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A VARIABLE MESH FINITE DIFFERENCE SCHEME FOR TWO-PARAMETERS SINGULARLY PERTURBED BOUNDARY VALUE PROBLEMS

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Abstract. A variable mesh second order finite difference method is established for the solution of two-parameter singularly perturbed boundary value problem. In this method, the derivatives in the problem are replaced by higher order finite differences on the nonuniform mesh to get the discretization equation. This equation is solved efficiently using the tridiagonal solver. The convergence of the proposed method is analysed, and the method gives second-order uniform convergence. Test examples are illustrated and maximum absolute errors in comparison to the other methods in the literature are shown to justify the method.

1. Introduction

Singularly perturbed boundary value problems (SPBVP) with multi parameters are recognisable in engineering and science. In general, such problems arise in diverse area of applied mathematics such as fluid mechanics, aerodynamics, quantum mechanics, reaction-diffusion process, elasticity and many other areas. Particularly, these type of problems occur in transport phenomena of chemical reactor theory, biology and lubrication theory [1],[2]. The character of the two-parameter problem asymptotically examined by [2]in which the ratio of two parameters has a significant role in the solution. It is known that as perturbation parameter goes to zero, the analytical solution of SPBVP approaches to a discontinuous limit and boundary or interior layers appear. This gives there are significant difficulties in numerical computation of singularly perturbed equations due to a minimal amount of perturbation parameter and occurrence of boundary or interior layers. Hence, it is necessary to develop efficient numerical methods for such problems, whose accuracy is independent of the perturbation value ε_1 , i.e., the techniques are ε_1 -uniformly convergent. In verity of papers and books, the authors described various methods to solve SPBV problems [3],[7], citenay85, [11].

Collocation method based on B-Spline for a two-parameter singularly perturbed convection-diffusion is derived and demonstrated [8]. [13] proposed a method of a quadratic spline collocation for two parameter SPBVP. [9] used a uniform Ritz-Galerkin method on shishkin mesh to solve the boundary value problem with two

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parameters. An approximate method is derived by [6]for solving two parameters SPBV problems where the boundary layers near the both end points. [5] proposed finite difference, finite element and B-spline collocation methods very nicely for solving a class of SPBVP with two parameters. [14] constructed an exponential spline method to solve semilinear SPBVP with multi parameters on shishkin mesh. [4]derived a numerical method of second order monotone for a two-parameter SPBVP affecting the convection and diffusion terms.

A second order convergence numerical method to solve the two parameters SPBVP on the non-uniform mesh is proposed in this paper. In section 2, we have given description of the problem. The solution of the problem with proposed numerical method is discussed in section 3. The method is analysed for convergence in section 4. Test examples, results and graphs are given in next section. The discussions and conclusions are given in the final section.

2. Description of the method

Consider a model singular perturbed differential equation of the form

$$L[y(x)] \equiv \varepsilon_1 y''(x) + \varepsilon_2 a(x) y'(x) + b(x) y(x) = f(x), \quad 0 \leqslant x \leqslant 1 \tag{2.1}$$

with

$$\begin{aligned} y(0) &= \alpha_0 \\ y(1) &= \alpha_1 \end{aligned} \tag{2.2}$$

where ε_1 $(0 < \varepsilon_1 \ll 1)$,ε_2 $(\varepsilon_2 \ll 1)$ are very small positive parameters and the functions $a(x)$, $b(x)$ and $f(x)$ are functions which are sufficiently smooth and satisfying the condition $b(x) \leq -\theta < 0$ where θ is a positive constant.

The zeroes of the characteristic equation can describe the solution of Eq. (2.1)

$$\varepsilon_1 \lambda(x)^2 + \varepsilon_2 a(x) \lambda(x) + b(x) = 0$$

which gives two functions

$$\lambda_1(x) = -\frac{\varepsilon_2 a(x)}{2\varepsilon_1} - \sqrt{\left(\frac{\varepsilon_2 a(x)}{2\varepsilon_1}\right)^2 + \frac{b(x)}{\varepsilon_1}}$$

$$\lambda_2(x) = -\frac{\varepsilon_2 a(x)}{2\varepsilon_1} + \sqrt{\left(\frac{\varepsilon_2 a(x)}{2\varepsilon_1}\right)^2 + \frac{b(x)}{\varepsilon_1}}$$

Put $\theta_1 := \max\limits_{x\in[0,1]} \lambda_1 < -\frac{\varepsilon_2}{\varepsilon_1} \leq 0$, $\theta_2 := \min\limits_{x\in[0,1]} \lambda_2(x)$.
The decay of the solution in the boundary layer region is defined by θ_1 and θ_2.
For

$$\frac{\varepsilon_1}{{\varepsilon_2}^2} \leq 1, \;\; |\theta_1| = O\left(\frac{\varepsilon_2}{\varepsilon_1}\right) \text{ and } |\theta_2| = O\left(\frac{1}{\varepsilon_2}\right),$$

for

$$\frac{\varepsilon_2^2}{\varepsilon_1} \leq 1, \ \ |\theta_1| = O\left(\frac{1}{\sqrt{\varepsilon_1}}\right) \text{ and } |\theta_2| = O\left(\frac{1}{\sqrt{\varepsilon_1}}\right).$$

The term $e^{-\theta_1 x}$ governs the layer at left end $x = 0$, and the layer is governed using $e^{-\theta_2(1-x)}$ at right end $x = 1$.
From [4], we have

$$\begin{aligned} \theta_1 =& \frac{\sqrt{\alpha_2 \tilde{a}}}{\sqrt{\varepsilon_1}}, \ \frac{\varepsilon_2^2}{\varepsilon_1} \leq \frac{\alpha_2}{\tilde{a}} \\ & \frac{\tilde{a}\varepsilon_2}{\varepsilon_1}, \ \frac{\varepsilon_2^2}{\varepsilon_1} \geq \frac{\alpha_2}{\tilde{a}} \\ \theta_2 =& \frac{\sqrt{\alpha_2 \tilde{a}}}{2\sqrt{\varepsilon_1}}, \ \frac{\varepsilon_2^2}{\varepsilon_1} \leq \frac{\alpha_2}{\tilde{a}} \\ & \frac{\alpha_2}{2\varepsilon_2}, \ \frac{\varepsilon_2^2}{\varepsilon_1} \geq \frac{\alpha_2}{\tilde{a}} \end{aligned}$$

where $\tilde{a} = \min\limits_{x\in[0,1]} a(x)$ and $\alpha_2 = \min\limits_{x\in[0,1]} \frac{b(x)}{a(x)}$.

3. Numerical Scheme

Let the domain [0, 1] be divided into N subintervals with variable mesh size $h_i = x_i - x_{i-1}$ for $i = 1$ to N and $h_{i+1} = \sigma_i h_i$. To start the computational implementation, we have to determine the value of h_1. Denote $R = x_N - x_0$. Then

$$\begin{aligned} R &= (x_N - x_{N-1}) + (x_{N-1} - x_{N-2}) + ... + (x_1 - x_0) \\ &= h_N + h_{N-1} + ... + h_1 \\ &= \ (\sigma_1 + \sigma_1\sigma_2 + ... + \sigma_1\sigma_2\sigma_3....\sigma_{N-1})\, h_1 \end{aligned}$$

Then $h_1 = \frac{R}{(\sigma_1+\sigma_1\sigma_2+....+\sigma_1\sigma_2\sigma_3....\sigma_{N-1})}$ determines the value of the starting step length with which we can compute the subsequent step lengths h_2,h_3, etc. In singular perturbation problems, if the layer is at the left end boundary x=0, then a large collection of nodal points nearer to this point are required. Likewise, when the layer is at the right end boundary then a large cluster of nodal points at this boundary is needed. The following process achieves this distribution of nodal points.

Choose $\sigma_i = \sigma =$ constant for $i = 1, 2, ..., N$. Then the step length h_1 reduces to $h_1 = \frac{R(1-\sigma)}{(1-\sigma^N)}$. If the boundary layer is at left end point, then we choose $\sigma > 1$. It guarantees that more number of nodal points exist near to the left end boundary. If the boundary layer exists at the right end, then choose $\sigma < 1$ which ensures a collection of large number of nodal points near the right end boundary. When the boundary layer is at the both end points of the interval then take $\sigma > 1$ in the left half, $\sigma < 1$ in the right half of the respective intervals. Then we have a symmetric mesh with more number of nodal points at both ends of the domain.

Since, the two parameter problem Eq. (2.1) possesses boundary layers at the ends $x = 0$ and $x = 1$, discretise the interval [0, 1] into $[0,\ x_m]$ and $[x_m,\ 1]$. The layer is near the end $x = 0$ in the interval $[0,\ x_m]$ and the layer will be near the end $x = 1$ in the interval $[x_m,\ 1]$. In these two subintervals, we apply the following numerical scheme on non-uniform mesh. Consider the non uniform higher order finite difference approximation of first and second derivatives:

$$y'_i = \tilde{y}'_i - \frac{\sigma_i h_i^2}{6} y'''_i - \frac{({\sigma_i}^2 - \sigma_i) h_i^3}{24} y_i^{(iv)} + \tau_1(i) \tag{3.1}$$

$$y''_i = \tilde{y}''_i + \frac{(1-\sigma_i)h_i}{3} y'''_i - \frac{({\sigma_i}^2 - \sigma_i + 1) h_i^2}{12} y_i^{(iv)} + \tau_2(i) \tag{3.2}$$

where, $\tilde{y}'_i = \frac{y_{i+1} - {\sigma_i}^2 y_{i-1} + ({\sigma_i}^2 - 1) y_i}{\sigma_i (1+\sigma_i) h_i}$, $\quad \tilde{y}''_i = \frac{2[y_{i+1} + \sigma_i y_{i-1} - (1+\sigma_i) y_i]}{\sigma (1+\sigma) h_i^2}$

$\tau_1(i) = -\frac{({\sigma_i}^4 + \sigma_i)}{120(1+\sigma_i)} h_i^4 y_i^v$ and $\tau_2(i) = -\frac{({\sigma_i}^4 - \sigma_i)}{(1+\sigma_i)} h_i^3 y_i^v$

Computing y'''_i and y_i^{iv} using Eq. (2.1), replacing them in Eqs.(3.1) and (3.2), we get

$$\begin{aligned} y'_i &= \tilde{y}'_i - C_i \{ \frac{f'_i - \varepsilon_2 a_i y''_i - k_3(i) y'_i - b'_i y_i}{\varepsilon_1} \} \\ &- D_i \{ \frac{f''_i - \varepsilon_2 a_i (f'_i - \varepsilon_2 a_i y''_i - k_3(i) y'_i - b'_i y_i) - k_4(i) y''_i - k_5(i) y'_i - b''_i y_i}{\varepsilon_1} \} \end{aligned} \tag{3.3}$$

$$\begin{aligned} y''_i &= \tilde{y''}_i + A_i \left(\frac{f'_i - \varepsilon_2 a_i y''_i - k_3(i) y'_i - b'_i y_i}{\varepsilon_1} \right) \\ &- B_i \left(\frac{f''_i - \varepsilon_2 a_i (f'_i - \varepsilon_2 a_i y''_i - k_3(i) y'_i - b'_i y_i) - k_4(i) y''_i - k_5(i) y'_i - b''_i y_i}{\varepsilon_1} \right) \end{aligned} \tag{3.4}$$

where $A_i = \frac{(1-\sigma_i)h_i}{3}$, $\quad B_i = \frac{({\sigma_i}^2 - \sigma_i + 1) h_i^2}{12}$, $\quad C_i = \frac{\sigma_i h_i^2}{6}$, $\quad D_i = \frac{({\sigma_i}^2 - \sigma_i) h_i^3}{24}$

$k_1(i) = \sigma_i (1+\sigma_i) h_i^2$, $\quad k_2(i) = \sigma_i (1+\sigma_i) h_i$, $\quad k_3(i) = (\varepsilon_2 a'_i + b_i)$,
$k_4(i) = (2\varepsilon_2 a'_i + b_i)$, $\quad k_5(i) = (\varepsilon_2 a''_i + 2b'_i)$

Now inserting Eqs. (3.3) and (3.4) in Eq. (2.1) and simplifying we get the following tridiagonal relation

$$E_i y_{i-1} + F_i y_i + G_i y_{i+1} = R_i, \quad i = 1, 2, ..., N-1 \tag{3.5}$$

where $E_i = \left(2L_i \sigma_i - {\sigma_i}^2 M_i h_i \right)$, $\quad G_i = (2L_i + M_i h_i)$,

$F_i = \left(M_i ({\sigma_i}^2 - 1) h_i - 2L_i (1+\sigma_i) + N_i \sigma_i (1+\sigma_i) h_i^2 \right)$

$L_i = \varepsilon_1 - \varepsilon_2 a_i A_i - \frac{\varepsilon_2 {a_i}^2 B_i}{\varepsilon_1} + B_i k_4(i) + \frac{\varepsilon_2 {a_i}^2 C_i}{\varepsilon_1} - \frac{\varepsilon_2 {a_i}^3 D_i}{{\varepsilon_1}^2} + \frac{\varepsilon_2 a_i D_i k_4(i)}{\varepsilon_1}$

$M_i = -A_i k_3(i) - \frac{\varepsilon_2 a_i B_i k_3(i)}{\varepsilon_1} + B_i k_5(i) + \varepsilon_2 a_i + \frac{\varepsilon_2 a_i C_i k_3(i)}{\varepsilon_1} - \frac{\varepsilon_2 {a_i}^2 D_i k_3(i)}{{\varepsilon_1}^2} + \frac{\varepsilon_2 a_i D_i k_5(i)}{\varepsilon_1}$

$N_i = -\frac{\varepsilon_2 a_i B_i {b'}_i}{\varepsilon_1} + B_i {b''}_i + \frac{\varepsilon_2 a_i C_i {b'}_i}{\varepsilon_1} - \frac{\varepsilon_2 {a_i}^2 D_i {b'}_i}{{\varepsilon_1}^2} + \frac{\varepsilon_2 a_i D_i {b''}_i}{\varepsilon_1} + b_i - A_i {b'}_i$

$R_i = \left(f_i - (A_i + \frac{\varepsilon_2 a_i B_i}{\varepsilon_1} - \frac{\varepsilon_2 a_i C_i}{\varepsilon_1} + \frac{\varepsilon_2 {a_i}^2 D_i}{{\varepsilon_1}^2}) {f'}_i - (-B_i - \frac{\varepsilon_2 a_i D_i}{\varepsilon_1}) {f''}_i \right) k_1(i)$

The solution of the system of tridiagonal Eq.(3.5) is obtained by using the tridiagonal solver Thomas algorithm.

4. Convergence analysis

Truncation error in the proposed scheme is

$$T_i(h_i) = \frac{\sigma_i(\sigma_i+1)(\sigma_i-1)^2 a_i \varepsilon_2}{9} {y'''}_i \; h_i^4 + O(h_i^5) \tag{4.1}$$

Consider the tridiagonal system Eq.(3.5) in matrix form

$$UY = V \tag{4.2}$$

where $U = [u_{ij}]$for$1 \leq i, j \leq N-1$ is a tridiagonal matrix with $u_{i,i+1} = G_i, u_{i,i} = F_i, u_{i,i-1} = E_i$and $V = (v_i)$ is a column vector with $v_i = R_i$
We also have

$$U\bar{Y} - T_i(h_i) = V \tag{4.3}$$

where $\bar{Y} = (\bar{y}_0, \bar{y}_1, ..., \bar{y}_N\ )^t$ represents original solution
and $T_i\,(h_i) = (T_0(h_0), T_1(h_1), ..., T_N(h_N))^t$ is the local truncation error.
From Eq. (4.2) and Eq. (4.3), it is clear that

$$U\left(\bar{Y}\ - Y\right) = T_i(h_i) \tag{4.4}$$

So that the error equation is

$$UE = T_i(h_i) \tag{4.5}$$

where $E = \bar{Y}\ - Y = (e_0, e_1, ..., e_N)^t$
Let S_i be the sum of elements of the i^{th} row of matrix U, then we have

$S_i = \sum_{j=1}^{N-1} m_{i\ \mathrm{j}} = -2\sigma_i\varepsilon_1 + \left(\varepsilon_2 a_i {\sigma_i}^2 + \frac{2\varepsilon_2 a_i \sigma_i (1-\sigma_i)}{3}\right) h_i + O(h_i^2)$ for $i = 1$

$$S_i = \sum_{j=1}^{N-1} m_{i\ \mathrm{j}} = \sigma_i(\sigma_i+1) b_i h_i^2 + \left(\frac{\sigma_i(\sigma_i+1)(\sigma_i-1){b'}_i}{3}\right) h_i^3 + O(h_i^4)$$
$$= \beta_i h_i^2 + O(h_i^3) \text{ for } i = 2, 3, ..., N-2 \text{ where } \beta_i = \sigma_i(\sigma_i+1) b_i$$

$S_i = \sum_{j=1}^{N-1} m_{i\ \mathrm{j}} = -2\varepsilon_1 + \left(\frac{(1-\sigma_i)\varepsilon_2 a_i}{3} - \varepsilon_2 a_i\right) h_i + O(h_i^2)\ for\ i = N-1$

Since $0 < \varepsilon_1 \ll 1$, U^{-1} exists and it has non-negative elements. So that from Eq. (4.5), it has

$$E = U^{-1}T_i(h_i) \tag{4.6}$$

and

$$\|E\| \leq \left\|U^{-1}\right\| . \|T(h)\| \tag{4.7}$$

Let $\bar{m}_{ki}$ be the $(ki)^{th}$ element of U^{-1} . Since $\bar{m}_{ki} \geq 0$, from the theory of matrices, we have

$$\sum_{i=1}^{N-1} \bar{m}_{ki}\ S_{\rm i} = 1\ ,\ k = 1,\ 2,\ ....,\ N\text{-}1 \tag{4.8}$$

Therefore,

$$\sum_{i=1}^{N-1} \bar{m}_{ki} \leq \frac{1}{\min\limits_{1\leq\ i\ \leq N-1} S_i} = \frac{1}{\beta_i} \leq \frac{1}{|\beta_i|} \tag{4.9}$$

We define $\left\|\ U^{-1}\right\| = \max\limits_{1\leq k\leq N-1} \sum\limits_{i=1}^{N-1} \mid \bar{m}_{ki} \mid$ and $\|\ T\left(h\right)\ \| = \max\limits_{1\leq i\leq N-1} \mid T_i\left(h_i\right)\ \mid$

From Eq. (4.1), Eq. (4.6) and Eq. (4.9), we get

$$e_j = \sum_{i=1}^{N-1} \bar{m}_{ki}T_{\rm i}(h_i),\ j = 1,\ 2,\ 3,\ \ldots,\ N\text{-}1$$

which implies that

$$e_j \leq k_i h_i^2 \text{ for } j = 1,2,...,N\text{-}1 \tag{4.10}$$

here $k_i = \frac{(\sigma_i-1)^2\varepsilon_2 a_i}{9b_i}{y'''}_i$ is a constant independent of h.

Therefore, using Eq. (4.10), we have $\|E\| = O\left(h_i^2\right)$

Thus the order of convergence of the method on variable mesh is two.

5. Numerical examples

Four test problems are considered to illustrate the proposed method computationally. The maximum absolute errors in the solution of the problems are computed using the principle $E_{N,\varepsilon_1,\varepsilon_2} = \max\limits_{0\leq i\leq N} |y\left(x_i\right) - y_i|$. Here $y\left(x_i\right)$ is an exact solution and the computed solution is given by y_i .

Example 1: $\varepsilon_1 y'' + \varepsilon_2 y' - y = 1$ with boundary conditions $y(0) = y(1) = 1$.

$y(x) = -1 + \frac{(2e^{k_2}-2)e^{k_1(x-1)}}{e^m-1} + \frac{(2e^{-k_1}-2)e^{k_2 x}}{e^m-1}$ is the exact solution,

where $k_1 = \frac{-\varepsilon_2+\sqrt{\varepsilon_2^2+4\varepsilon_1}}{2\varepsilon_1}$,$k_2 = \frac{-\varepsilon_2-\sqrt{\varepsilon_2^2+4\varepsilon_1}}{2\varepsilon_1}$, $m = \frac{-\sqrt{\varepsilon_2^2+4\varepsilon_1}}{\varepsilon_1}$

Tables 1 and 2 show the maximum absolute errors for different values ε_1 and ε_2 . The boundary layer behaviour in the solution is shown in Fig 1.

Example 2: $\varepsilon_1 y'' + \varepsilon_2 y' - y = -x$ with boundary conditions $y(0) = 1$ and $y(1) = 0$. Exact solution of the problem is

$$y(x) = \frac{(1+\varepsilon_2) + (1-\varepsilon_2)e^{k_2}}{e^{k_2} - e^{k_1}} e^{k_1 x} + \frac{(1+\varepsilon_2) + (1-\varepsilon_2)e^{k_1}}{e^{k_2} - e^{k_1}} e^{k_2 x} + x + \varepsilon_2$$

where $k_1 = \frac{-\varepsilon_2 - \sqrt{\varepsilon_2{}^2 + 4\varepsilon_1}}{2\varepsilon_1}$ and $k_2 = \frac{-\varepsilon_2 + \sqrt{\varepsilon_2{}^2 + 4\varepsilon_1}}{2\varepsilon_1}$
Tables 3 and Table 4 represent the maximum absolute errors in comparison with the results of [9] for different values ε_1 and ε_2. The boundary layer behaviour in the solution is shown in Fig 2.
Example 3: $\varepsilon_1 y'' - \varepsilon_2 y' - y = -\cos(\pi x)$ with $y(0) = y(1) = 0$.
Actual solution is given by $y(x) = \rho_1 \cos(\pi x) + \rho_2 \sin(\pi x) + k_1 e^{\lambda_1 x} + k_2 e^{-\lambda_2 (1-x)}$

here $\rho_1 = \frac{\varepsilon_1 \pi^2 + 1}{\varepsilon_2{}^2 \pi^2 + (\varepsilon_1 \pi^2 + 1)^2}$, $\rho_2 = \frac{\varepsilon_2 \pi}{\varepsilon_2{}^2 \pi^2 + (\varepsilon_1 \pi^2 + 1)^2}$, $k_1 = \frac{-\rho_1 (1 + e^{-\lambda_2})}{1 - e^{\lambda_1 - \lambda_2}}$
$k_2 = \frac{\rho_1 (1 + e^{\lambda_1})}{1 - e^{\lambda_1 - \lambda_2}}$, $\lambda_1 = \frac{\varepsilon_2 - \sqrt{\varepsilon_2{}^2 + 4\varepsilon_1}}{2\varepsilon_1}$, $\lambda_2 = \frac{\varepsilon_2 + \sqrt{\varepsilon_2{}^2 + 4\varepsilon_1}}{2\varepsilon_1}$

The comparison of the maximum absolute errors are presented in Tables 5 and 6 for diverse values ε_1 , ε_2. The boundary layer behaviour is shown graphically in Fig 3.
Example 4: $\varepsilon_1 y'' - \varepsilon_2 (1+x) y' - y = -x$ with $y(0) = 1,\ y(1) = 0$.
Since the exact solution is not available, the maximum absolute errors are calculated for different values ε_1 , ε_2 and N using double mesh principle [3], $E_{N,x,\varepsilon_2} = \max\limits_{0 \le i \le N} \left| y_i^N - y_i^{2N} \right|$, where y_i^N and y_i^{2N} are the numerical solutions with N and $2N$ intervals. These results are shown in Tables 7 and 8. The layer behaviour in the solution of the example is shown in Fig 4.

6. Discussions and Conclusion

A finite difference method is proposed for two parameters SPBVP on non-uniform mesh which converges uniformly. Using the higher order finite difference approximations to the first and second derivatives of the problem on geometric mesh, the discretization equation is acquired. To demonstrate the proposed scheme, it is implemented on four examples. Numerical results are compared with the results of other methods available in the literature to justify the method. We observed that, the proposed method gives better results. From the graphical representation of the solution of the examples, we noticed that, as perturbation parameter ε_1 decreases for fixed ε_2, the width of the layer at both the ends decreases.

Table 1. The maximum absolute errors in solution of Example 1 for $\varepsilon_2 = 10^{-4}$

$\epsilon_1 \rightarrow$	10^{-1}	10^{-2}	10^{-3}	10^{-4}
$N = 2^5$	2.2780(−7)	1.4573(−5)	1.4171(−3)	5.1920(−2)
$N = 2^6$	1.4241(−8)	9.1344(−7)	9.0653(−5)	7.6892(−3)
$N = 2^7$	8.902(−10)	5.7257(−8)	5.7050(−6)	5.4533(−4)
$N = 2^8$	5.576(−11)	3.5800(−9)	3.5720(−7)	3.5211(−5)
$N = 2^9$	3.778(−12)	2.252(−10)	2.2341(−8)	2.2372(−6)
$N = 2^{10}$	2.635(−12)	1.742(−11)	1.4068(−9)	1.4001(−7)

Table 2. The maximum absolute errors in solution of Example 1 for $\varepsilon_1 = 10^{-2}$

$\epsilon_2 \rightarrow$	10^{-3}	10^{-4}	10^{-5}	10^{-6}
$N = 2^5$	1.4631(−5)	1.4573(−5)	1.4568(−5)	1.4567(−5)
$N = 2^6$	9.1707(−7)	9.1344(−7)	9.1306(−7)	9.1303(−7)
$N = 2^7$	5.7504(−8)	5.7257(−8)	5.7232(−8)	5.7229(−8)
$N = 2^8$	3.5954(−9)	3.5800(−9)	3.5784(−9)	3.5782(−9)
$N = 2^9$	2.262(−10)	2.252(−10)	2.251(−10)	2.251(−10)
$N = 2^{10}$	1.747(−11)	1.742(−11)	1.744(−11)	1.740(−11)

Table 3. Comparison of point wise error in the solution of Example 2 for $\varepsilon_1 = 10^{-3}$

$\epsilon_2 \downarrow$	$N = 64$	$N = 128$	$N = 256$	$N = 512$	$N = 1024$
			Results by proposed method		
10^{-2}	4.7468(−5)	3.1158(−6)	3.2860(−7)	2.7576(−7)	2.8150(−7)
10^{-3}	4.6064(−5)	3.0132(−6)	3.5177(−7)	3.1377(−7)	3.2167(−7)
10^{-4}	4.5451(−5)	2.9746(−6)	3.5375(−7)	3.1782(−7)	3.2599(−7)
			Results by[9]		
10^{-2}	3.6590(−3)	1.1005(−3)	2.7573(−4)	6.8812(−5)	1.7196(−5)
10^{-3}	3.0262(−3)	7.4023(−4)	1.8406(−4)	4.5953(−5)	1.1484(−5)
10^{-4}	2.9008(−3)	7.0989(−4)	1.7654(−4)	4.4076(−5)	1.1015(−5)

Table 4. Comparison of point wise error in the solution of Example 2 for $\varepsilon_2 = 10^{-4}$

$\epsilon_1 \downarrow$	$N = 64$	$N = 128$	$N = 256$	$N = 512$	$N = 1024$
			Results by proposed method		
10^{-1}	3.0416(−9)	1.7044(−9)	3.4089(−9)	6.8461(−9)	1.3721(−8)
10^{-2}	4.5591(−7)	3.0933(−8)	1.2412(−8)	2.4900(−8)	5.0228(−8)
10^{-3}	4.5335(−5)	2.8609(−6)	1.9503(−7)	8.2871(−8)	1.6720(−7)
			Results by[9]		
10^{-1}	1.5725(−5)	3.9408(−6)	9.8514(−7)	2.4628(−7)	6.1570(−8)
10^{-2}	2.8064(−4)	7.0125(−5)	1.7522(−5)	4.3807(−6)	1.0952(−6)
10^{-3}	2.9008(−3)	7.0989(−4)	1.7654(−4)	4.4076(−5)	1.1015(−5)

Table 5. Comparison of maximum absolute error in the solution of example 3for $\varepsilon_1 = 10^{-2}$, $N = 128$

$\varepsilon_2 \downarrow$	Our method	[14]	[12]	[8]
10^{-3}	2.5236(−8)	4.1924(−5)	6.0243(−6)	8.3832(−5)
10^{-4}	2.5080(−8)	4.1296(−5)	6.1827(−7)	8.2686(−5)
10^{-5}	2.5064(−8)	4.1232(−5)	1.1455(−7)	8.2572(−5)
10^{-6}	2.5062(−8)	4.1226(−5)	7.2269(−8)	8.2561(−5)
10^{-7}	2.5062(−8)	4.1225(−5)	6.8266(−8)	8.2559(−5)

Table 6. Comparison of maximum absolute error in the solution of example 3for $\varepsilon_1 = 10^{-4}$, $N = 128$

$\varepsilon_2 \downarrow$	Our method	[14]	[12]	[8]
10^{-3}	2.7811(−4)	4.7598(−3)	6.2154(−3)	9.4446(−3)
10^{-4}	2.7240(−4)	4.2856(−3)	1.8330(−3)	9.0436(−3)
10^{-5}	2.7148(−4)	4.2295(−3)	1.1412(−3)	9.0036(−3)
10^{-6}	2.7139(−4)	4.2238(−3)	1.3699(−3)	8.9996(−3)
10^{-7}	2.7138(−4)	4.2232(−3)	1.3650(−3)	8.9992(−3)

Table 7. Comparison of maximum absolute error in the solution of Example 4 for $\varepsilon_2 = 10^{-4}$ with the result in [5]

ϵ_1	FDM	FEM	BS	Our method
		$N = 128$		
10^{-1}	2.3641(−3)	3.9655(−6)	4.0386(−6)	1.1799(−8)
10^{-2}	3.8548(−3)	7.0093(−5)	6.9830(−5)	4.4486(−8)
10^{-3}	4.0221(−3)	7.0802(−4)	6.9993(−4)	2.6982(−4)
10^{-4}	6.2962(−3)	5.5556(−3)	1.3252(−3)	2.5549(−4)
		$N = 256$		
10^{-1}	2.3641(−3)	3.9655(−6)	4.0386(−6)	1.1799(−8)
10^{-2}	1.9270(−3)	1.7574(−5)	1.7521(−5)	3.7983(−8)
10^{-3}	1.9599(−3)	1.7563(−4)	1.7652(−4)	2.1341(−7)
10^{-4}	3.0398(−3)	2.4116(−3)	5.7425(−4)	1.6557(−5)

Table 8. Comparison of maximum absolute error in the solution of Example 4 for $\varepsilon_1 = 10^{-2}$ with the result in [5]

ϵ_2	FDM	FEM	BS	Our method
		$N = 128$		
10^{-3}	3.8579(−3)	7.0792(−5)	7.0951(−5)	4.4095(−8)
10^{-4}	3.8548(−3)	7.0093(−5)	6.9830(−5)	4.4486(−8)
10^{-5}	3.8545(−3)	7.0034(−5)	7.0037(−5)	4.4465(−8)
10^{-6}	6.2962(−3)	5.5556(−3)	1.3252(−3)	4.4463(−8)
		$N = 256$		
10^{-3}	1.9282(−3)	1.8311(−5)	1.7731(−5)	3.7978(−8)
10^{-4}	1.9270(−3)	1.7574(−5)	1.7521(−5)	3.7983(−8)
10^{-5}	1.9269(−3)	1.7505(−5)	1.7500(−5)	3.7983(−8)
10^{-6}	3.0398(−3)	2.4116(−3)	5.7425(−4)	3.7983(−8)

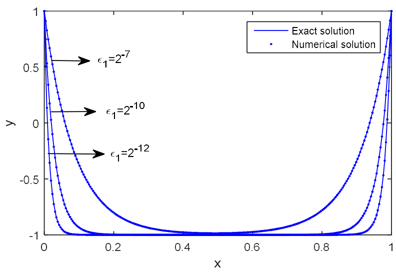

Figure 1. Graphical representation of the solution for for Example1 with $\varepsilon_2 = 2^{-8}$

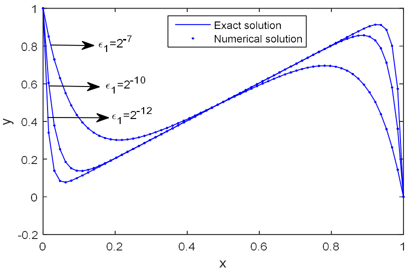

Figure 2. Graphical representation of the solution Example 2 with $\varepsilon_2 = 2^{-8}$

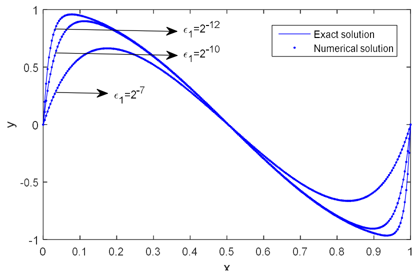

Figure 3. Graphical representation of the solution solution for Example 3 with $\varepsilon_2 = 2^{-8}$

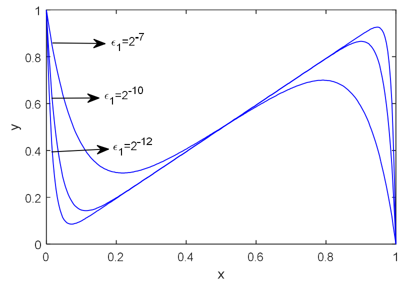

Figure 4. Graphical representation of the numerical for Example 4 with $\varepsilon_2 = 2^{-8}$

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