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Modelling the Critical Characteristic Area for the Control of Tuberculosis in Densely Populated Communities

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Abstract: A density-dependent mathematical model for the dynamics of tuberculosis is analyzed. We use the data from internally-displaced peoples' camps to illustrate the role of overcrowding in the propagation of a contagious epidemic that would otherwise have been localized. The critical characteristic area per individual required for eradication of tuberculosis is obtained as 0.25 square kilometers. The effect of R_0 on the different epidemiological classes shows an increase in number of infecteds for higher values of R_0. This study also shows that a reduction in the contact rate and/or progression rate results into a decrease in size of the required critical characteristic area thereby easing the task of eradicating tuberculosis. Stability analysis of the endemic and disease-free equilibria has been carried out, and a forward bifurcation exists at the bifurcation point $R_0 = 1$. Results from the study also show that much as we can try to reduce overcrowding, in un-avoidable circumstances such as lack of vast tracts of land for resettling internally displaced people, measures to reduce on the contact and progression rates should be applied.

Keywords: Basic Reproductive Number, forward bifurcation, critical characteristic area, progression and contact rates

1. INTRODUCTION

The impact of infectious diseases on humans is enormous, both in terms of suffering and in terms of social and economic consequences. Tuberculosis is an airborne communicable disease caused by *Mycobacterium tuberculosis*, or the *tubercle bacillus*. It is spread by tiny airborne particles *(droplet nuclei)* expelled by a person who has infectious tuberculosis. If another person inhales air containing these droplet nuclei, transmission may occur. Infection begins with the multiplication of *tubercle bacilli* in alveolar macrophages, some of which spread through the bloodstream. However, the immune system response usually prevents the development of the disease. Persons who are infected but who do not have tuberculosis disease are *asymptomatic* and not infectious. Such persons usually have a positive reaction to the tuberculin skin test. About 10% of infected persons will develop tuberculosis disease at some time in their lives, but the risk is considerably higher for persons who are immunosuppressed, especially those with HIV infection [1].

Despite the treatment and control strategies being used world wide to check on the high incidence levels of tuberculosis, it is still one of the major causes of high mortality and morbidity in developed and developing countries. Tuberculosis cannot be controlled effectively unless problems of prevention, diagnosis, treatment, and poor health systems are addressed. There is also a need to address the issue of limited knowledge on the biology of tuberculosis (mechanisms of transmission, life cycle of the transmitting vector and effectiveness of the existing drugs in fighting this vector) and the host-parasite relationship. In Uganda, the disease incidence is on the increase as per the available records [2] and among others, overcrowding and unsanitary conditions play

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a major role in explaining the observed trends in Uganda and the world over. In order to control the tuberculosis incidence, these issues must be addressed.

Mathematical modelling has become an essential tool in studying disease dynamics. The models have become invaluable management tools for epidemiologists, both shedding light on mechanisms underlying observed dynamics as well as making quantitative predictions on the effectiveness of the different control measures. The effective management and control of infections is increasingly being done with substantial input from mathematical models, which are used not only to provide information on the nature of the infection itself, through estimates of key parameters, but also to make predictions about the likely outcome of alternative courses of action [3].

Modelling of tuberculosis has existed since its first model by Waaler *et al*. [4] and ReVelle *et al.* [5] who expounded on Waaler's work through the explanation of why the infection rate depends linearly on the prevalence. Many studies have been carried out on the epidemiology of tuberculosis by different researchers [6].

One of the characteristics of internally displaced peoples' camps wherever they are created is the limited land visa-a-vis the big population that is displaced. An equally difficult condition to counteract this problem is the problem of securing a big land in cases of when these camps are a result of civil conflict. Because such resettlements are associated with emergencies, there are no sanitation facilities, sudden influx of population results in disease rapid transmissions. There is need therefore, of studying the issue of critical characteristic area in disease dynamics. However, previous models on tuberculosis have been analyzed under the consideration of prevalence rate of the disease in its transmission. When modelling tuberculosis in areas with high population densities such as refugee camps and internally displaced peoples' camps, prevalence rate may not clearly bring out the role of population density in explaining the dynamics of tuberculosis.

In this paper, we use a mathematical model as in [7] to determine the critical characteristic area required for control of tuberculosis in densely populated areas. We determine the critical characteristic areas for different R_0 and carry out a bifurcation behaviour analysis of the model. We also analyze the stability of the equilibrium points and investigate the effect of variation of the contact and progression rates on the magnitude of the critical characteristic area.

2. MODEL FORMULATION

In the model, the total population is subdivided into four epidemiological classes; susceptibles ($S(t)$), latent/ exposed, $L(t)$, infectious, $I(t)$; recovered/treated, $T(t)$ and $N(t) = S(t) + L(t) + I(t) + T(t)$ is the total population size. The density of infecteds (the number of infecteds per unit area) was considered by the incorporation of parameter, A, in the transmission term. The underlying assumption is that the rate of infection of the susceptibles is proportional to the number of infecteds per unit area. This is in contrast with other models at population level, where the rate of infection is dependent on the disease prevalence [5, 6, 8]. The model assumes a constant recruitment rate, μ into the susceptible population, through birth by the people in the camps. Other parameters used include; the per capita natural mortality rate μ, the tuberculosis-induced mortality rate d, the probability that a susceptible individual becomes infected by one infectious individual per contact per unit time β_1, the probability that a treated individual becomes infected by one infectious individual per contact per unit time β_2, the rate of progression to active tuberculosis k, recovery rate of the latent class r_1, recovery rate of the infectious class r_2 and the per capita contact rate c.

The equations of the model are:

$$\frac{dS}{dt} = \Lambda - \mu S - \beta_1 c S \frac{I}{A} \tag{1}$$

$$\frac{dL}{dt} = \beta_1 c S \frac{I}{A} - (\mu + k + r_1)L + \beta_2 c T \frac{I}{A} \tag{2}$$

$$\frac{dI}{dt} = kL - (\mu + d + r_2)I \tag{3}$$

$$\frac{dT}{dt} = r_1 L + r_2 I - \mu T - \beta_2 c T \frac{I}{A} \tag{4}$$

We note here that the above model is different from that given in [8] in that it incorporates the number of infected per unit area, ($\frac{I}{A}$), instead of the prevalence, ($\frac{I}{N}$).

3 MODEL ANALYSIS

(a) Disease-free equilibrium point

In [7], we showed that the system has a disease-free equilibrium point at $E_0 = \left(\frac{\Lambda}{\mu}, 0, 0, 0\right)$ where $\frac{\Lambda}{\mu}$ is the asymptotic carrying capacity of the total population. Stability of E_0 ensures that tuberculosis will be eradicated and no epidemic is expected. If it is unstable, an epidemic is expected. The analysis shows that the condition for the stability of disease-free equilibrium point is

$$k\beta_1 c\left(\frac{\Lambda/\mu}{A}\right) - (\mu + k + r_1)(\mu + d + r_2) < 0$$

This condition can be written as

$$\left(\frac{A}{\Lambda/\mu}\right) > \left(\frac{k}{\mu + k + r_1}\right)\left(\frac{\beta_1 c}{\mu + d + r_2}\right) \tag{5}$$

where

$\frac{A}{\Lambda/\mu}$ is the area occupied per individual (characteristic area)

$\frac{k}{\mu+k+r_1}$ is the probability of survival from latent stage into the infectious stage

$\frac{\beta_1 c}{\mu+d+r_2}$ is the number of latent infections produced by a typical infectious individual during the mean infectious period

From equation (5), the condition for the stability of the disease-free equilibrium point, and hence eradication of tuberculosis is that the critical characteristic area per individual be greater than the product of the probability of survival from latent stage into the infectious stage and the number of latent infections produced by a typical infectious individual during the mean infectious period.

(b) Endemic equilibrium point

Let the endemic equilibrium point be $E_1 = (S^*, L^*, I^*, T^*)$, where $(S^*, L^*, I^*, T^*) \neq 0$. The Jacobian matrix of the system, when evaluated at E_1 is given by

$$J_{E_1} = \begin{pmatrix} -(\mu + \beta_1 c\,(\frac{I^*}{A}) & 0 & -\beta_1 c\left(\frac{S^*}{A}\right) & 0 \\ \beta_1 c\left(\frac{I^*}{A}\right) & -(\mu + k + r_1) & \beta_1 c\left(\frac{S^*}{A}\right) + \beta_2 c\left(\frac{T^*}{A}\right) & \beta_2 c\left(\frac{I^*}{A}\right) \\ 0 & k & -(\mu + d + r_2) & 0 \\ 0 & r_1 & r_2 - \beta_2 c\left(\frac{T^*}{A}\right) & -(\mu + \beta_2 c\left(\frac{I^*}{A}\right)) \end{pmatrix}$$

From the Jacobian, we have

$$det(J_{E_1} - \lambda I) = (\mu + \beta_1 c\frac{I^*}{A} + \lambda)\left[kdet\begin{pmatrix} \beta_1 c\frac{S^*}{A} + \beta_2 c\frac{T^*}{A} & \beta_2 c\frac{I^*}{A} \\ r_2 - \beta_2 c\frac{T^*}{A} & -(\mu + \beta_2 c\frac{I^*}{A} + \lambda) \end{pmatrix} + \right.$$

$$\left. (\mu + d + r_2 + \lambda) det\begin{pmatrix} -(\mu + k + r_1 + \lambda) & \beta_2 c\frac{I^*}{A} \\ r_1 & -(\mu + \beta_2 c\frac{I^*}{A} + \lambda) \end{pmatrix} \right]$$

$$-\beta_1 c\frac{I^*}{A}\left(\mu + \beta_2 c\frac{I^*}{A} + \lambda\right) det\begin{pmatrix} 0 & -\beta_1 c\frac{S^*}{A} \\ k & -(\mu + d + r_2 + \lambda) \end{pmatrix} = 0$$

Simplifying this gives the characteristic equation as $\lambda^3 + a_2\lambda^2 + a_1\lambda + a_0 = 0$, where

$$a_0 = (\mu + d + r_2)(\mu + k + r_1)\left(\mu + \beta_2 c\frac{I^*}{A}\right) - (\mu + d + r_2) r_1 \beta_2 c\frac{I^*}{A}$$

$$-\left(\beta_1 c\frac{I^*}{A}\right)\left(\beta_1 c\frac{S^*}{A}\right)\left(\mu + \beta_2 c\frac{I^*}{A}\right) - \left(\mu + \beta_1 c\frac{I^*}{A}\right)\left(\mu + \beta_2 c\frac{I^*}{A}\right)\left(\beta_1 c\frac{S^*}{A} + \beta_2 c\frac{T^*}{A}\right)$$

$$-\left(\mu + \beta_1 c\frac{I^*}{A}\right)\left(\beta_2 c\frac{I^*}{A}\left(r_2 - \beta_2 c\frac{T^*}{A}\right)\right)$$

$$a_1 = -\left(\mu + \beta_2 c\frac{I^*}{A}\right)\left(\beta_1 c\frac{S^*}{A} + \beta_2 c\frac{T^*}{A}\right) - \left(\mu + \beta_1 c\frac{I^*}{A}\right)\left(\beta_1 c\frac{S^*}{A} + \beta_2 c\frac{T^*}{A}\right)$$

$$+\left(\beta_2 c\frac{I^*}{A}\left(r_2 - \beta_2 c\frac{T^*}{A}\right)\right) + \left((\mu + k + r_1) + \left(\mu + \beta_2 c\frac{I^*}{A}\right)\right)(u + d + r_2)$$

$$+2(\mu + k + r_1)\left(\mu + \beta_2 c\frac{I^*}{A}\right) - r_1\beta_2 c\frac{I^*}{A} - \beta_1 c\frac{I^*}{A}\beta_1 c\frac{S^*}{A}$$

$$a_2 = (\mu + d + r_2) + (\mu + k + r_1) + \left(\mu + \beta_2 c\frac{I^*}{A}\right) + \left(\beta_1 c\frac{S^*}{A} + \beta_2 c\frac{T^*}{A}\right)$$

Routh-Hurwitz stability criterion [9] requires that for the polynomial $P(\lambda) = \lambda^3 + a_2\lambda^2 + a_1\lambda + a_0$ to have negative real parts, then $a_0 > 0$, $a_1 > 0$, $a_2 > 0$ and $a_1a_2 > a_0$. Clearly, $a_2 > 0$ since all parameters are positive and $a_0 > 0$ and $a_1 > 0$ and consequently $a_1a_2 > a_0$ for smaller values of S^*, I^* and T^*. Accordingly, the endemic equilibrium is stable. However, there is a change in this stability that may occur for higher values of I^*, S^*,L^* and T^* i.e the endemic equilibrium may be stable for small values and unstable for higher values of I^* [10]. Our model demonstrates this because for S^*; I^* and $T^* \gg$ other parameters, we have $a_1a_2 < a_0$ implying instability of the endemic equilibrium.

(c) The Basic Reproductive Number and Characteristic Area

The net and basic reproductive numbers are among the most widely applied concepts in infectious disease epidemiology. A net reproduction number; the average number of secondary infectious cases resulting from each case in a given population, is conventionally associated with an increase in incidence, and the basic reproduction number R0, defined analogously for a "totally susceptible" population, provides a standard measure of the "transmission potential" of an infection [11].

For this model, the basic reproductive number R_0, is obtained as (see [7] for derivation)

$$R_0 = \frac{kc(\beta_1 + \beta_2)\left(\frac{\Lambda/\mu}{A}\right)}{(\mu + k + r_1)(\mu + d + r_2)}$$

or

$$R_0 = \left(\frac{\Lambda/\mu}{A}\right)\left(\frac{(\beta_1 + \beta_2)c}{\mu + d + r_2}\right)\left(\frac{k}{\mu + k + r_1}\right) \tag{6}$$

where $\left(\frac{\Lambda/\mu}{A}\right)$ is the density of the susceptible population, $\beta_1 c$ and $\beta_2 c$ are the effective transmission rates, $\frac{1}{\mu+d+r_2}$ is the effective infectious period, $\frac{(\beta_1+\beta_2)c}{\mu+d+r_2}$ is the number of latent infections produced by a typical infectious individual during the mean infectious period, and $\frac{k}{\mu+k+r_1}$ is the probability of survival from latent stage into the infectious stage.

(a) When $R_0 < 1$, we have

$$\left(\frac{\Lambda/\mu}{A}\right)\left(\frac{(\beta_1 + \beta_2)c}{\mu + d + r_2}\right)\left(\frac{k}{\mu + k + r_1}\right) < 1$$

from which we obtain

$$A > \left(\frac{\Lambda}{\mu}\right)\left(\frac{(\beta_1 + \beta_2)c}{\mu + d + r_2}\right)\left(\frac{k}{\mu + k + r_1}\right) \tag{7}$$

(b) When $R_0 > 1$, we have

$$A > \left(\frac{\Lambda}{\mu}\right)\left(\frac{(\beta_1 + \beta_2)c}{\mu + d + r_2}\right)\left(\frac{k}{\mu + k + r_1}\right) \tag{8}$$

From equations (7) and (8) above, we observe that the size of the area occupied, A is vital in determining tuberculosis dynamics. In order to eradicate tuberculosis (equation (7)), the area occupied should be greater than the product of the asymptotic carrying capacity of the population, the number of latent infections produced by a typical infectious individual during the mean infectious period and the probability of survival from latent stage into the infectious stage.

Preventive measures targeting reducing any parts of the components of the basic reproductive number will help halt the tuberculosis epidemic. Some of these measures include; increasing the size of the area occupied, A, lowering the progression rate, k by use of preventive drugs and vaccines, and increasing the recovery rate by ensuring timely diagnosis and treatment.

4. NUMERICAL RESULTS

This section presents the numerical results of the study. We tabulate; the estimated values of R_0 with varying characteristic area sizes, the estimated critical characteristic area sizes with varying contact rates and progression rates and the recommended area sizes for selected camps. Graphs on the effect of varying R_0 on the different epidemiological groups are plotted and a bifurcation diagram showing a forward bifurcation at $R_0 = 1$ is included. The parameter values used in this section are extracted from [7] and are summarized in Table 1

Table 1
Parameter Values for the Model

Parameter	*Symbol*	*Value*
Natural mortality rate of individuals	μ	0.0222
Recovery rate for the latents	r_1	1.5
Recovery rate for the infectious	r_2	1.5
Recruitment rate	Λ	1,500
Per capita rate of infection of the Susceptibles	β_1	2.0
Per capita rate of infection of the Treated	β_2	2.0
Per capita disease-induced mortality rate	d	0.365
Progression rate to active tuberculosis	k	0.00396
Contact rate	c	100

4.1 Estimating Values of R_0 Corresponding to Different Camp Area Sizes

The basic reproductive number is vital in predicting trends in disease dynamics. Since we are dealing with a model that excludes reinfection, having R_0 value below unity (i.e $R_0 < 1$) implies that there will be no epidemic [13].

Table 2 gives the different values of the basic reproductive number, R_0 that are obtained for different characteristic area sizes by substituting the parameter values given in Table 1 into the expression of R_0 given by equation (6). The asymptotic carrying capacity of the population is $\frac{\Lambda}{\mu} = \frac{1500}{0.0222} = 67568$. Therefore, for every 67,568 individuals, we require to have an area of 18,579 square kilometers in order to keep the basic reproductive number at below unity. This implies that, the critical characteristic area (i.e. the area occupied per individual) required to eradicate tuberculosis in such internally displaced peoples' camps is $\approx 2.5 \times 10^{-1}$ square kilometers per individual.

Table 2
Variation of R_0 with Characteristic area Occupied per Individual

Characteristic Area (square kilometers)	*Value of R_0*
0.018	15.00
0.027	10.00
0.039	7.00
0.069	4.00
0.272	0.99

The above critical characteristic area requirement of 0.25 square kilometers may require vast tracts of land that may not be available given the current land use activities and the associated scarce resources such as providing perimeter security. Under these circumstances, the model needs to look at other useful and implementable control interventions that will help reduce R_0. Other conditions besides overcrowding such as poverty, malnutrition and poor access to health care, play a big role in the transmission of tuberculosis.

4.2 The Effect of Varying the Contact Rate on the Critical Characteristic Area

An effective contact is defined as any kind of contact between two individuals such that, if one individual is infectious and the other susceptible, then the susceptible becomes infected. Whether or not a particular kind of contact will be effective depends on the infectious agent and its route of transmission.

To study this effect, contact rates of $c = 100, 10, 1, 0.5, 0.25$ have been used and their corresponding critical characteristic areas per individual determined. These are summarized in Table 3. From Table 3 it is noted that the required critical characteristic area size decreases as the contact rate reduces.

Table 3
The Critical Characteristic Area Sizes for Different Contact Rates, *c*

Contact Rate, c	*Critical characteristic Area (square kilometers)*
100	0.25
10	0.025
1	0.0025
0.5	0.00125
0.25	0.000625

4.3 Recommended Area Sizes for Selected Camps with Varying Contact Rates

Using the population figures in the camps quoted as of July 2004 in World food programme and Norwegian Refugee Council Report [14], we summarize the area requirements in square kilometers for selected internally displaced peoples' camps in Gulu district (Uganda), required to eradicate tuberculosis, for different contact rates,c. From Table 4, it is noted that the critical characteristic area is smaller for smaller values of contact rate. For a camp with 10*;* 726*;* if the $c = 10$, an area of 268.2 square kilometers is required for eradication of tuberculosis, where as if the contact rate is reduced to $c = 0.25$, the area required is only 6.705 square kilometers.

Much as 268.2 square kilometers may be a lot of land which may not be easily got, 6.705 square kilometers is more affordable. Therefore, eradication of tuberculosis, where there is a problem of land limitation, can be

Table 4
Recommended Area Sizes for Selected Camps

Camp	*Population*	*A, (c = 10)*	*A, (c = 1)*	*A, (c = 0.5)*	*A, (c = 0.25)*
Palenga	10,726	268.2	26.82	13.41	6.705
KochGoma	11,306	282.7	28.27	14.14	7.068
Unyama	13,282	332.1	33.21	16.61	8.30
Paicho	13,413	335.3	33.53	16.77	8.38
Alero	16,651	416.3	41.63	20.82	10.41

done using techniques that reduce contact rate. These include isolation of the infectious individuals and limiting the number of visits by the uninfected and the use of protective wear such as gloves and protective masks to reduce on the effective contacts.

4.4 Bifurcation Behaviour of the Model

Mathematical models that give rise to multiple steady-states such as tuberculosis models show bifurcation phenomena. A bifurcation in general is a set of parameter values at which an equilibrium, or fixed point, of the system being considered changes stability and/or appears/disappears. In epidemiology, bifurcation phenomena are associated with threshold parameters, the most common of which is R_0 [13].

According to Singer and Kirschner [12], many epidemiological models have defined a threshold condition that indicates whether an infection introduced into a population will be eliminated or become endemic. In models with only two steady states and a transcritical bifurcation, $R_0 > 1$ implies that the endemic state is stable (i.e the infection persists), and $R_0 \leq 1$ implies that the disease-free state is stable (i.e the infection is eliminated).

Using the parameter values in Table 1 and the expression for R_0 in equation (6), we plot I^* against R_0 to observe the behaviour of the bifurcation that results. From Figure 1, the mathematical description of the bifurcation behaviour involves a transcritical bifurcation that brings about an exchange in stability between the disease-free and the endemic equilibrium at the bifurcation point $R_0 = 1$. The R_0 threshold behaviour features a forward bifurcation, in which the endemic equilibrium exists only if $R_0 > 1$, so that there is no possibility of an endemic state when $R_0 < 1$.

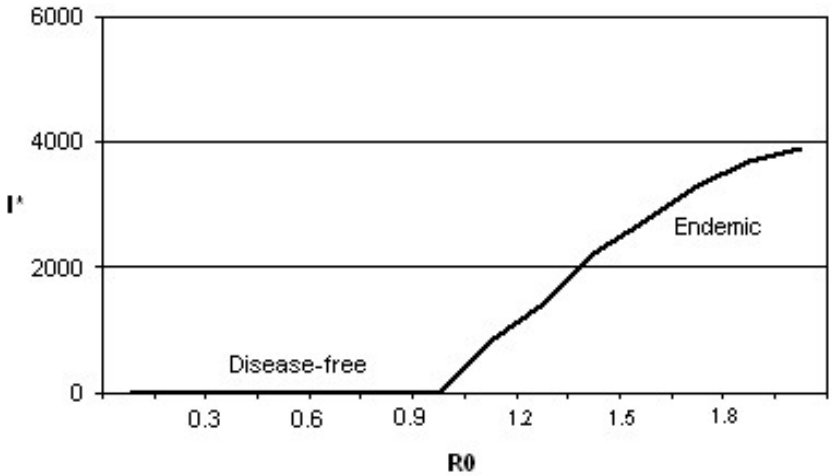

Figure 1: The Bifurcation Diagram Showing a Forward Bifurcation

In a system with a forward bifurcation, if parameters change and cause R_0 to rise slightly above one, a small endemic state results; that is, the endemic level at equilibrium is a continuous function of R_0 [13]. Therefore, we conclude here that strategies to reduce R_0 to below unity can successfully wipe out tuberculosis since the bifurcation behaviour shows a forward bifurcation; implying that there is no possibility of an endemic state when $R_0 < 1$.

4.5 The Effect of R_0 on the Different Epidemiological Classes

Almost one parameter, the basic reproductive number, does it all in predicting disease dynamics. It is a fundamental parameter governing the spread of diseases, and is also related to the long term behaviour and the level of vaccination necessary for eradication. It tells us whether a population is at risk from a given disease and it is a measure of growth potential of an epidemic [15, 16].

Here, results of simulation of the effect of R_0 on the different epidemiological classes are presented. This effect is studied using $R_0 = 0.99, 4.00, 5.00, 6.00, 7.00, 10.00$ and 15.00. We observe in Figure 2 that the number of susceptibles increases for all values of R_0 in the first phases and a sharp decline occurs suddenly for all cases of $R_0 > 0.99$. However, there is a delay in the sudden drop depending on the magnitude of R_0. The sudden decline occurs faster for higher values of R_0 and the time lag increases as the magnitude reduces. There is an all time increment in the number of susceptibles whenever R_0 goes below unity (i.e. $R_0 = 0.99$). This is as a result of the lower infection rate whenever $R_0 < 1$. The higher the magnitude of R_0, the more the number of susceptibles joining the latent and later the infectious stage and the consequence of this are the noted sudden declines in the susceptible population. Figure 3 depicts the exact opposite of Figure 2 in the trend. It is observed that the population of the latently infected individuals is increasing with exception of when $R_0 = 0.99$. For all the cases of $R_0 > 1$, there is a sharp increase in the number of latently infected individuals, with the sudden change in pattern occurring earlier for high values of R_0. There is no increase in the number of latently infected when $R_0 = 0.99$ because in this case, as expected, the disease is dying out since $R_0 < 1$.

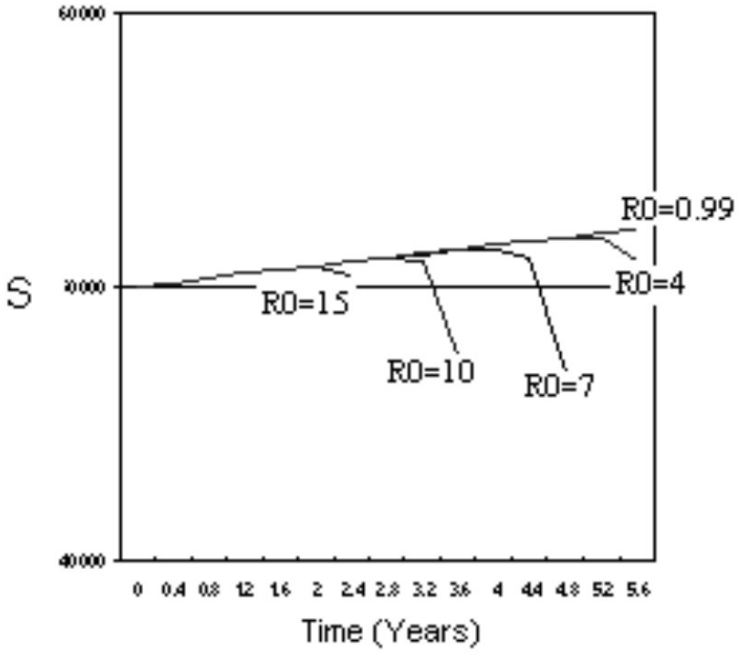

Figure 2: The Effect of R_0 on the Susceptible Population

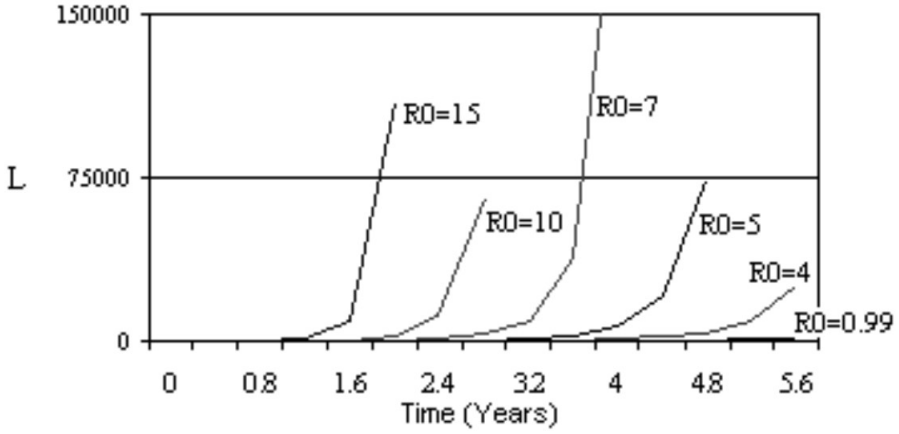

Figure 3: The Effect of R_0 on the Latently Infected Population

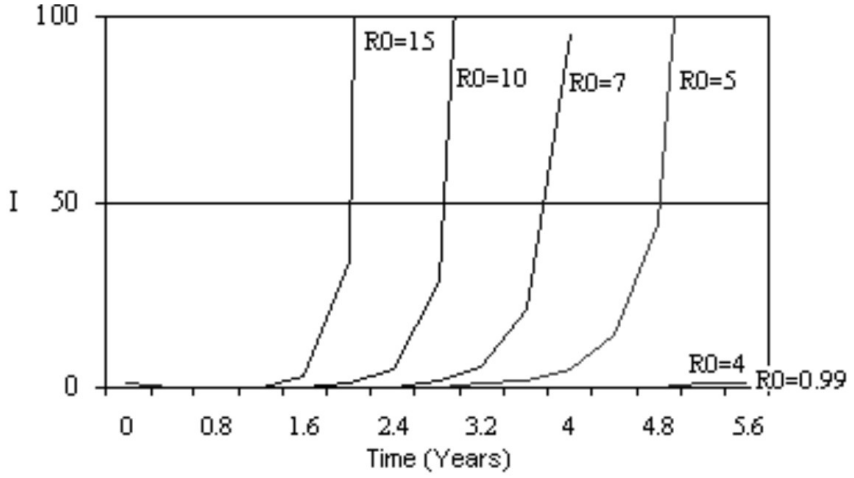

Figure 4: The Effect of R_0 on the Infectious Population

The observed pattern in Figure 4 is exactly the same as that in Figure 3. This is because the more the latently infected individuals, the greater the number of individuals that progresses to the infectious stage, provided there are no variations in the treatment administered. In Figure 5, there is an increment in the number of treated individuals for all cases of R_0. However, the number of treated individuals does not increase for $R_0 = 0.99$. This is because we do not expect any latently infected or infectious individuals to be treated when $R_0 < 1$.

5. DISCUSSION AND CONCLUSIONS

We have analyzed a density-dependent mathematical model for the dynamics of tuberculosis. The basic reproductive number R_0 and its role in; determining the critical characteristic area, determining the bifurcation behaviour have been shown. Stability analysis of the disease-free and the endemic equilibrium point have been carried out and numerical results have been obtained.

We have shown that the model has a transcritical bifurcation at the point in the parameter space specified by $R_0 = 1$ and it is a forward bifurcation whereby there is a change in stability from the disease-free equilibrium (when $R_0 < 1$) to the endemic equilibrium (when $R_0 > 1$). There is no possibility of endemicity when $R_0 < 1$. A

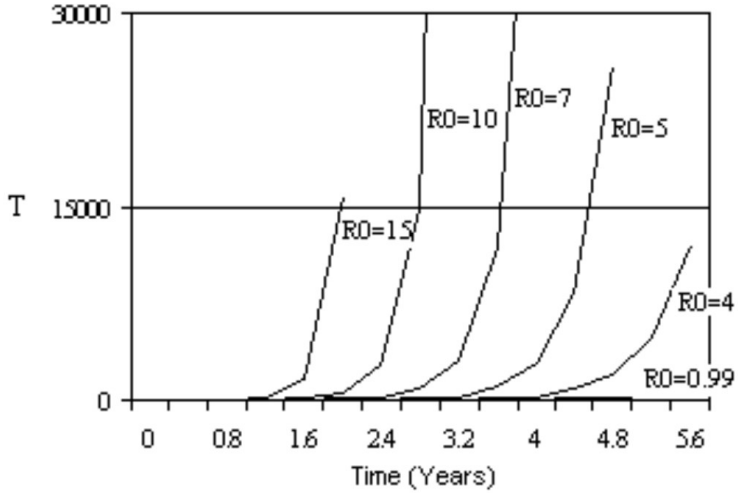

Figure 5: The Effect of *R*0 on the Treated Population

critical characteristic area per individual, for eradication of tuberculosis has been obtained (using R_0) and analysis of the role of R_0 in determining the level of the tuberculosis incidence carried out.

Results of the stability analysis for the disease-free and the endemic equilibria show that the disease-free equilibrium point is stable if the critical characteristic area per individual is greater than the product of the probability of survival from latent stage into the infectious stage and the number of latent infections produced by a typical infectious individual during the mean infectious period. The endemic equilibrium is stable for small values of L^* and changes stability for higher values. Stability of endemic equilibrium means that tuberculosis can not be wiped out and neither can it wipe out the population. However, its instability implies that, if no control measures are put in place, tuberculosis has a potential of wiping out the population.

Given the current situation in Uganda's camps (giving the parameter values used), results from this study recommend a critical characteristic area per individual of at least 2.5×10^{-1} square kilometers in settlement areas to wipe out tuberculosis. However, due to limitation on land availability and other resources, the vast tracts of land required by this condition may not be got because such resentments are associated with emergencies. With this in mind, the possible alternative strategies have been identified and analyzed in relation their effect on the magnitude of critical characteristic area required for eradication of tuberculosis.

This study has established that a reduction in the contact rate, which can be done through isolation of the infected individuals and use of protective wear such as gloves and masks, and a decrease in the progression rate from the latent stage to the infectious stage of tuberculosis through use of stronger drugs and vaccines, and ensuring timely and proper diagnosis and treatment would help in lowering the critical characteristic area size required to implementable levels in terms of land requirement.

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