as follows : In Section 2 we detail the DDFV formulation. In Section 3 we present a priori analyses of the method and the convergence of the method is proved in section 4. Finally in Section 5 we present a number of numerical results obtained on different two-dimensional meshes.

2. The Discrete Duality Finite Volume schemes

In order to define a DDFV scheme, as for instance in [17, 18], we need to introduce three different meshes – the primal mesh, the dual mesh and the diamond mesh – and some associated notations.

2.1. Definition of the meshes. $\mathcal{T}$ is the mesh adopted for the schema 2D-DDFV, this mesh is composed the tow meshes primal $\mathfrak{M}$ and dual $\mathfrak{M}^*$.

Let Ω be a polygonal open bounded connected subset of IR^d with $d \in IN^*$, and $\partial\Omega = \overline{\Omega} - \Omega$ its boundary .

2.1.1. *Primal Mesh.* The primal mesh is defined as the triplet $(\mathfrak{M}, \mathcal{E}, P)$ (see figure (2)), where

- $\mathfrak{M}$ is a finite family of nonempty open disjoint subset $\mathcal{K}$ of Ω (the control volume optimal) such that $\overline{\Omega} = \cup_{\mathcal{K}\in\mathfrak{M}}\overline{\mathcal{K}}$, $for\ all\ \mathcal{K} \in \mathfrak{M}$,with $\partial K = \overline{\mathcal{K}} - \mathcal{K}$ be the boundary of $\mathcal{K}$, let $|\mathcal{K}| > 0$ is the measure of $\mathcal{K}$ and let $d_{\mathcal{K}}$ the diameter of $\mathcal{K}$.
- $\mathcal{E}$ is the set of the edges σ of this mesh, m_σ is the measure of σ, $\mathcal{E}_{ext}$ is the subset of edges of interior of Ω. For all $\mathcal{K} \in \mathfrak{M}$ and $\sigma \in \mathcal{K}$ (subset of edges of $\mathcal{K}$) , we denote by $n_{\mathcal{K},\sigma}$ the unite vector normal to σ outward to $\mathcal{K}$.
- P is the subset of points of Ω indexed by $\mathfrak{M}$, we denote $P = \{(x_{\mathcal{K}})_{\mathcal{K}\in\mathfrak{M}}; x_{\mathcal{K}} \in \mathcal{K}\}$, ($x_{\mathcal{K}}$ is the barycentre of $\mathcal{K}$) we than denote by $D_{\mathcal{K},\sigma}$ the cone withe vertex $x_{\mathcal{K}}$ and basis $\mathcal{K}$

2.1.2. *Dual Mesh.* The dual mesh is defined as the triplet $(\mathfrak{M}^*, \mathcal{E}^*, P^*)$ (see figure (2)), where

- $\mathfrak{M}^*$ is a finite family of nonempty open disjoint subset $\mathcal{K}^*$ of Ω (the control volume dual) such that $\overline{\Omega} = \cup_{\mathcal{K}^*\in\mathfrak{M}^*}\overline{\mathcal{K}^*}$, $for\ all\ \mathcal{K}^* \in \mathfrak{M}^*$,with $\partial\mathcal{K}^* = \overline{\mathcal{K}^*} - \mathcal{K}^*$ be the boundary of $\mathcal{K}^*$, let $m_{\mathcal{K}^*} = |\mathcal{K}^*| > 0$ is the measure of $\mathcal{K}^*$ and let $d_{\mathcal{K}^*}$ the diameter of $\mathcal{K}^*$.

- $\mathcal{E}^*$ is the set of the edges σ^* of this mesh, m_{σ^*} is the measure of σ^*, $\mathcal{E}^*_{int}$ is the subset of edges of interior of Ω. For all $\mathcal{K}^* \in \mathfrak{M}^*$ and $\sigma^* \in \mathcal{K}^*$ (subset of edges of $\mathcal{K}^*$) , we denote by $n_{\mathcal{K}^*,\sigma^*}$ the unite vector normal to σ^* outward to $\mathcal{K}^*$.
- P^* is the subset of points of Ω indexed by $\mathfrak{M}^*$, we denote $\{P^* = (x_{\mathcal{K}^*})_{\mathcal{K}^* \in \mathfrak{M}^*}; x_{\mathcal{K}^*} \in \mathcal{K}^*\}$, ($x_{\mathcal{K}^*}$ is the barycentre of $\mathcal{K}^*$) we than denote by $D_{\mathcal{K}^*,\sigma^*}$ the cone withe vertex $x_{\mathcal{K}^*}$ and basis $\mathcal{K}^*$

2.1.3. *Diamond Mesh.*

- $\mathfrak{D}_{\mathcal{K}} = \{\mathcal{D}_{\sigma,\sigma^*} \in \mathfrak{D} / \sigma \in \mathcal{E}_{\mathcal{K}}\}$
- $\mathfrak{D}_{\mathcal{K}^*} = \{\mathcal{D}_{\sigma,\sigma^*} \in \mathfrak{D} / \sigma^* \in \mathcal{E}_{\mathcal{K}^*}\}$
- $m_{\mathcal{D}}$ measure of the diamond.
- For a diamond cell $\mathcal{D}$ recall that $(x_{\mathcal{K}}, x_{\mathcal{K}^*}, x_{\mathcal{L}}, x_{\mathcal{L}^*})$ are the vertices of $\mathcal{D}_{\sigma,\sigma^*}$.
- τ the unite vector parallel to σ, oriented from $\mathcal{K}^*$ to $\mathcal{L}^*$.
- τ^* the unite vector parallel to σ^*, oriented from $\mathcal{K}$ to $\mathcal{L}$.
- $\alpha_{\mathcal{D}}$ the angle between τ and τ^*.
- $d_{\mathcal{D}}$ the diameter of $\mathcal{D}_{\sigma,\sigma^*}$.

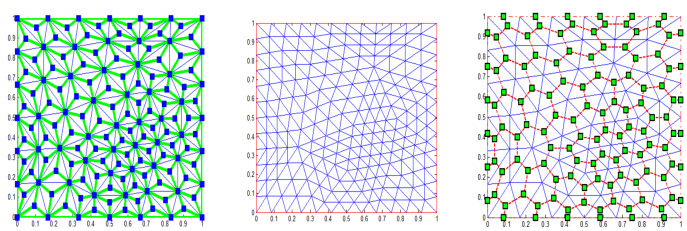

FIGURE 2. In the middle primal mesh $\mathfrak{M}$, in the right the dual mesh $\mathfrak{M}^*$ and in the left the mesh Diamond $\mathfrak{D}$.

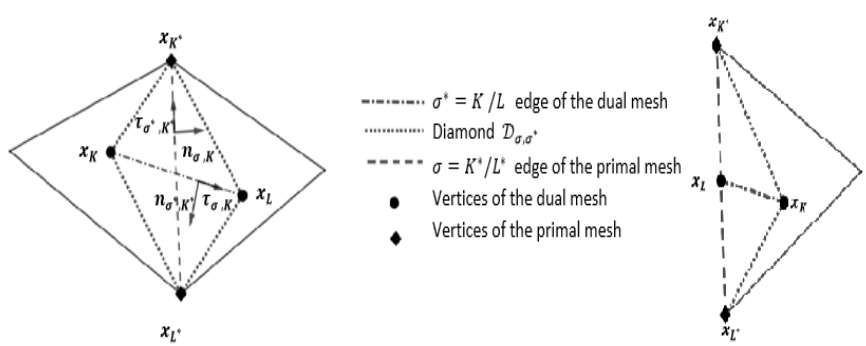

FIGURE 3. interior Diamond and exterior.

2.2. Discrete operators and duality formula. Let $f^{\mathcal{T}} = ((f_{\mathcal{K}})_{\mathcal{K}\in\mathfrak{M}}, (f_{\mathcal{K}^*})_{\mathcal{K}^*\in\mathfrak{M}^*})$ and $h^{\mathcal{T}} = ((h_{\mathcal{K}})_{\mathcal{K}\in\mathfrak{M}}, (h_{\mathcal{K}^*})_{\mathcal{K}^*\in\mathfrak{M}^*})$ the discrete functions such that

$$\begin{cases} f^{\mathcal{T}} = \frac{1}{2}(f^{\mathfrak{M}} + f^{\mathfrak{M}^*}), \\ h^{\mathcal{T}} = \frac{1}{2}(h^{\mathfrak{M}} + h^{\mathfrak{M}^*}), \end{cases}$$

with

$$\begin{cases} f^{\mathfrak{M}} = \sum_{\mathcal{K}\in\mathfrak{M}} f_{\mathcal{K}}\xi_{\mathcal{K}} \text{ and } f^{\mathfrak{M}^*} = \sum_{\mathcal{K}^*\in\mathfrak{M}^*} f_{\mathcal{K}^*}\xi_{\mathcal{K}^*}, \\ h^{\mathfrak{M}} = \sum_{\mathcal{K}\in\mathfrak{M}} h_{\mathcal{K}}\xi_{\mathcal{K}} \text{ and } h^{\mathfrak{M}^*} = \sum_{\mathcal{K}^*\in\mathfrak{M}^*} h_{\mathcal{K}^*}\xi_{\mathcal{K}^*}. \end{cases}$$

Definition 2.1. $C_0(\overline{\Omega})$ is the set of continuous function which vanish on $\partial\Omega$, we define the interpolation

$$\mathcal{P}_{\mathcal{T}} : C_0(\overline{\Omega}) \to H^{\mathcal{T}} \tag{2.1}$$

$$\varphi \to \mathcal{P}_{\mathcal{T}}\varphi = (\varphi(x_{\mathcal{K}})_{\mathcal{K}\in\mathfrak{M}}, \varphi(x_{\mathcal{K}^*})_{\mathcal{K}^*\in\mathfrak{M}^*}). \tag{2.2}$$

Definition 2.2. We define

- $IR^{\mathcal{T}}$ is a linear space of scalar fields constant on the cells of $\overline{\mathfrak{M}}$ and $\overline{\mathfrak{M}^*}$

$$\begin{cases} IR^{\mathcal{T}} = \{u_{\mathcal{T}} = ((u_{\mathcal{K}})_{\mathcal{K}\in\overline{\mathfrak{M}}}, (u_{\mathcal{K}^*})_{\mathcal{K}^*\in\overline{\mathfrak{M}^*}}), \text{ with } u_{\mathcal{K}} \in IR, \\ for\ all\ \mathcal{K} \in \overline{\mathfrak{M}} \text{ and } u_{\mathcal{K}^*} \in IR;\ for\ all\ \mathcal{K}^* \in \overline{\mathfrak{M}^*}\}. \end{cases} \tag{2.3}$$

- $(IR^2)^{\mathcal{D}}$ is a linear space of vector fields constant on the cells of $\mathcal{D}$.

$$(IR^2)^{\mathcal{D}} = \{\xi_{\mathcal{D}} = ((\xi_{\mathcal{D}})_{\mathcal{D}\in\overline{\mathfrak{D}}};\ \text{with } \xi_{\mathcal{D}} \in IR^2;\ for\ all\ \mathcal{D} \in \overline{\mathfrak{D}}\}. \tag{2.4}$$

Definition 2.3. for $\mathcal{D} \in \mathfrak{D}$ we have:

$$\nabla^{\mathfrak{D}} : IR^{\mathcal{T}} \to (IR^2)^{\mathcal{D}}$$

$$u_{\mathcal{T}} \to \begin{cases} \nabla^{\mathcal{D}} u_{\mathcal{T}}.\tau_{\mathcal{K}^*,\mathcal{L}^*} = \frac{u_{\mathcal{L}^*} - u_{\mathcal{K}^*}}{m_\sigma} \\ \nabla^{\mathcal{D}} u_{\mathcal{T}}.\tau_{\mathcal{K},\mathcal{L}} = \frac{u_{\mathcal{L}} - u_{\mathcal{K}}}{m_{\sigma^*}}. \end{cases}$$

So for $\mathcal{D} \in \mathfrak{D}$:

$$\nabla^{\mathcal{D}} u_{\mathcal{T}} = \frac{1}{sin(\alpha_{\mathcal{D}})}\left(\frac{u_{\mathcal{L}} - u_{\mathcal{K}}}{m_\sigma^*} n_{\sigma,\mathcal{K}} + \frac{u_{\mathcal{L}^*} - u_{\mathcal{K}^*}}{m_\sigma} n_{\sigma^*,\mathcal{K}^*}\right),$$

with $m_{\mathcal{D}} = \frac{1}{2} m_\sigma m_{\sigma^*} sin(\alpha_{\mathcal{D}})$ we have:

$$\nabla^{\mathcal{D}} u_{\mathcal{T}} = \frac{1}{2m_{\mathcal{D}}}\left((u_{\mathcal{L}} - u_{\mathcal{K}}) m_\sigma n_{\sigma,\mathcal{K}} + (u_{\mathcal{L}^*} - u_{\mathcal{K}^*}) m_{\sigma^*} n_{\sigma^*,\mathcal{K}^*}\right). \tag{2.5}$$

Let $\xi : \Omega \to IR^2$ be a regular function, Using the Green formula we get

$$\int_{\mathcal{K}} div(\xi(x))dx = \sum_{\sigma\in\partial\mathcal{K}} \int_\sigma \xi(s).n_{\sigma,\mathcal{K}} ds,\ for\ all\ \mathcal{K}\mathfrak{M}, \tag{2.6}$$

and

$$\int_{\mathcal{K}^*} div(\xi(x))dx = \sum_{\sigma\in\partial\mathcal{K}^*} \int_\sigma \xi(s).n_{\sigma,\mathcal{K}^*} ds,\ for\ all\ \mathcal{K}^*\mathfrak{M}^*, \tag{2.7}$$

Than the discrete divergence $div^{\mathcal{T}}$ is defined by

Definition 2.4. The discrete divergence operator $div^{\mathcal{T}}$is a mapping from $(IR^2)^{\mathfrak{D}}$ to $IR^{\mathcal{T}}$ defined for all $\xi \in (IR^2)^{\mathfrak{D}}$ by

$$div^{\mathcal{T}}\xi_{\mathfrak{D}} = \left(div^{\mathfrak{M}}\xi_{\mathfrak{D}}, div^{\partial\mathfrak{M}}\xi_{\mathfrak{D}}, div^{\mathfrak{M}^*}\xi_{\mathfrak{D}}, div^{\partial\mathfrak{M}^*}\xi_{\mathfrak{D}}\right),$$

with $div^{\mathfrak{M}}\xi_{\mathfrak{D}} = (div_{\mathcal{K}}\xi_{\mathfrak{D}})_{\mathcal{K}\in\mathfrak{M}}$, $div^{\partial\mathfrak{M}}\xi_{\mathfrak{D}} = 0$, $div^{\mathfrak{M}^*}\xi_{\mathfrak{D}} = (div_{\mathcal{K}^*}\xi_{\mathfrak{D}})_{\mathcal{K}^*\in\mathfrak{M}^*}$, $div^{\partial\mathfrak{M}^*}\xi_{\mathfrak{D}} = (div_{\mathcal{K}^*}\xi_{\mathfrak{D}})_{\mathcal{K}^*\in\partial\mathfrak{M}^*}$ such that

$$div_{\mathcal{K}}\xi_{\mathfrak{D}} = \frac{1}{m_{\mathcal{K}}}\sum_{\mathcal{D}\in\mathfrak{D}_{\mathcal{K}},\mathcal{D}=\mathcal{D}_{\sigma,\sigma^*}} m_\sigma \xi_{\mathcal{D}}.n_{\sigma,\mathcal{K}},\ for\ all\ \mathcal{K}\in\mathfrak{M}$$

$$div_{\mathcal{K}^*}\xi_{\mathfrak{D}} = \frac{1}{m_{\mathcal{K}^*}}\sum_{\mathcal{D}\in\mathfrak{D}_{\mathcal{K}^*},\mathcal{D}=\mathcal{D}_{\sigma,\sigma^*}} m_{\sigma^*} \xi_{\mathcal{D}}.n_{\sigma^*,\mathcal{K}^*},\ for\ all\ \mathcal{K}^*\in\mathfrak{M}^*$$

$$div_{\mathcal{K}^*}\xi_{\mathfrak{D}} = \frac{1}{m_{\mathcal{K}^*}}\Big(\sum_{\mathcal{D}\in\mathfrak{D}_{\mathcal{K}^*},\mathcal{D}=\mathcal{D}_{\sigma,\sigma^*}} m_{\sigma^*} \xi_{\mathcal{D}}.n_{\sigma^*,\mathcal{K}^*} + \sum_{\mathcal{D}\in\mathfrak{D}_{\mathcal{K}^*}\cap\mathfrak{D}_{ext},\mathcal{D}=\mathcal{D}_{\sigma,\sigma^*}} \frac{m_\sigma}{2}\xi_{\mathcal{D}}.n_{\sigma,\mathcal{K}}\Big),\ for\ all\ \mathcal{K}^*\in\partial\mathfrak{M}^*$$

In order to show the duality property between the discrete gradient and the discrete divergence we define the scalar products $< .,. >_{\mathcal{T}}$ on $IR^{\mathcal{T}}$ and $< .,. >_{\mathfrak{D}}$ on $(IR^2)^{\mathfrak{D}}$ by

$$< v_{\mathcal{T}}, u_{\mathcal{T}} >_{\mathcal{T}} = \frac{1}{2}\left(\sum_{\mathcal{K}\mathfrak{M}} m_{\mathcal{K}}u_{\mathcal{K}}v_{\mathcal{K}} + \sum_{\mathcal{K}^*\mathfrak{M}^*} m_{\mathcal{K}^*}u_{\mathcal{K}^*}v_{\mathcal{K}^*}\right),\ for\ all\ u_{\mathcal{T}}, v_{\mathcal{T}} \in IR^{\mathcal{T}}. \tag{2.8}$$

$$< \xi_{\mathfrak{D}}, \varphi_{\mathfrak{D}} >_{\mathfrak{D}} = \sum_{\mathcal{D}\in\mathfrak{D}} m_{\mathcal{D}}\xi_{\mathcal{D}}.\varphi_{\mathcal{D}},\ for\ all\ \xi_{\mathfrak{D}}, \varphi_{\mathfrak{D}} \in \left(IR^2\right)^{\mathfrak{D}}. \tag{2.9}$$

The corresponding norms are denoted by $\|.\|_{2,\mathcal{T}}$ and $\|.\|_{2,\mathfrak{D}}$

$$\|.\|_{2,\mathcal{T}} = \left(\frac{1}{2}\sum_{\mathcal{K}\mathfrak{M}} m_{\mathcal{K}}|u_{\mathcal{K}}|^2 + \frac{1}{2}\sum_{\mathcal{K}^*\mathfrak{M}^*} m_{\mathcal{K}^*}|u_{\mathcal{K}^*}|^2\right)^{1/2},\ for\ all\ u_{\mathcal{T}} \in IR^{\mathcal{T}} \tag{2.10}$$

$$\|\xi_{\mathfrak{D}}\|_{2,\mathfrak{D}} = \left(\sum_{\mathcal{D}\in\mathfrak{D}} m_{\mathcal{D}}|\xi_{\mathcal{D}}|^2\right)^{1/2},\ for\ all\ \xi_{\mathfrak{D}} \in \left(IR^2\right)^{\mathfrak{D}}. \tag{2.11}$$

$$\|u_{\mathcal{T}}\|_{\infty,\mathcal{T}} = \max\left(\max_{\mathcal{K}\in\overline{\mathfrak{M}}}|u_{\mathcal{K}}|, \max_{\mathcal{K}^*\in\overline{\mathfrak{M}^*}}|u_{\mathcal{K}^*}|\right),\ for\ all\ u_{\mathcal{T}} \in IR^{\mathcal{T}}, \tag{2.12}$$

$$\|\xi_{\mathfrak{D}}\|_{\infty,\mathfrak{D}} = \max_{\mathcal{D}\in\mathfrak{D}}|\xi_{\mathcal{D}}|,\ for\ all\ \xi_{\mathfrak{D}} \in (IR^2)^{\mathfrak{D}}. \tag{2.13}$$

Theorem 2.5 (see [18])**.** *For all* $(\xi_{\mathfrak{D}}, v_{\mathcal{T}}) \in (IR^2)^{\mathfrak{D}} \times IR^{\mathcal{T}}$ *we have*

$$< div^{\mathcal{T}}\xi_{\mathfrak{D}}, v_{\mathcal{T}} >_{\mathcal{T}} = - < \xi_{\mathfrak{D}}, \nabla^{\mathfrak{D}}v_{\mathcal{T}} >_{\mathfrak{D}}. \tag{2.14}$$

2.3. The numerical schemes. Let $(\mathcal{T},\mathcal{D})$ be a DDFV mesh of Ω, and $\delta t > 0$ be a time step we set $N_T = \frac{T}{\delta t}$, and we define $t_n = n\delta t$, for $n = \{0, ..., N_T\}$.

First we discrete all the data of the problem, so let $\mathcal{P}_{\mathcal{K}}$ (respectively $\mathcal{P}_{\mathcal{K}^*}$ the L^2 projection over a interior primal cell (resp. a dual cell), and $h_0^{\mathcal{T}} = ((\mathcal{P}_{\mathcal{K}}h_0)_{\mathcal{K}\in\mathfrak{M}}, 0, (\mathcal{P}_{\mathcal{K}^*}h_0)_{\mathcal{K}^*\in\mathfrak{M}^*})$ in a similarly way for all $n \geq 1$ we define $(I_s^n, I_f^n) \in (IR^{\mathcal{T}})^2$:

$$\begin{cases} I^n_{s,\mathcal{T}} = \frac{1}{\delta t}\int_{t_{n-1}}^{t_n} \left((\mathcal{P}_{\mathcal{K}} I_s(.,t))_{\mathcal{K}\in\mathfrak{M}}, 0, (\mathcal{P}_{\mathcal{K}^*} I_s(.,t))_{\mathcal{K}^*\in\overline{\mathfrak{M}^*}}\right) dt, \\ \\ I^n_{f,\mathcal{T}} = \frac{1}{\delta t}\int_{t_{n-1}}^{t_n} \left((\mathcal{P}_{\mathcal{K}} I_f(.,t))_{\mathcal{K}\in\mathfrak{M}}, 0, (\mathcal{P}_{\mathcal{K}^*} I_f(.,t))_{\mathcal{K}^*\in\overline{\mathfrak{M}^*}}\right) dt. \end{cases} \tag{2.15}$$

The numerical solution will be given by $(h_{\mathcal{T}}^{n+1}, f_{\mathcal{T}}^n) \in IR^{\mathcal{T}} \times IR^{\mathcal{T}}$ at each time step we have

$$div^{\mathcal{T}}(H_2 K_{\mathcal{D}} \nabla^{\mathfrak{D}} f^n_{\mathcal{T}}) - div^{\mathcal{T}}(T_s(h^n) K_{\mathcal{D}} \nabla^{\mathfrak{D}} h^n_{\mathcal{T}}) = I^n_{s,\mathcal{T}} + I^n_{f,\mathcal{T}} \tag{2.16}$$

$$\frac{h^{n+1}_{\mathcal{T}} - h^n_{\mathcal{T}}}{dt} + div^{\mathcal{T}}(T_s(h^n) K_{\mathcal{D}} \nabla^{\mathfrak{D}} h^{n+1}_{\mathcal{T}}) - div^{\mathcal{T}}(T_s(h^n) K_{\mathcal{D}} \nabla^{\mathfrak{D}} f^n_{\mathcal{T}}) = -I^n_{s,\mathcal{T}}. \tag{2.17}$$

With

$$K_{\mathcal{D}} = \frac{1}{m_{\mathcal{D}}} \int_{\mathcal{D}} K(x)dx \tag{2.18}$$

2.4. The propriety of the scheme. We denoted $\alpha_{\mathcal{T}} \in]0, \frac{\pi}{2}[$ such that

$$sin(\alpha_{\mathcal{T}}) := min_{\mathcal{D}\in\mathfrak{D}} |sin(\alpha_{\mathcal{D}})|,$$

for the mesh direct and

$$sin(\alpha_{\mathcal{T}}) := min_{\mathcal{D}\in\mathfrak{D}}(|sin(\alpha_{\mathcal{K}})|, |sin(\alpha_{\mathcal{L}})|),$$

for the barycentre dual mesh so we have

$$\mathcal{N}_{\mathcal{T}} = sup_{x\in\Omega} Card\left(\mathcal{D} \text{ such that } x \in \widehat{\mathcal{D}\cup\mathcal{K}}, \mathcal{D} \in \mathfrak{D}_{\mathcal{K}}, \mathcal{K} \in \mathfrak{M}\right) + sup_{x\in\Omega} Card\left(\mathcal{D} \text{ such that } x \in \widehat{\mathcal{D}\cup\mathcal{K}^*}, \mathcal{D} \in \mathfrak{D}_{\mathcal{K}^*}, \mathcal{K}^* \in \mathfrak{M}^* \cup \partial\mathfrak{M}^*\right).$$

with $\widehat{I}$ is the envelope convex of the set I.
Hens, the regularity of the mesh is given by

$$reg(\mathcal{T}) = \max(\frac{1}{sin(\alpha_{\mathcal{T}})}, \mathcal{N}, \mathcal{N}^*, \mathcal{N}_{\mathcal{T}}, \max_{\mathcal{D}\in\mathfrak{D}} \max_{\mathfrak{L}\in\Omega_{\mathcal{D}}} \frac{d_{\mathcal{D}}}{\min\limits_{\sigma\in\partial\mathfrak{L}} m_\sigma}, \max_{\mathcal{K}\in\mathfrak{M}} \left(\frac{d_{\mathcal{K}}}{\sqrt{m_{\mathcal{K}}}}\right), \max_{\mathcal{K}^*\in\mathfrak{M}^*} \left(\frac{d_{\mathcal{K}^*}}{\sqrt{m_{\mathcal{K}^*}}}\right), \max_{\mathcal{K}\in\mathfrak{M}} \max_{\mathcal{D}\in\mathfrak{D}_{\mathcal{K}}} \left(\frac{d_{\mathcal{K}}}{d_{\mathcal{D}}}\right), \max_{\mathcal{K}^*\in\mathfrak{M}^*} \max_{\mathcal{D}\in\mathfrak{D}_{\mathcal{K}^*}} \left(\frac{d_{\mathcal{K}^*}}{.} d_{\mathcal{D}}\right)).$$

With $\mathcal{N}$ is the number maximum of control volume $\mathcal{K} \in \mathfrak{M}$ and $\mathcal{N}^*$ is the number maximum of control volume $\mathcal{K}^* \in \mathfrak{M}^*$ and there exists $C > 0$, dependant uniqueness on $reg(\mathcal{T})$ such that

$$\frac{d_{\mathcal{D}}}{\sqrt{m_{\mathcal{D}}}} \leq C, \ for \ all \ \mathcal{D} \in \mathfrak{D} \text{ and } d_{\mathcal{D}} \leq C min(m_\sigma, m_{\sigma^*}). \tag{2.19}$$

Remark 2.6. Let

$diam(\widehat{\mathcal{K}}) \leq 2d_{\mathcal{K}},\ for\ all\ \mathcal{K} \in \mathfrak{M}$ and $diam(\widehat{\mathcal{K}^*}) \leq 2d_{\mathcal{K}^*},\ for\ all\ \mathcal{K}^* \in \mathfrak{M}^* \cup \partial\mathfrak{M}^*$.

That's implies

$$diam(\widehat{\mathcal{D}}) \leq 2d_{\mathcal{D}},\ for\ all\ \mathcal{D} \in \mathfrak{D}.$$

Now, we cite the Poincaré inequality, this inequality is recalled in lemma 2.7 we refer to [27] or [28] for its proof.

Lemma 2.7 (Poincaré inequality)**.** *Let Ω be an open bounded connected polygonal domain of IR^2 and $\mathcal{T}$ a DDFV mesh of Ω. There exists $C > 0$ depending only on Ω and on ψ, such that for all $u_{\mathcal{T}} \in IR^{\mathcal{T}}$ with $\sum_{\mathcal{K}\in\mathfrak{M}} m_{\mathcal{K}} u_{\mathcal{K}} = \sum_{\mathcal{K}^*\in\mathfrak{M}^*} m_{\mathcal{K}^*} u_{\mathcal{K}^*} = 0$, we have*

$$\|u_{\mathcal{T}}\|_{2,\mathcal{T}} \leq \frac{C}{sin(\alpha_{\mathcal{T}})} \|\nabla u_{\mathcal{T}}\|_{2,\mathfrak{D}}. \tag{2.20}$$

Lemma 2.8 (see [29])**.** *For all $(\xi_{\mathcal{D}}, v_{\mathcal{T}}) \in (IR^2)^{\mathcal{D}} \times IR^{\mathcal{T}}$ we have*

$$< div^{\mathcal{T}} \xi_{\mathcal{D}}, v_{\mathcal{T}} >_{\mathcal{T}} = -(\xi_{\mathcal{D}}, \nabla^{\mathcal{D}} v_{\mathcal{T}})_{\mathcal{D}} + < \gamma^{\mathcal{D}}(\xi_{\mathcal{D}}).n, \gamma^{\mathcal{T}}(v^{\mathcal{T}}) >_{\partial\Omega} . \tag{2.21}$$

We recall some proprieties of DDFV scheme, the proof of this lemmas can be found in [30].

Lemma 2.9. *Let $\mathcal{T}$ be a family of mesh of Ω in the sense of definition* (2.1)*, and let $u^{\mathcal{T}} \in IR_0^{\mathcal{T}}$, such that*

- *There exists $C_1 > 0$ with $\|\nabla^{\mathcal{T}} u^{\mathcal{T}}\|_{L^2(\Omega)} \leq C_1$.*

Than, there exists a subsequence, denoted by $u^{\mathcal{T}}$ for simplicity, and a function $u \in H_0^1(\Omega)$ such that $u^{\mathcal{T}}$ converge to u in $L^2(\Omega)$ and such that the gradient $\nabla^{\mathcal{T}} u^{\mathcal{T}}$ weakly converge to ∇u in $(L^2(\Omega))^2$ as $\frac{h}{\mathcal{T}}$ tend to 0*.*

Proposition 2.10 (See [29])**.** *Let $(\mathcal{T}_m)_m$ be a sequence of DDFV meshes satisfying $size(\mathcal{T}_m \to 0wwhen\ m \to \infty$ and (RT). Let $(\delta t_m)_{m\geq 1}$ be a sequence of time steps such that $T/\delta t_m$ is an integer and $\delta t_m \to O$ when $m \to \infty$. We consider a sequence of functions $(v_m)_m$ with $v_m = v_{h_m,\delta t_m} \in H_{\mathcal{T}_m,\delta t_m}$ when $m \to \infty$ such that*

- *$v_m \to v$ weakly in $L^2((0,T)\times\Omega)$ (respectivly weakly$-*$ in $L^\infty(0,T;L^2(\Omega))$);*
- *$\nabla^{h_m} v_m \to \xi$ weakly in $(L^2((0,T)\times\Omega))^2$ (respectivly weakly$-*$ in $L^\infty(0,T;L^2(\Omega))$);*

then, we have

$$\nabla v = \xi \text{ and } v \in L^2(0,T;H^1(\Omega)) \text{ (respectively } L^\infty(0,T;H^1(\Omega))\text{)}.$$

Lemma 2.11 (See [18])**.** *Let $\mathcal{T}$ be a mesh of Ω in the sense of definition* (2.1)*. Then, for any function $\phi \in C_c^\infty(\Omega)$ the discret gradient $\nabla^{\mathcal{T}} \mathcal{P}_{\mathcal{T}} \phi$ strongly converge in $L^2(\Omega))^2$ to $\nabla\phi$ as $\frac{h}{\mathcal{T}}$ tend to* 0*.*

3. A priori estimate

Lemma 3.1. *Let Ω be an open bounded connected polygonal domain of IR^2 and let $\mathcal{D}$ be a DDFV mesh of Ω in the sense of the definition* (2.1)*. Assume* (1.2)*-*(1.8) *hold and that the scheme* (2.16) *and* (2.17) *has a solution $(f^n, h^{n+1})_{1\leq n\leq N_T}$. Then, there exists $C_1 > 0$ depending only on Ω, $\alpha_{\mathcal{D}}$, h_0, γ_B and α_A, and $C_2 > 0$*

depending only on Ω, $\alpha_{\mathcal{D}}$, h_0, γ_B *and* α_A *such that we have for all* $n \in [0, ..., N-1]$ *with* $1 \leq N \leq N_T$.

$$\begin{cases} \|\nabla f^n\|_{2,\mathfrak{D}} \leq C1, \\ ||h_{\mathcal{D}}^N||_{2,\mathcal{T}} + \sum_{n=0}^{N-1} \alpha_A \|\nabla h^{n+1}\|_{2,\mathfrak{D}}^2 \leq C2. \end{cases} \tag{3.1}$$

Proof. **Step 1:** multiplying (2.16) by f^n we get

$$< div^{\mathcal{T}}(T_s(h^n)K_{\mathcal{D}}\nabla^{\mathcal{D}}h^n), f^n >_{\mathfrak{D}} - < div^{\mathcal{T}}(H_2K_{\mathcal{D}}\nabla^{\mathcal{D}}f^n), f^n >_{\mathfrak{D}} = < I_s + I_f, f^n >_{\mathcal{T}} .$$

Then

$$- < T_s(h_{\mathcal{D}}^n K_{\mathcal{D}}\nabla^{\mathcal{D}}h^n, \nabla^{\mathcal{D}}f^n >_{\mathfrak{D}} + < H_2K_{\mathcal{D}}\nabla^{\mathcal{D}}f^n, \nabla^{\mathcal{D}}f^n >_{\mathfrak{D}} = < I_s + I_f, f^n >_{\mathcal{T}} .$$

That's give

$$< H_2K_{\mathcal{D}}(x)\nabla^{\mathcal{D}}f^n, \nabla^{\mathcal{D}}f^n >_{\mathfrak{D}} = < I_s + I_f, f^n >_{\mathcal{T}} + < T_s(h_{\mathcal{D}}^n)K_{\mathcal{D}}\nabla^{\mathcal{D}}h^n, \nabla^{\mathcal{D}}f^n >_{\mathfrak{D}} .$$

Then, using hypothesis (1.7) , (1.8) and Cauchy-Schwarz inequality, we have

$$\Lambda_A H_2 \|\nabla f^n\|_{2,\mathfrak{D}}^2 \leq \|I_s^n + I_f^n\|_{2,\mathcal{T}} \|f^n\|_{2,\mathcal{T}} + \Lambda_B \|\sqrt{T_s(h^n)}\nabla h^n\|_{2,\mathfrak{D}} \|\nabla f^n\|_{2,\mathfrak{D}}.$$

Applying now the discrete Poincaré inequality (2.20), we get

$$\Lambda_A H_2 \|\nabla f^n\|_{2,\mathfrak{D}} \leq \|I_s^n + I_f^n\|_{2,\mathcal{T}} + \Lambda_B \|\sqrt{T_s(h^n)}\nabla h^n\|_{2,\mathfrak{D}}.$$

As

$$\max_{1 \leq n \leq N_T} \|I_s^n + I_f^n\|_{2,\mathcal{T}} \leq \|I_s + I_f^n\|_{L^\infty_{(0,T;L^2(\Omega))}}.$$

Then

$$\|\nabla f^n\|_{2,\mathfrak{D}} \leq \frac{C}{\Lambda_A H_2 sin(\alpha_{\mathcal{T}})} \|I_s + I_f^n\|_{L^\infty_{(0,T;L^2(\Omega))}} + \frac{\Lambda_B}{\Lambda_A H_2} \|\sqrt{T_s(h^n)}\nabla h^n\|_{2,\mathfrak{D}}. \tag{3.2}$$

and

$$\|\nabla f^n\|_{2,\mathfrak{D}} \leq \frac{C}{\Lambda_A H_2 sin(\alpha_{\mathcal{T}})} \|I_s + I_f^n\|_{L^\infty_{(0,T;L^2(\Omega))}} + \frac{\Lambda_B}{\Lambda_A H_2} \|\sqrt{T_s(h^n)}\|_{L^\infty(\Omega)} \|\nabla h^n\|_{2,\mathfrak{D}}. \tag{3.3}$$

Step 2: multiplying (2.17) by h^{n+1} we get

$$m_{\mathcal{K}}(h_{\mathcal{K}}^{n+1} - h_{\mathcal{K}}^n)h_{\mathcal{K}}^{n+1} - \delta t < div(T_s(h^n)K_{\mathcal{D}}\nabla h^{n+1}), h_{\mathcal{K}}^{n+1} >_{\mathfrak{D}} +$$
$$\delta t < div(T_s(h^n)K_{\mathcal{D}}\nabla f^n), h_{\mathcal{K}}^{n+1} >_{\mathfrak{D}} = -\delta t < I_s, h_{\mathcal{K}}^{n+1} >_{\mathcal{T}} .$$

Let us analyze successively the different terms in this equality. The relation

$$(a - b)a \geq \frac{1}{2}(a^2 - b^2), \tag{3.4}$$

ensures

$$m_{\mathcal{K}}(h_{\mathcal{K}}^{n+1} - h_{\mathcal{K}}^n)h_{\mathcal{K}}^{n+1} \geq m_{\mathcal{K}}\frac{1}{2}((h_{\mathcal{K}}^{n+1})^2 - (h_{\mathcal{K}}^n)^2). \tag{3.5}$$

Summing over $n = 0, ..., N-1$ with $1 \leq N \leq N_T$ the formula (3.5), we get

$$\sum_{n=0}^{N-1} m_{\mathcal{K}}(h_{\mathcal{K}}^{n+1} - h_{\mathcal{K}}^n)h_{\mathcal{K}}^{n+1} \geq m_{\mathcal{K}}((h_{\mathcal{K}}^N)^2 - (h_{\mathcal{K}}^0)^2).$$

Then

$$\sum_{\mathcal{K}\in\mathcal{M}}\sum_{n=0}^{N-1} m_{\mathcal{K}}(h_{\mathcal{K}}^{n+1}-h_{\mathcal{K}}^{n})h_{\mathcal{K}}^{n+1} \geq \frac{1}{2}\sum_{\mathcal{K}\in\mathcal{M}} m_{\mathcal{K}}((h_{\mathcal{K}}^{N})^2-(h_{\mathcal{K}}^{0})^2).$$

But the hypothesi (1.3) en h_0 ensure

$$\sum_{\mathcal{K}\in\mathcal{M}}\sum_{n=0}^{N-1} m_{\mathcal{K}}(h_{\mathcal{K}}^{n+1}-h_{\mathcal{K}}^{n})h_{\mathcal{K}}^{n+1} \geq ||h_{\mathcal{T}}^{N}||_{2,\mathcal{T}}-\delta. \tag{3.6}$$

Thanks to the hypothesis (1.7) and (3.6) lead to

$$\begin{cases} -\delta t\sum_{n=0}^{N-1} < div(T_s(h^n)K_{\mathcal{D}}\nabla h^{n+1}), h_{\mathcal{T}}^{n+1} >_{\mathfrak{D}} = \delta t\sum_{n=0}^{N-1} < T_s(h^n)K_{\mathcal{D}}\nabla h^{n+1}, \nabla h_{\mathcal{T}}^{n+1} >_{\mathfrak{D}} \\ \geq \Lambda_A\sum_{n=0}^{N-1} \|T_s(h^n)\|_{L^\infty}\|\nabla h_{\mathcal{T}}^{n+1}\|_{2,\mathcal{D}}^2. \end{cases} \tag{3.7}$$

$$\begin{cases} \delta t\sum_{n=0}^{N-1} < div(T_s(h_{\mathcal{T}}^n)K_{\mathcal{D}}\nabla f_{\mathcal{T}}^n), h_{\mathcal{T}}^n >_{\mathcal{D}} = -\delta t\sum_{n=0}^{N-1} < T_s(h_{\mathcal{T}}^n)K_{\mathcal{D}}\nabla f_{\mathcal{T}}^n, \nabla h_{\mathcal{T}}^n >_{\mathcal{D}} \\ \geq -\Lambda_B\delta t\sum_{n=0}^{N-1} \|T_s(h_{\mathcal{T}}^n)\|_{L^\infty}\|\nabla f_{\mathcal{T}}^n\|_{2,\mathcal{D}}\|\nabla h_{\mathcal{T}}^n\|_{2,\mathcal{D}} \end{cases} \tag{3.8}$$

and

$$\sum_{n=0}^{N-1}\delta t\sum_{\mathcal{D}\in\mathfrak{D}} I_{s,\mathcal{T}}^n f_{\mathcal{T}}^n \leq \sum_{n=0}^{N-1}\delta t\|I_{s,\mathcal{T}}^n\|_{2,\mathcal{T}}\|f_{\mathcal{T}}^n\|_{2,\mathcal{T}} \tag{3.9}$$

Finally we have

$$\begin{cases} ||h_{\mathcal{T}}^N||_{2,\mathcal{T}}-\delta+\Lambda_A\sum_{n=0}^{N-1}\|T_s(h^n)\|_{L^\infty}\|\nabla h_{\mathcal{T}}^{n+1}\|_{2,\mathcal{D}}^2 \\ -\Lambda_B\delta t\sum_{n=0}^{N-1}\|T_s(h_{\mathcal{T}}^n)\|_{L^\infty}\|\nabla f_{\mathcal{T}}^n\|_{2,\mathcal{D}}\|\nabla h_{\mathcal{T}}^n\|_{2,\mathcal{D}} \\ \leq \sum_{n=0}^{N-1}\delta t\|I_{s,\mathcal{T}}^n\|_{2,\mathcal{T}}\|f_{\mathcal{T}}^n\|_{2,\mathcal{T}} \end{cases} \tag{3.10}$$

$$\begin{cases} -\delta t < div(T_s(h^n)K_{\mathcal{D}}\nabla h_{\mathcal{T}}^{n+1}), h_{\mathcal{T}}^{n+1} >_{\mathfrak{D}} +\delta t < div(H_2K_{\mathcal{D}}\nabla f^n), h_{\mathcal{T}}^{n+1} >_{\mathfrak{D}} \geq \\ ||h_{\mathcal{T}}^N||_{2,\mathcal{T}}-\sum_{n=0}^{N-1}\Lambda_A\|\nabla h_{\mathcal{T}}^{n+1}\|_{2,\mathfrak{D}}\|\nabla f^n\|_{2,\mathfrak{D}}+\sum_{n=0}^{N-1}\alpha_A\|\nabla h_{\mathcal{T}}^{n+1}\|_{2,\mathfrak{D}}^2-\delta. \end{cases} \tag{3.11}$$

Using (3.11) and (3.6) , we have

$$||h_{\mathcal{T}}^N||_{2,\mathcal{T}}-\sum_{n=0}^{N-1}\Lambda_A\|\nabla h_{\mathcal{T}}^{n+1}\|_{2,\mathfrak{D}}\|\nabla f^n\|_{2,\mathfrak{D}}+\sum_{n=0}^{N-1}\alpha_A\|\nabla h_{\mathcal{T}}^{n+1}\|_{2,\mathfrak{D}}^2 \leq \sum_{n=0}^{N-1}\|I_s\|_{2,\mathcal{T}}\|h_{\mathcal{T}}^{n+1}\|_{2,\mathcal{T}}+\delta.$$

As

$$\sum_{n=0}^{N-1}\|I_s\|_{2,\mathcal{T}}\|h_{\mathcal{T}}^{n+1}\|_{2,\mathcal{T}} \le T\|I_s\|_{L^\infty(0,T;L^2(\Omega))} \sup_{n\in[0,\dots,N-1]}\|h_{\mathcal{T}}^{n+1}\|_{2,\mathcal{T}}$$
$$\le T\|I_s\|^2_{L^\infty(0,T;L^2(\Omega))} + \sup_{n\in[0,\dots,N-1]}\|h_{\mathcal{T}}^{n+1}\|^2_{2,\mathcal{T}}.$$

Then

$$||h_{\mathcal{T}}^N||_{2,\mathcal{T}}+\sum_{n=0}^{N-1}\|\nabla h_{\mathcal{T}}^{n+1}\|_{2,\mathfrak{D}}(\alpha_A\|\nabla h_{\mathcal{T}}^{n+1}\|_{2,\mathfrak{D}}-\Lambda_A\|\nabla f^n\|_{2,\mathfrak{D}}) \le T\|I_s\|^2_{L^\infty(0,T;L^2(\Omega))}+\sup_{n\in[0,\dots,N-1]}\|h_{\mathcal{T}}^{n+1}\|^2_{2,\mathcal{T}}+\delta.$$

Using (3.4) we have

$$||h_{\mathcal{T}}^N||_{2,\mathcal{T}}+\sum_{n=0}^{N-1}\frac{\alpha_A}{2}\|\nabla h_{\mathcal{T}}^{n+1}\|^2_{2,\mathfrak{D}}-\sum_{n=0}^{N-1}\frac{\Lambda_A^2}{2\alpha_A}\|\nabla f^n\|^2_{2,\mathfrak{D}} \le T\|I_s\|^2_{L^\infty(0,T;L^2(\Omega))}+\sup_{n\in[0,\dots,N-1]}\|h_{\mathcal{T}}^{n+1}\|^2_{2,\mathcal{T}}+\delta,$$

Finally we have

$$||h_{\mathcal{T}}^N||_{2,\mathcal{T}}+\sum_{n=0}^{N-1}\frac{\alpha_A}{2}\|\nabla h_{\mathcal{T}}^{n+1}\|^2_{2,\mathfrak{D}} \le \sum_{n=0}^{N-1}\frac{\Lambda_A^2}{2\alpha_A}\|\nabla f^n\|^2_{2,\mathfrak{D}}+T\|I_s\|^2_{L^\infty(0,T;L^2(\Omega))}+\sup_{n\in[0,\dots,N-1]}\|h^{n+1}\|^2_{2,\mathcal{T}}+\delta, \tag{3.12}$$

The estimates (3.1) are is a consequence of (3.12) and (3.6), Than there exists an $C1$ and $C2$ such that

$$\|\nabla f^n\|_{2,\mathfrak{D}} \le C1.$$

$$||h_{\mathcal{T}}^N||_{2,\mathcal{T}} + \sum_{n=0}^{N-1}\alpha_A\|\nabla h_{\mathcal{T}}^{n+1}\|^2_{2,\mathfrak{D}} \le C2.$$

□

4. Convergence analysis

Lemma 4.1 (see [29]). *Let $\mathcal{T}$ be a mesh of Ω in the sense of Definition 2.1. Then, for any $(u^{\mathcal{T}}, v^{\mathcal{T}}) \in H_0^{\mathcal{T}} \times H_0^{\mathcal{T}}$, we have*

$$\int_0^T\int_\Omega -div(\Lambda(x)\nabla u(x))v(x)dx = \sum_{n=0}^{N-1}\sum_{\mathcal{D}\in\mathfrak{D}} m_{\mathcal{D}}\Lambda_{\mathcal{D}}(\nabla^{\mathcal{D}}u^n, \nabla^{\mathcal{D}}v^n), \tag{4.1}$$

where $\Lambda_{\mathcal{D}} = \int_{\mathcal{D}}\Lambda(x)dx$.

Remark 4.2. The DDFV approximation to the first equation of the problem (1.1) is given as the solution of the following equation

$$\begin{cases} f \in H_0^{\mathcal{T}}, \\ \int_0^T\int_\Omega H_2K(x)\nabla^{\mathcal{T}}f\nabla^{\mathcal{T}}vdx = \int_0^T\int_\Omega gv, \ for\ all\ v\in H_0^{\mathcal{T}}. \end{cases} \tag{4.2}$$

With $g = I_S + I_f - div(A(x)h)$.

Convergence analysis of the freshwater hydraulic head.

Theorem 4.3. *Let Ω be an open bounded connected polygonal domain of IR^2 and $T > 0$. Assume hypothesis ((1.2)-(1.8)) hold, let $(\mathcal{T}_m)$ be a sequence of DDFV meshes such that $size(\mathcal{T}_m) \to 0$ wile the regularity verified $\theta_m = reg(\mathcal{T}_m)$*

$$\exists \theta > 0 \text{ such that } \theta_m \leq \theta. \tag{4.3}$$

Let $(\delta t_m)_{m\leq 1}$ be a sequence of time steps such that $T/\delta t_m$ is an integre and $\delta t_m \to 0$ if $m \to \infty$.

Then, there exists $\overline{f} \in F$ such that the sequences (f_m) defined by the scheme (2.16)-(2.17) have the following convergence result when $m \to \infty$

- *$f_m \to \overline{f}$ strongly in $L^2(0, T; L^2(\Omega))$.*
- *$\nabla^m f_m \to \nabla\overline{f}$ strongly in $(L^2(0, T; L^2(\Omega)))^2$.*

Proof. **Step 1:**Let $\mathcal{T}_m$ be a sequence of meshes such that $l_{\mathcal{T}_m} = size(\mathcal{T}_m)$ tends to 0 as $n \to \infty$, and $(\delta_m)_m$ be a sequence of time steps such that T/δ_m is an integer and $\delta_m \to 0$ when $m \to \infty$. Let h fixed in $L^2(0, T; L^2(\Omega))$. Thanks to Lemma 2.9 and Lemma 3.1 there exists a subsequence (again denoted $f_{\mathcal{T}_m}$ and $f_{\mathcal{T}_m} \in H_0^1(\Omega)$) such that

$$\begin{cases} f_{\mathcal{T}_m} \to \overline{f}, & \text{strongly in } L^2(0, T; L^2(\Omega)) \\ \nabla^{\mathcal{T}_m} f_{\mathcal{T}_m} \to \nabla\overline{f}, & \text{weakly in } L^2(0, T; L^2(\Omega))^2. \end{cases} \tag{4.4}$$

Let $\varphi \in \mathcal{C}_c^\infty(\Omega)$ be given (for approximate $\overline{f} \in H_0^1(\Omega)$), $v = \mathcal{P}\varphi$ in the second equation of (1.10) we get

$$\begin{cases} \int_0^T \int_\Omega H_2 K(x) \nabla^{\mathcal{T}_m} f_{\mathcal{T}_m} \nabla^{\mathcal{T}_m} \mathcal{P}_{\mathcal{T}_m}\varphi & = \int_0^T \int_\Omega (I_s + I_f) \mathcal{P}_{\mathcal{T}_m}\varphi dx + \\ & \int_0^T \int_\Omega T_s(h) K(x) \nabla h \nabla^{\mathcal{T}_m} \mathcal{P}_{\mathcal{T}_m}\varphi. \end{cases} \tag{4.5}$$

Since

$$\nabla^{\mathcal{T}_m} f_{\mathcal{T}_m} \to \nabla\overline{f} \text{ weakly in } L^2(0, T; L^2(\Omega))^2.$$

and the consistency propert of the discrete gradient we have

$$\nabla^{\mathcal{T}_m} \mathcal{P}_{\mathcal{T}_m}\varphi \to \nabla\varphi.$$

This implies that

$$\lim_{l_{\mathcal{T}_m}\to 0} \int_0^T \int_\Omega H_2 K(x) \nabla^{\mathcal{T}_m} f_{\mathcal{T}_m} \nabla^{\mathcal{T}_m} \mathcal{P}_{\mathcal{T}_m}\varphi = \int_0^T \int_\Omega \nabla\overline{f} H_2 K(x) \nabla\varphi dx, \tag{4.6}$$

and we have

$$\begin{cases} \lim_{l_{\mathcal{T}_m}\to 0} \int_0^T \int_\Omega T_s(h) K(x) \nabla h \nabla^{\mathcal{T}_m} \mathcal{P}_{\mathcal{T}_m}\varphi + \lim_{l_{\mathcal{T}_m}\to 0} \int_0^T \int_\Omega (I_s + I_f) \nabla^{\mathcal{T}_m} \mathcal{P}_{\mathcal{T}_m}\varphi = \\ \int_0^T \int_\Omega T_s(h) K(x) \nabla h \nabla\varphi + \int_0^T \int_\Omega (I_s + I_f) \nabla\varphi. \end{cases}$$

Which concludes the proof that

$$\int_0^T \int_\Omega \nabla\overline{f} H_2 K(x) \nabla\varphi = \int_0^T \int_\Omega \nabla h T_s(h) K(x) \nabla\varphi + \int_0^T \int_\Omega (I_s + I_f) \nabla\varphi.$$

Than, $\overline{f}$ is the unique solution of (1.1), and we get that the whole family $f^{\mathcal{T}_m}$ converge to $\overline{f}$ as $l_m \to 0$.

Step 2: Let us first prove that $J_n = \int_0^T \int_\Omega (\nabla^{\mathcal{T}_m} f^{\mathcal{T}_m}(x) - \nabla \overline{f}(x))^2 dx$ tend to 0 as $l_{\mathcal{T}_m} \to 0$. For that, using Cauchy-Schwartz inequality, we have

$$\int_0^T \int_\Omega (\nabla^{\mathcal{T}_m} f^{\mathcal{T}_m}(x) - \nabla \overline{f}(x))^2 dx \leq 3(J_1^{\mathcal{T}_m} + J_2^{\mathcal{T}_m} + J_3^{\mathcal{T}_m}),$$

with

$$\begin{aligned}
J_1^{\mathcal{T}_m} &= \int_0^T \int_\Omega (\nabla^{\mathcal{T}_m} f^{\mathcal{T}_m}(x) - \nabla^{\mathcal{T}_m} \mathcal{P}_{\mathcal{T}_m}\varphi(x))^2 dx,\\
J_2^{\mathcal{T}_m} &= \int_0^T \int_\Omega (\nabla^{\mathcal{T}_m} \mathcal{P}_{\mathcal{T}_m}\varphi(x) - \nabla\varphi(x))^2 dx,\\
J_3^{\mathcal{T}_m} &= \int_0^T \int_\Omega (\nabla\varphi(x) - \nabla \overline{f}(x))^2 dx.
\end{aligned}$$

Using the consistency property of discrete gradient recall in [30] we have

$$\lim_{l_{\mathcal{T}_m} \to 0} J_2^{\mathcal{T}_m} = 0.$$

Let us take $a_{\mathcal{T}}(f_{\mathcal{T}_m}, f_{\mathcal{T}_m}) = \int_0^T \int_\Omega \nabla^{\mathcal{T}_m} f_{\mathcal{T}_m} H_2 K(x) \nabla^{\mathcal{T}_m} f_{\mathcal{T}_m}$, $a_{\mathcal{T}}$ is coercive so we have

$$\begin{cases} J_1^{\mathcal{T}_m} & \leq C a_{\mathcal{T}}(f_{\mathcal{T}_m} - \mathcal{P}_{\mathcal{T}_m}\varphi, f_{\mathcal{T}_m} - \mathcal{P}_{\mathcal{T}_m}\varphi) \\ & \leq C\left[a_{\mathcal{T}}(f_{\mathcal{T}_m}, f_{\mathcal{T}_m}) - 2a_{\mathcal{T}}(f_{\mathcal{T}_m}, \mathcal{P}_{\mathcal{T}_m}\varphi) + a_{\mathcal{T}}(\mathcal{P}_{\mathcal{T}_m}\varphi, \mathcal{P}_{\mathcal{T}_m}\varphi)\right], \end{cases} \tag{4.7}$$

as $f_{\mathcal{T}_m}$ is the solution of (4.2), we deduce that

$$a_{\mathcal{T}}(f_{\mathcal{T}_m}, f_{\mathcal{T}_m}) = \int_0^T \int_\Omega ((I_s + I_f) f_{\mathcal{T}_m} + T_s(h) K(x) \nabla h \nabla f_{\mathcal{T}_m})\, dx, \tag{4.8}$$

$$a_{\mathcal{T}}(f_{\mathcal{T}_m}, \mathcal{P}_{\mathcal{T}_m}\varphi) = \int_0^T \int_\Omega ((I_s + I_f) \mathcal{P}_{\mathcal{T}_m}\varphi + T_s(h) K(x) \nabla h \nabla \mathcal{P}_{\mathcal{T}_m}\varphi)\, dx, \tag{4.9}$$

It results that

$$\lim_{l_{\mathcal{T}_m} \to 0} a_{\mathcal{T}}(f_{\mathcal{T}_m}, f_{\mathcal{T}_m}) = \int_0^T \int_\Omega \left((I_s + I_f)\overline{f} + T_s(h) K(x) \nabla h \nabla \overline{f}\right) dx, \tag{4.10}$$

$$\lim_{l_{\mathcal{T}_m} \to 0} a_{\mathcal{T}}(f_{\mathcal{T}_m}, \mathcal{P}_{\mathcal{T}_m}\varphi) = \int_0^T \int_\Omega ((I_s + I_f)\mathcal{P}\varphi + T_s(h) K(x) \nabla h \nabla \mathcal{P}\varphi)\, dx, \tag{4.11}$$

in (4.2) take $v = \mathcal{P}_{\mathcal{T}_m}\varphi$ as test function, we obtian

$$\int_0^T \int_\Omega \nabla^{\mathcal{T}_m} f_{\mathcal{T}_m} H_2 K(x) \nabla^{\mathcal{T}_m} \mathcal{P}_{\mathcal{T}_m}\varphi dx = \int_0^T \int_\Omega \left((I_s + I_f)\mathcal{P}_{\mathcal{T}_m}\varphi + \nabla h T_s(h) K(x) \nabla^{\mathcal{T}_m} \mathcal{P}_{\mathcal{T}_m}\varphi\right) dx. \tag{4.12}$$

The gradient $\nabla^{\mathcal{T}_m} f_{\mathcal{T}_m}$ converge weakly to $\nabla\overline{f}$ and thanks to consistency property of the discrete gradient we have $\nabla^{\mathcal{T}_m} \mathcal{P}_{\mathcal{T}_m}\varphi$ converges to $\nabla\varphi$, we deduce

$$\lim_{l_{\mathcal{T}_m} \to 0} a_{\mathcal{T}}(\mathcal{P}_{\mathcal{T}_m}\varphi, \mathcal{P}_{\mathcal{T}_m}\varphi) = \int_0^T \int_\Omega \nabla\varphi H_2 K(x) \nabla\varphi dx, \tag{4.13}$$

$$\lim_{l_{\mathcal{T}_m} \to 0} a_{\mathcal{T}}(f_{\mathcal{T}_m}, \mathcal{P}_{\mathcal{T}_m}\varphi) = \int_0^T \int_\Omega \nabla\overline{f} H_2 K(x) \nabla\varphi dx, \tag{4.14}$$

using the formula (4.14) and (4.10) we have

$$\begin{cases} 2 \lim_{l_{\mathcal{T}_m} \to 0} a_{\mathcal{T}_m}(f_{\mathcal{T}_m}, \mathcal{P}_{\mathcal{T}_m}\varphi) = \\ \int_0^T \int_\Omega ((I_s + I_f)\varphi(x) + \nabla h T_s(h) K(x) \nabla\varphi(x)) dx + \int_0^T \int_\Omega \nabla \overline{f}(x) H_2 K(x) \nabla\varphi(x) dx. \end{cases} \tag{4.15}$$

Summing the limits (4.10),(4.13) and (4.15) in (4.7) we obtain that

$$\begin{cases} \lim_{l_{\mathcal{T}_m} \to 0} J_1^{\mathcal{T}_m} \leq \int_0^T \int_\Omega ((I_s + I_f)(\overline{f}(x) - \varphi(x)) + \nabla h T_s(h) K(x) \nabla(\overline{f}(x) - \varphi(x))) dx + \\ C \int_0^T \int_\Omega \nabla(\varphi(x) - \overline{f}(x)) H_2 K(x) \nabla\varphi(x) dx, \end{cases} \tag{4.16}$$

then we have,

$$\lim_{l_{\mathcal{T}_m} \to 0} J_1^{\mathcal{T}_m} \leq C_1 \|\overline{f} - \varphi\|_{L^2(0,T;L^2(\Omega))} + C_2 \|\nabla\varphi - \nabla\overline{f}\|_{L^2(0,T;L^2(\Omega))^2}. \tag{4.17}$$

Let $\epsilon > 0$, we may choose φ such that $\|\overline{f} - \varphi\|_{L^2(0,T;L^2(\Omega))^2} \leq \epsilon$ and $\|\nabla\varphi - \nabla\overline{f}\|_{L^2(0,T;L^2(\Omega))} \leq \epsilon$ and we may then choose $l_{\mathcal{T}_m}$ small enough so that $J_1^{\mathcal{T}_m} \leq \epsilon$. This completes the proof that

$$\lim_{l_{\mathcal{T}_m} \to 0} J_n = 0. \tag{4.18}$$

$\square$

Convergence analysis of the deft of the interface.

Theorem 4.4. *Let Ω be an open bounded connected polygonal domain of IR^2 and $T > 0$. Assume hypothesis (1.2)-(1.8) hold, let $(\mathcal{T}_m)$ be a sequence of DDFV meshes such that $l_m = size(\mathcal{T}_m) \to 0$ wile the regularity verified $\theta_m = reg(\mathcal{T}_m)$*

$$\exists \theta > 0 \text{ such that } \theta_m \leq \theta. \tag{4.19}$$

Let $(\delta t_m)_{m \leq 1}$ be a sequence of time steps such that $T/\delta t_m$ is an integer and $\delta t_m \to 0$ if $m \to \infty$. The sequence $(h_m)_m$ defined by the scheme (2.16)-(2.17) is relatively compact in $L^1(0, T; L^1(\Omega))$. Let us note by $\overline{h}$ its limit up to a subsequence. Then, $\overline{h}$ lies in $L^2(0, T; H^1(\Omega))$. Furthermore, up to a subsequence, we have, when $m \to \infty$ the function $\overline{h}$ satisfy (1.10).

Proof. Let $\varphi \in C^\infty([0, T] \times \Omega)$.

$$\begin{cases} \sum_{n=0}^{N-1} \delta t \partial h_{\mathcal{T}_m}^n + \sum_{n=0}^{N-1} \delta t div^{\mathcal{T}_m}(T_s(h_{\mathcal{T}}^n) K_{\mathcal{D}_m} \nabla^{\mathcal{D}_m} h_{\mathcal{T}_m}^{n+1}) - \\ \sum_{n=0}^{N-1} \delta t div^{\mathcal{T}_m}(T_s(h_{\mathcal{T}}^n) K_{\mathcal{D}_m} \nabla^{\mathcal{D}_m} f_{\mathcal{T}m}^n) = -\sum_{n=0}^{N-1} \delta t (I_{S,\mathcal{T}_m}^n). \end{cases}$$

multiplying by φ we obtain

$$T_0 + T_1 - T_2 = -T_3,$$

with

$$\begin{cases} T_0 = \sum_{n=0}^{N-1} \delta t < \partial h^n_{\mathcal{T}_m}, \varphi^n_{\mathcal{T}_m} >_{\mathcal{T}_m}, \\ T_1 = \sum_{n=0}^{N-1} \delta t < div^{\mathcal{T}_m}(T_s(h^n_{\mathcal{T}})K_{\mathcal{D}_m}\nabla^{\mathcal{D}_m} h^{n+1}_{\mathcal{T}_m}), \varphi^n_{\mathcal{T}_m} >_{\mathcal{T}_m}, \\ T_2 = \sum_{n=0}^{N-1} \delta t < div^{\mathcal{T}_m}(T_s(h^n_{\mathcal{T}})K_{\mathcal{D}_m}\nabla^{\mathcal{D}_m} f^n_{\mathcal{T}_m}), \varphi^n_{\mathcal{T}_m} >_{\mathcal{T}_m}, \\ T_3 = \sum_{n=0}^{N-1} < I^n_{s,\mathcal{T}_m}, \varphi^n_{\mathcal{T}_m} >_{\mathcal{T}_m} . \end{cases}$$

We have

$$\begin{aligned} T_0 &:= \sum_{n=0}^{N-1} \delta t < \partial h^n_{\mathcal{T}_m}, \varphi^n_{\mathcal{T}_m} >_{\mathcal{T}_m} \\ &= -\sum_{n=0}^{N-1} \delta t < h^n_{\mathcal{T}_m}, \frac{\varphi^{n+1}_{\mathcal{T}_m} - \varphi^n_{\mathcal{T}_m}}{\delta t} >_{\mathcal{T}_m} - < h^0_{\mathcal{T}_m}, \varphi^0_{\mathcal{T}_m} >_{\mathcal{T}_m}, \end{aligned}$$

since $\varphi^N_{\mathcal{T}} = 0$. That's give

$$T_0 = -\int_0^T \int_\Omega h_{\mathcal{T}_m}(s,x)\frac{\varphi(s+\delta t,x) - \varphi(s,x)}{\delta t} dxds - \int_\Omega h_0(x)\varphi_{\mathcal{T}_m}(\delta t, x)dx.$$

The function φ is smooth and then we have the uniform convergence of $\frac{\varphi(.+\delta t,.)-\varphi(.,.)}{\delta t}$ and $\varphi_m(\delta t, .)$ respectively to $\partial_t\varphi$ and $\varphi(0, .)$. Using Lemma 2.9 we have

$$T_0 \to -\int_0^T \int_\Omega \overline{h}\partial\varphi - \int_\Omega h_0\varphi(0, .). \tag{4.20}$$

We define $\Psi_m = \nabla^m \varphi_m$. Using lemma 2.8 we have

$$\begin{aligned} T_1 &:= \sum_{n=0}^{N-1} \delta t < div^{\mathcal{T}_m}(T_s(h^n_{\mathcal{T}})K_{\mathcal{D}_m}\nabla^{\mathcal{D}_m} h^{n+1}_{\mathcal{T}_m}), \varphi^n_{\mathcal{T}_m} >_{\mathcal{T}_m} \\ &= \sum_{n=0}^{N-1} \delta t < T_s(h^n_{\mathcal{T}})K_{\mathcal{D}_m}\nabla^{\mathcal{D}_m} h^{n+1}_{\mathcal{T}_m}, \nabla^{\mathcal{D}_m}\varphi^n_{\mathcal{T}_m} >_{\mathfrak{D}_m} \end{aligned}$$

we deduce

$$T_1 = \int_0^T \int_\Omega \nabla^{\mathcal{D}_m} h_{\mathcal{T}_m}({}^t A_{\mathcal{T}_m}(h_{\mathcal{T}_m}, .)\Psi_m).$$

Because φ is smooth, we have Ψ_m uniformed converge to $\nabla\varphi$. Using lemma 2.9 we have

$$T_1 \to \int_0^T \int_\Omega \nabla\overline{h}.({}^t A(\overline{h}, .)\nabla\varphi) = \int_0^T \int_\Omega A(\overline{h}, .)\nabla(\overline{h}).\nabla\varphi. \tag{4.21}$$

Similarly we have

$$T_2 := \sum_{n=0}^{N-1} \delta t < div^{\mathcal{T}_m}(T_s(h^n_{\mathcal{T}})K_{\mathcal{D}_m}\nabla^{\mathcal{D}_m} f^n_{\mathcal{T}_m}), \varphi^n_{\mathcal{T}_m} >_{\mathcal{T}_m}$$
$$= \sum_{n=0}^{N-1} \delta t < T_s(h^n_{\mathcal{T}})K_{\mathcal{D}_m}\nabla^{\mathcal{D}_m} f^n_{\mathcal{T}_m}, \nabla^{\mathcal{D}_m}\varphi^n_{\mathcal{T}_m} >_{\mathfrak{D}_m},$$

we deduce

$$T_2 = \int_0^T \int_\Omega \nabla^{\mathcal{D}_m} f_{\mathcal{T}_m}({}^t A_{\mathcal{T}_m}(h_{\mathcal{T}_m}, .)\Psi_m).$$

φ is smooth, then we have Ψ_m uniformed converge to $\nabla\varphi$. Using lemma 2.9 we have

$$T_2 \to \int_0^T \int_\Omega \nabla\overline{f}.({}^t A(\overline{h}, .)\nabla\varphi) = \int_0^T \int_\Omega A(\overline{h}, .)\nabla(\overline{f}).\nabla\varphi. \tag{4.22}$$

We have

$$T_3 = \sum_{n=0}^{N-1} < I^n_{S,\mathcal{T}_m}, \varphi^n_{\mathcal{T}_m} >_{\mathcal{T}_m}$$
$$= \frac{1}{2}\int_0^T \int_\Omega I^{\mathfrak{M}}_{S,m}(s,x)\varphi_{m,\mathfrak{M}}(s,x)dxds + \frac{1}{2}\int_0^T \int_\Omega I^{\mathfrak{M}^*}_{S,m}(s,x)\varphi_{m,\mathfrak{M}^*}(s,x)dxds,$$

$\varphi_{m,\mathfrak{M}}$ and $\varphi_{m,\mathfrak{M}^*}$ uniformly convergent to φ and the weak convergence of $I^{\mathfrak{M}}_{S,m}$ and $I^{\mathfrak{M}^*}_{S,m}$ to I_s in $L^2((0,T)\times\Omega)$ imply that

$$T_3 \to \int_0^T \int_\Omega I_S \varphi.$$

Passing to the limit in each term, we have proved (1.10). □

5. Numerical convergence of the DDFV scheme

In this section, we illustrate the behavior of the DDFV scheme by applying it to the system (1.1). In all the test cases, the spatial domain is $\Omega = (0,1)\times(0,1)$ and the time period is $[0,1]$.

In order to compute the numerical order of convergence of the scheme, we introduce a sequence of triangular meshes. We present in Figure 4 the meshes obtained for i = 1 and i = 3. Let us also mention that, even though many choices are possible, we always assume in this paper that $x_{\mathcal{K}}$ is the barycenter of $\mathcal{K} \in M$.

To test the numerical solution obtained using the DDFV method, we compare the numerical results with the analytical solution proposed by Keulegan [31]. We consider a confined aquifer of uniform thickness with a vertical interface at $x = 0$, the salt water being in the part $x < 0$ and the fresh water in the part $x > 0$. At $t = 0$, the grid is removed And the interface begins to move due to the difference in density. The interface is then described by a linear profile pivoting around a

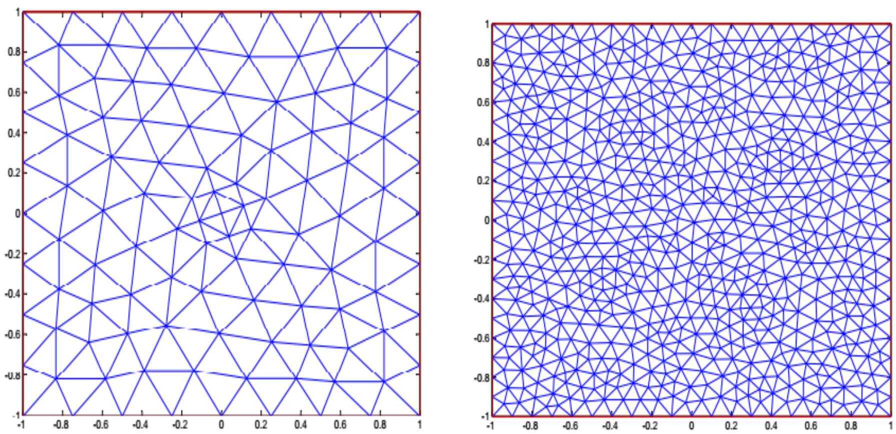

FIGURE 4. Triangular meshes with a refinement level $i = 1$ on the left and $i = 3$ on the right.

fixed axis $(0; D/2)$. Keulegan gave an analytical solution for the movement of the elevation of the interface:

$$h(x,t) = -\frac{D}{2}(1 + \frac{x}{L(t)}).$$

The location of the interface intersection with the bottom of the aquifer is given for $h = -D$ by:

$$L(t) = \sqrt{\frac{D\alpha K t}{\phi}}.$$

Where the parameter $\alpha = \frac{p_s}{p_f} - 1$ characterizes the density contrast. The initial position of the interface was fixed at $l = 50m$, position corresponding to time t_0. In (Figures 5 and 6) we trace the exact linear solution and the numerical solution obtained with the DDFV method at the three times $t = 5$ days, $t = 10$ days and $t = 15$ days.

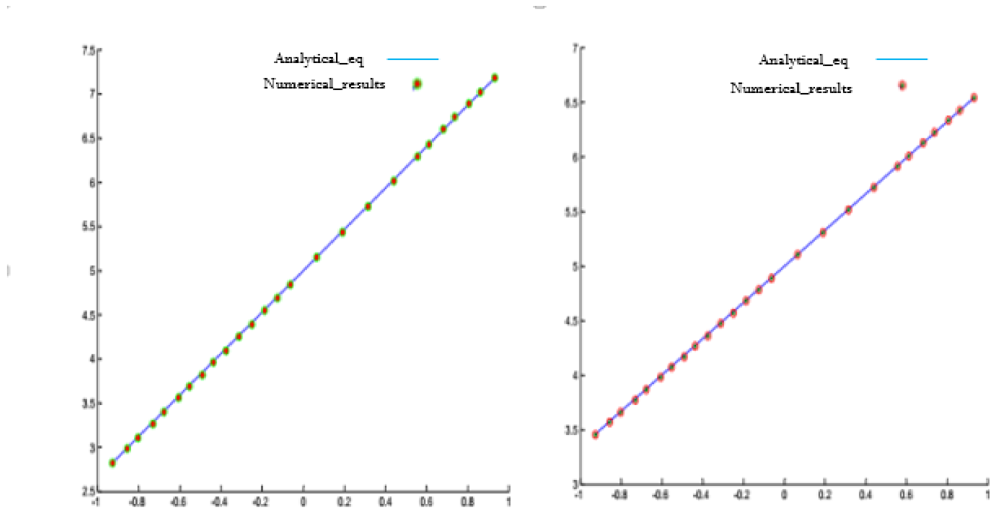

FIGURE 5. test 2. Exact solution and numerical solution at $t = 5$ days (right) and $t = 10$ days(left).

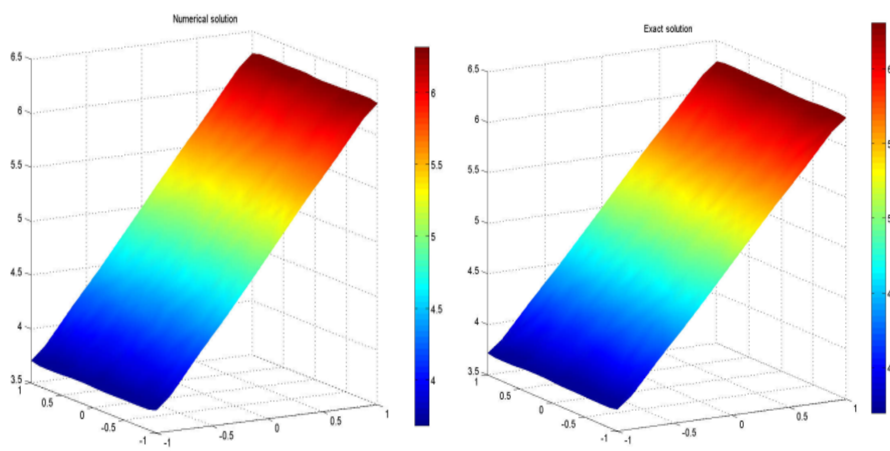

FIGURE 6. test 2. Exact solution and numerical solution at $t = 15$.

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