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CONTROLLABILITY OF NON-AUTONOMOUS SEMILINEAR NEUTRAL EQUATIONS WITH IMPULSES AND NON-LOCAL CONDITIONS

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Abstract. In this work we study the controllability of a control system governed by a non-autonomous semilinear neutral equations with impulses and non-local conditions. The idea is to see under which conditions the controllability of the associated system of ordinary differential equations implies the controllability of the semilinear system of neutral equations with impulses and non-local conditions. This is done by imposing some conditions on the non-linear terms that appear in the system. First, we prove the approximate controllability assuming that the associated system of linear ordinary differential equations is exactly controllable over every small interval, which allows us to use a technique developed by A.E. Bashirov et al. avoiding fixed point theorems to prove approximate controllability; then assuming different conditions on the nonlinear terms of system, allows us to apply Banach Fixed Point Theorem to prove exact controllability.

1. Introduction and Preliminaries

In this work we study the controllability of a control system governed by a semilinear neutral differential equation with impulses and non-local conditions; the idea is to see that under certain conditions the controllability of the associated linear system of ordinary differential equations implies the controllability of the semilinear system of neutral differential equations with impulses and non-local conditions. This is done by imposing some conditions on the non-linear terms that involve the system, and applying a direct approach developed by A.E. Bashirov et al.[1–3] to avoid fixed point theorems to prove approximate controllability; then assuming different conditions on the nonlinear terms of system, allows us to apply Banach Fixed Point Theorem to prove exact controllability. At this point it is good to mention that there is a wide literature on the controllability of linear equations of neutral type, there is even an algebraic condition for controllability of such equations that extends the well-known Kalman's condition for autonomous systems of linear ordinary differential equations ([15–17]). However, for semilinear neutral equations, the literature is limited, there are few works on the existence of solutions, to mention(see [6, 7]), and recently, in [14] the controllability of neutral differential equation with impulses on time scales has been studied. As far as we know, this is the first time that the controllability of neutral equations with impulses and nonlocal conditions simultaneously has been studied, which reveals the novelty of this work, showing that neutral differential equations are just perturbations of ordinary differential equations from a controllability point of view. Without further ado, the system that we will study here is

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given as follows:

$$
(1.1)\quad \begin{cases} \dfrac{d}{dt}[z(t) - f_{-1}(t, z_t)] = A_0(t)z(t) + B(t)u(t) + f_1(t, z_t), \quad t \neq t_k, \quad t \in [0, \tau], \\ z(\theta) + h(z_{\tau_1}, z_{\tau_2}, \dots, z_{\tau_q})(\theta) = \eta(\theta), \quad \theta \in [-r, 0], \\ z(t_k^+) = z(t_k^-) = J_k(t_k, z(t_k), u(t_k)), \quad k = 1, 2, \dots, p, \end{cases}
$$

where $A_{-1}(t), A_0(t), A_1(t)$ are $n \times n$ continuous matrices, $B(t)$ a $n \times m$ continuous matrix and u belongs to $L^2(0, \tau; \mathbb{R}^m)$, the functions f_{-1}, f_1, h are smooth enough, and $0 < t_1 < t_2 < \cdots < t_p < \tau$, $0 < \tau_1 < \tau_2, \cdots < \tau_q < r < \tau$. Here, $z_t : [-r, 0] \longrightarrow \mathbb{R}^n$, $z_t(\theta) = z(t + \theta)$, and $\eta \in \mathcal{PW}_r$ the phase Banach space defined as follows

$$
\begin{aligned}
\mathcal{PW}_r = \big\{ \eta : [-r, 0] \longrightarrow \mathbb{R}^n : \eta &\text{ is continuous except in a finite number of points } \theta_k, \\
k = 1, 2, \dots, p, &\text{where the side limits } \eta(\theta_k^+), \eta(\theta_k^-) \text{ exist and } \eta(\theta_k) = \eta(\theta_k^+) \big\},
\end{aligned}
$$

endowed with the norm

$$
\|\eta\|_r = \sup_{t \in [-r, 0]} \|\eta(t)\|_{\mathbb{R}^n}.
$$

Also, we shall introduce some notation, define a natural Banach spaces for the solutions of problem (1.1) will take place and present motivation for our main theorems in the next sections. We begin defining the following Banach space

$$
\begin{aligned}
\mathcal{PW}_p = \big\{ \eta : [-r, \tau] \longrightarrow \mathbb{R}^n : \eta|_{[-r,0]} \in \mathcal{PW}_r &\text{ and } \eta|_{[0,\tau]} \text{ is continuous} \\
&\text{except in a finite number of points } t_k, k = 1, 2, \dots, p, \text{where} \\
&\text{the side limits exist } \eta(t_k^-), \eta(t_k^+) = \eta(t_k) \big\}
\end{aligned}
$$

equipped with the norm

$$
\|\eta\|_p = \sup_{t \in [-r, \tau]} \|\eta(t)\|_{\mathbb{R}^n}.
$$

We will also consider

$$
\mathbb{R}^{qn} = \underbrace{\mathbb{R}^n \times \mathbb{R}^n \times \cdots \times \mathbb{R}^n}_{q-\text{times}} = \prod_{k=1}^{q} \mathbb{R}^n
$$

endowed with the norm

$$
\|y\|_q = \sum_{i=1}^{q} \|y_i\|_{\mathbb{R}^n}.
$$

Analogously, we define the Banach space

$$
\begin{aligned}
\mathcal{PW}_{qp} = \big\{ \eta : [-r, 0] \longrightarrow \mathbb{R}^{qn} : \eta &\text{ is continuous except in a finite number of points} \\
\theta_k, k = 1, 2, \dots, p, &\text{where the side limits exist } \eta(\theta_k^-), \eta(\theta_k^+) = \eta(\theta_k) \big\}
\end{aligned}
$$

equipped with the norm

$$
\|\eta\|_{qp} = \sup_{t \in [-r, 0]} \|\eta(t)\|_q = \sup_{t \in [-r, 0]} \left(\sum_{i=1}^{q} \|\eta_i(t)\|_{\mathbb{R}^n} \right).
$$

The functions involved in system (1.1) are defined in these spaces

$$
f_{-1}, f_1 : [0, \tau] \times \mathcal{PW}_r \longrightarrow \mathbb{R}^n, \quad h : \mathcal{PW}_{qp} \longrightarrow \mathcal{PW}_r, \quad J_k : [0, \tau] \times \mathbb{R}^n \longrightarrow \mathbb{R}^n.
$$

Corresponding to the nonlinear system (1.1), we have the linear autonomous system of ordinary differential equations($A_0(t) = A_0, B(t) = B$ constants)

$$
(1.2)\quad \begin{cases} z'(t) = A_0 z(t) + Bu(t), \quad t \in [t_0, \tau], \\ z(t_0) = z_0. \end{cases}
$$

The idea is to prove that under some conditions the controllability of the linear system (1.2) implies the controllability of the nonlinear system (1.1). Indeed, it is well known that the autonomous linear system (1.2) is exactly controllable on $[0,\tau]$ if, and only if, the Kalman's Rank condition holds

$$\mathrm{Rank}[B : A_0B : \dots A_0^{n-1}B] = n.$$

REMARK 1.1. *Since Kalman's Algebraic condition does not depend on time, then system* (1.2) *is controllable on any interval, particularly, on* $[t_0,\tau]$ *with* $t_0<\tau$.

Also, it is well known that the following autonomous neutral linear system

$$\begin{cases} \dfrac{d}{dt}[z(t)-A_{-1}z(t-r)] = A_0z(t)+A_1z(t-r)+Bu(t), \quad t\in[0,\tau], \\ z(\theta)=\eta(\theta), \quad \theta\in[-r,0]. \end{cases} \tag{1.3}$$

is exactly controllable if, and only if, the following rank conditions hold:

$$\mathrm{Rank}\,(\Delta(\lambda)) = n, \quad \mathrm{Rank}\,[B : A_0B : \dots A_0^{n-1}B] = n$$

where $\Delta(\lambda)=\lambda I-\lambda A_{-1}e^{-\lambda r}-A_0-A_1e^{-\lambda r}$.
We assume the reader is familiar with the concepts of exact controllability and approximate controllability(See [4, 8–11]). Without further ado we have the following result on approximate controllability

LEMMA 1.1. *If system* (1.2) *is controllable, then system* (1.3) *is approximately controllable on* $[0,\tau]$.

Proof. Suppose that system (1.2) is exactly controllable. Then, from Remark 1.1 it is exactly controllable on any interval $[t_0,\tau]$, with $0\le t_0<\tau$. Therefore, for any initial statez_0 and a final statez_1 there exists a control $u_{t_0}\in L^2(t_0,\tau;\mathbb{R}^m)$ such that the corresponding solution of the initial value problem (1.2) satisfies that $y(\tau)=z_1$. Moreover, u_{t_0} can be taken as follows

$$u_{t_0}(t) = B^*e^{A^*(\tau-t)}\mathfrak{Y}_{t_0}^{-1}\left(z_1-e^{A_0(\tau-t_0)}z_0\right),$$

with

$$\mathfrak{Y}_{t_0} = \int_{t_0}^{\tau} e^{A_0(\tau-\theta)}BB^*e^{A_0^*(\tau-\theta)}d\theta.$$

(see [4, 5]). On the other hand, the solution of the initial value problem (1.3) is given by

$$\begin{aligned} z(t) =& A_{-1}z(t-r)+e^{A_0t}[\eta(0)-A_{-1}\eta(-r)] \\ &+\int_{t_0}^{t} e^{A_0(t-\theta)}[A_0A_{-1}+A_1]z(\theta-r)d\theta+\int_0^t e^{A_0(t-\theta)}Bu(\theta)d\theta. \end{aligned}$$

Let η, z_1 be the initial and the final state for system (1.3). Consider any control $u\in L^2(0,\tau;\mathbb{R}^m)$ fixed and the corresponding solution $z(t)$ of (1.3) evaluated at $t=\tau-d$

$$\begin{aligned} z(\tau-d) =& A_{-1}z(\tau-d-r)+e^{A_0(\tau-d)}[\eta(0)-A_{-1}\eta(-r)] \\ &+\int_0^{\tau-d} e^{A_0(\tau-d-\theta)}[A_0A_{-1}+A_1]z(\theta-r)d\theta+\int_0^{\tau-d} e^{A_0(\tau-d-\theta)}Bu(\theta)d\theta. \end{aligned} \tag{1.4}$$

Consider $0<d<\min\{r,\tau-r,\epsilon/M\}$ small enough, with

$$M = \max_{0\le\theta\le\tau}\left\{\left\|e^{A_0(\tau-d)}\left(A_0A_{-1}+A_1\right)\right\|\|z(\theta)\|\right\},$$

and define the control

$$u^d(t) = \begin{cases} u(t), & \text{if } 0\le t\le\tau-d \\ u_{\tau-d}(t), & \text{if } \tau-d<t\le\tau \end{cases}$$

where

$$u_{\tau-d}(t) = B^* e^{A_0^*(\tau-t)} \mathfrak{Y}_{\tau-d}^{-1}(z_1 - e^{A\tau} z_0)$$

and

$$z_0 = e^{-A_0 d} A_{-1} z(\tau - d) - A_{-1} z(\tau - d - r) + A_1 z(\tau - d). \tag{1.5}$$

Consider $z^d(t, \eta, u^d) = z^d(t)$ the corresponding solution of (1.3) of the control u^d, which we evaluate at $t = \tau$.

$$\begin{aligned} z^d(\tau) =& A_{-1} z^d(\tau - r) + e^{A_0 \tau}[\eta(0) - A_{-1}\eta(-r)] \\ &+ \int_0^\tau e^{A_0(\tau-\theta)}[A_0 A_{-1} + A_1] z^d(\theta - r) ds + \int_0^\tau e^{A_0(\tau-\theta)} B u^d(\theta) d\theta. \end{aligned}$$

Since $d < r$ implies $\tau - r < \tau - d$, then we have that $z^d(\tau - r) = z(\tau - r)$. So, $z^d(\tau)$ can be written as follows

$$\begin{aligned} z^d(\tau) =& A_{-1} z(\tau - d - r) + e^{A_0(\tau-d+d)}[\eta(0) - A_{-1}\eta(-r)] \\ &+ \int_0^{\tau-d} e^{A_0(\tau-d+d-\theta)}[A_0 A_{-1} + A_1] z(\theta - r) d\theta + \int_0^{\tau-d} e^{A_0(\tau-d+d-\theta)} B u(\theta) d\theta \\ &+ \int_{\tau-d}^\tau e^{A_0(\tau-\theta)}[A_0 A_{-1} + A_1] z(\theta - r) d\theta + \int_{\tau-d}^\tau e^{A_0(\tau-\theta)} B u_{\tau-d}(\theta) d\theta. \end{aligned}$$

Then

$$\begin{aligned} z^d(\tau) =& A_{-1} z(\tau - d - r) + e^{A_0 d}\Big\{ e^{A_0(\tau-d)}[\eta(0) - A_{-1}\eta(-r)] \\ &+ \int_0^{\tau-d} e^{A_0(\tau-d-\theta)}[A_0 A_{-1} + A_1] z(\theta - r) d\theta + \int_0^{\tau-d} e^{A_0(\tau-d-\theta)} B u(\theta) d\theta \Big\} \\ &+ \int_{\tau-d}^\tau e^{A_0(\tau-\theta)}[A_0 A_{-1} + A_1] z^d(\theta - r) d\theta + \int_{\tau-d}^\tau e^{A_0(\tau-\theta)} B u_{\tau-d}(\theta) d\theta. \end{aligned}$$

Therefore,

$$\begin{aligned} z^d(\tau) =& A_{-1} z(\tau - d - r) - e^{A_0 d} A_{-1} z(\tau - d - r) + e^{A_0 d}\Big\{ A_{-1} z(\tau - d - r) \\ &+ e^{A_0(\tau-d)}[\eta(0) - A_{-1}\eta(-r)] + \int_0^{\tau-d} e^{A_0(\tau-d-\theta)}[A_0 A_{-1} + A_1] z(\theta - r) d\theta \\ &+ \int_0^{\tau-d} e^{A_0(\tau-d-\theta)} B u(\theta) d\theta \Big\} + \int_{\tau-d}^\tau e^{A_0(\tau-\theta)}[A_0 A_{-1} + A_1] z^d(\theta - r) d\theta \\ &+ \int_{\tau-d}^\tau e^{A_0(\tau-\theta)} B u_{\tau-d}(\theta) d\theta. \end{aligned}$$

Thus,

$$\begin{aligned} z^d(\tau) =& A_{-1} z(\tau - d - r) - e^{A_0 d} A_{-1} z(\tau - d - r) + e^{A_0 d} z(\tau - d) \\ &+ \int_{\tau-d}^\tau e^{A_0(\tau-\theta)}[A_0 A_{-1} + A_1] z^d(\theta - r) d\theta + \int_{\tau-d}^\tau e^{A_0(\tau-\theta)} B u_{\tau-d}(\theta) d\theta \\ =& A_{-1} z(\tau - d - r) + e^{A_0 d}\left(z(\tau - d) - A_{-1} z(\tau - d - r) \right) \\ &+ \int_{\tau-d}^\tau e^{A_0(\tau-\theta)} B u_{\tau-d}(\theta) d\theta + \int_{\tau-d}^\tau e^{A_0(\tau-\theta)}[A_0 A_{-1} + A_1] z^d(\theta - r) d\theta. \end{aligned}$$

If we consider

$$z_0 = e^{-A_0 d} z(\tau - d - r) + \left(z(\tau - d) - A_{-1} z(\tau - d - r) \right),$$

then the solution of the initial value problem (1.2), with $t_0 = \tau - d$, evaluated at τ takes the form

$$\begin{aligned} y(\tau) =& e^{A_0 d} z_0 + \int_{\tau-d}^{\tau} e^{A_0(t-\theta)} B u_{\tau-d}(\theta) d\theta \\ =& A_{-1} z(\tau - d - r) + e^{A_0 d}\left(z(\tau - d) - A_{-1} z(\tau - d - r)\right) + \int_{\tau-d}^{\tau} e^{A_0(\tau-\theta)} B u_{\tau-d}(\theta) d\theta \\ =& z_1. \end{aligned}$$

Hence,

$$\left\| z^d(\tau) - y_d(\tau) \right\| \le \int_{\tau-d}^{\tau} \left\| e^{A_0(\tau-d)} \right\| \| A_0 A_{-1} + A_1 \| \left\| z^d(\theta - r) \right\| d\theta.$$

From the way we choose $0 < d$, it turns out that $z^d(\theta - r) = z(\theta - r)$. Thus

$$\left\| z^d(\tau) - z_1 \right\| < \epsilon.$$

□

2. Exact Controllability of System (1.2)

In this section we shall prove that under certain conditions on the matrices A_0, A_{-1}, A_1 and B the exact controllability of the autonomous system of ordinary differential equations (1.2) implies the exact controllability of the neutral autonomous system (1.3). This will be done by applying Banach contraction mapping theorem. Specifically, we shall prove the following theorem

Lemma 2.1. *If system* (1.2) *is exactly controllable and the following condition holds*

$$\|A_{-1}\| \left(1 + \tau M_1 \|B\|\right) + \tau M_1 M_2 \left(1 + \tau \|B\| M_1^2 \|\mathfrak{Y}^{-1}\|\right) < 1,$$

where $M_1 = \sup_{0 \le \theta \le \tau} \left\| e^{A_0 \theta} \right\|$ *and* $M_2 = \|A_0 A_{-1} + A_1\|$, *then system* (1.3) *is exactly controllable on* $[0, \tau]$.

Proof. From [4, 5, 9–11] it is well known that system (1.2) is exactly controllable if, and only if, the Gramian matrix

$$\mathfrak{Y} = \int_0^{\tau} e^{A_0(\tau-\theta)} B B^* e^{A_0^*(\tau-\theta)} d\theta$$

is invertible, and a control steering system (1.2) from initial state z_0 to the final state z_1 is given by

$$u(t) = B^* e^{A_0^*(\tau-t)} \mathfrak{Y}^{-1} (z_1 - e^{A_0 \tau} z_0),$$

and the steering operator $\Gamma : \mathbb{R}^n \longrightarrow L^2(0, \tau; \mathbb{R}^m)$, defined by $\Gamma\xi = B^* e^{A_0^*(\tau-\cdot)} \mathfrak{Y}^{-1} \xi$, is a right inverse of the controllability operator $\mathfrak{C} : L^2(0, \tau; \mathbb{R}^m) \longrightarrow \mathbb{R}^n$ define by

$$\mathfrak{C} u = \int_0^{\tau} e^{A_0(\tau-\theta)} B u(\theta) d\theta.$$

i.e.,

$$\mathfrak{C}\Gamma = I_{\mathbb{R}^n} \quad \text{and} \quad u = \Gamma(z_1 - e^{A_0 \tau} z_0).$$

Suppose for a moment that system (1.3) is exactly controllable. So, for every $\eta \in C(-r, 0; \mathbb{R}^n)$ and $z_1 \in \mathbb{R}^n$ there exists $u \in C(0, \tau; \mathbb{R}^m)$ such that the corresponding solution $z(t) = z(t, \eta, u)$ of (1.3) satisfies $z(\tau) = z_1$, i.e.,

$$\begin{aligned} z_1 = z(\tau) =& A_{-1} z(\tau - r) + e^{A_0 \tau}[\eta(0) - A_{-1}\eta(-r)] \\ &+ \int_0^{\tau} e^{A_0(\tau-\theta)} [A_0 A_{-1} + A_1] z(\theta - r) d\theta + \int_0^{\tau} e^{A_0(\tau-\theta)} B u(\theta) d\theta. \end{aligned}$$

Then

$$\mathfrak{C}u = z_1 - A_{-1}z(\tau - r) - e^{A_0\tau}[\eta(0) - A_{-1}\eta(-r)] - \int_0^\tau e^{A_0(\tau-\theta)}[A_0A_{-1} + A_1]z(\theta - r)d\theta.$$

Now, if we consider the operator $\mathscr{L} : C(-r, \tau; \mathbb{R}^n) \longrightarrow \mathbb{R}^n$ defined by

$$\mathscr{L}(z) = z_1 - A_{-1}z(\tau - r) - e^{A_0\tau}[\eta(0) - A_{-1}\eta(-r)] - \int_0^\tau e^{A_0(\tau-\theta)}[A_0A_{-1} + A_1]z(\theta - r)d\theta,$$

then $\mathfrak{C}u = \mathscr{L}(z)$. So, we choose the control in the following way,

$$u = \Gamma\mathscr{L}(z). \tag{2.6}$$

This suggests that we must solve a fixed point problem, which is equivalent to the exact controllability of (1.3). Indeed, it is enough to show that the following operator has a fixed point: Let $K : C(-r, \tau; \mathbb{R}^n) \longrightarrow C(-r, \tau; \mathbb{R}^n)$ be given by

$$\begin{aligned}(Ky)(t) =& A_{-1}z(t - r) + e^{A_0t}[\eta(0) - A_{-1}\eta(-r)] \\ &+ \int_0^t e^{A_0(t-\theta)}[A_0A_{-1} + A_1]z(\theta - r)d\theta + \int_0^t e^{A_0(t-\theta)}B\Gamma\mathscr{L}(z)(\theta)d\theta.\end{aligned}$$

Now, we shall prove that K is a contraction mapping. In fact, let $z, y \in C(-r, \tau; \mathbb{R}^n)$ and consider

$$\begin{aligned}\|(Ky)(t) - (Kz)(t)\| \leq& \|A_{-1}\|\|y(t - r) - z(t - r)\| \\ &+ \int_0^t \left\|e^{A_0(t-\theta)}\right\|\|A_0A_{-1} + A_1\|\|z(\theta - r) - y(\theta - r)\|d\theta \\ &+ \int_0^t \left\|e^{A_0(t-\theta)}\right\|\|B\|\|\Gamma\mathscr{L}(y)(\theta) - \Gamma\mathscr{L}(z)(\theta)\|d\theta\end{aligned}$$

Then,

$$\|(Ky)(t) - (Kz)(t)\| \leq \tau M_1M_2\|z - y\| + \|A_1\|\|z - y\| + \tau M_1\|B\|\|\Gamma\mathscr{L}(y) - \Gamma\mathscr{L}(z)\|.$$

On the other hand, we have the following estimate

$$\|\mathscr{L}(y) - \mathscr{L}(z)\| \leq \|A_{-1}\|\|y - z\| + \tau M_1\|A_0A_{-1} + A_1\|\|y - z\|.$$

Thus

$$\begin{aligned}\|K(y) - K(z)\| \leq& \tau M_1M_2\|y - z\| + \|A_{-1}\|\|y - z\| + \\ &\tau M_1\|B\| \left(\|B\|M_1\left\|\mathfrak{Y}^{-1}\right\|\tau M_1\|A_0A_{-1} + A_1\|\|y - z\| + \|A_{-1}\|\|y - z\|\right) \\ \leq& \left(\|A_{-1}\| + \tau M_1M_2 + \tau^2\|B\|^2M_1^3\left\|\mathfrak{Y}^{-1}\right\|M_2 + \tau M_1\|B\|\|A_{-1}\|\right)\|y - z\| \\ \leq& \left[\|A_{-1}\|\left(1 + \tau M_1\|B\|\right) + \tau M_1M_2\left(1 + \tau\|B\|M_1^2\left\|\mathfrak{Y}^{-1}\right\|\right)\right]\|y - z\|\end{aligned}$$

Since $\|A_{-1}\|\left(1 + \tau M_1\|B\|\right) + \tau M_1M_2\left(1 + \tau\|B\|M_1^2\|\mathfrak{Y}^{-1}\|\right) < 1$, then K is a contraction mapping, and by applying Banach Fixed Point Theorem it has a fixed point. That is to say,

$$z = K(z).$$

Since $u = \Gamma\mathscr{L}(z)$, then

$$\begin{aligned}\mathfrak{C}u =& \mathscr{L}(z) \\ =& z_1 - A_{-1}z(\tau - r) - e^{A_0\tau}[\eta(0) - A_{-1}\eta(-r)] \\ &- \int_0^\tau e^{A_0(\tau-\theta)}[A_0A_{-1} + A_1]z(\theta - r)d\theta.\end{aligned}$$

3. Controllability of Non-Autonomous Linear Neutral Differential Equations.

In this section we shall study the controllability of the following non-autonomous linear neutral differential equation

$$\begin{cases} \dfrac{d}{dt}[z(t) - A_{-1}(t)z(t-r)] = A_0(t)z(t) + A_1(t)z(t-r) + B(t)u(t), \quad t \in [0,\tau], \\ z(\theta) = \eta(\theta), \quad \theta \in [-r, 0], \end{cases} \tag{3.7}$$

where $A_{-1}(t), A_0(t), A_1(t)$ are $n \times n$ continuous matrices, $B(t)$ a $n \times m$ continuous matrix and u belongs to $L^2(0, \tau; \mathbb{R}^m)$. Corresponding to system (3.7), we have the following non-autonomous linear system of ordinary differential equations

$$\begin{cases} z'(t) = A_0(t)z(t) + B(t)u(t), \quad t \in [t_0, \tau], \\ z(t_0) = z_0. \end{cases} \tag{3.8}$$

We are interested in showing that, under some conditions, the controllability of system (3.8) implies the controllability of system (3.7). In this regard, we note that, the solution of the initial value problem (3.7) is given by

$$\begin{aligned} z(t) =& A_{-1}(t)z(t-r) + S(t,0)\left[\eta(0) - A_{-1}(0)\eta(-r)\right] \\ &+ \int_0^t S(t,\theta)\left[A_0(\theta)A_{-1}(\theta) + A_1(\theta)\right])z(\theta - r)d\theta + \int_0^t S(t,\theta)B(\theta)u(\theta)d\theta, \end{aligned}$$

where $S(t,\theta) = \Phi(t)\Phi^{-1}(\theta)$ and

$$\begin{cases} \Phi'(t) = A_0(t)\Phi(t), \\ \Phi(0) = I. \end{cases} \tag{3.9}$$

Consider the following definitions and notation:

$$\begin{aligned} N_1 &= \sup_{0 \le \theta \le \tau} \|A_{-1}(\theta)\|, \quad M_1 = \sup_{0 \le \theta \le \tau} \|S(\tau, \theta)\|, \\ M_2 &= \sup_{0 \le \theta \le \tau} \|A_0(\theta)A_{-1}(\theta) + A_1(\theta)\|, \quad \|B\|_\infty = \sup_{0 \le \theta \le \tau} \|B(\theta)\|, \end{aligned}$$

and

$$\mathfrak{Y}_{t_0} = \int_{t_0}^{\tau} S(\tau,\theta)B(\theta)B^*(\theta)S^*(\tau,\theta)d\theta, \quad \mathfrak{Y} := \mathfrak{Y}_0. \tag{3.10}$$

The proof of the following Lemmas follows in the same way as Lemmas 1.1 and 2.1.

Lemma 3.1. *If system* (3.8) *is exactly controllable in any interval* $[t_0, \tau]$ *with* $0 < t_0 < \tau$, *then system* (3.7) *is approximately controllable on* $[0, \tau]$.

Lemma 3.2. *If system* (3.8) *is exactly controllable on* $[0, \tau]$ *and*

$$N_1\left(1 + \tau M_1\|B\|_\infty\right) + \tau M_1 M_2\left(1 + \tau\|B\|_\infty M_1^2\|\mathfrak{Y}^{-1}\|\right) < 1,$$

then system (3.7) *is exactly controllable on* $[0, \tau]$.

4. Controllability of a Semilinear Non-autonomous Neutral Differential Equation with Impulses and Non-local Conditions.

In this section we shall study the controllability of the semilinear non-autonomous neutral differential equation with impulses and non-local conditions (1.1). To be more specific, we will study the controllability of the following semilinear neutral differential equation with impulses and non-local conditions.

$$
\begin{cases}
\dfrac{d}{dt}[z(t)-f_{-1}(t,z_t)] = A_0(t)z(t)+B(t)u(t)+f_1(t,z_t), \quad t\neq t_k, \quad t\in[0,\tau],\\
z(\theta)+h(z_{\tau_1},z_{\tau_2},\dots,z_{\tau_q})(\theta)=\eta(\theta), \quad \theta\in[-r,0],\\
z(t_k^+)=z(t_k^-)=J_k(t_k,z(t_k)), \quad k=1,2,\dots,p,
\end{cases}
\tag{4.11}
$$

Since the functions f_{-1}, f_1 and g are smooth enough functions,(from [12]) the problem (4.11) admits a solution given by

$$
\begin{aligned}
z(t) =& f_{-1}(t,z_t)+S(t,0)\left[\eta(0)-h(z_{\tau_1},z_{\tau_2},\dots,z_{\tau_q})(0)-f_{-1}(0,\eta-h(z_{\tau_1},z_{\tau_2},\dots,z_{\tau_q}))\right]\\
&+\int_0^t S(t,\theta)A_0(\theta)f_{-1}(\theta,z_\theta)d\theta+\int_0^t S(t,\theta)f_1(\theta,z_\theta)d\theta\\
&+\int_0^t S(t,\theta)B(\theta)u(\theta)d\theta+\sum_{0<t_k<t} S(t,t_k)J_k(t_k,z(t_k)).
\end{aligned}
$$

Corresponding to the non-linear system (4.11), we have the linear system

$$
\begin{cases}
z'(t)=A_0(t)z(t)+B(t)u(t), \quad t\in[t_0,\tau],\\
z(t_0)=z_0.
\end{cases}
\tag{4.12}
$$

4.1. **Approximate controllability of** (4.11)**.** As in section 1, we shall prove that, under certain conditions, that the controllability of the non-autonomous linear system of ordinary differential equations (4.12), over any interval $[t_0,\tau]$ with $0<t_0<\tau$, implies the controllability of the non-autonomous semilinear neutral system of differential equations with impulses and nonlocal conditions (4.11). For which, we will assume the following hypotheses:

(*H*1) The functions f_{-1} and f_1 satisfy the following conditions

$$
\|f_{-1}(t,\eta)\|\leq\rho_{-1}\left(\|\eta(-r)\|\right), \quad \|f_1(t,\eta,\nu)\|\leq\rho_1\left(\|\eta(-r)\|\right),
$$

where $\rho_{-1},\rho_1:\mathbb{R}^+\longrightarrow\mathbb{R}^+$ are continuous.

(*H*1) The system (1.2) is exactly controllable in any interval $[t_0,\tau]$ with $0<t_0<\tau$.

The hypothesis **(*H*2)** can be satisfied in many cases; perhaps, when $A_0(t)=A$ and $B(t)=B$ are constant, and satisfy the Kalman's Rank Conditions, according to remark 1.1. But, also in [13] one can find an example of non-autonomous system satisfying hypothesis **(*H*2)**.

THEOREM 4.1. *Under the hypotheses* **(*H*1)** *and* **(*H*2)** *the non-autonomous semilinear neutral system of differential equations with impulses and nonlocal conditions* (4.11) *is approximately controllable on* $[t_0,\tau]$.

Proof. Given $\epsilon>0$, we consider any fixed control $u\in L^2(0,\tau;\mathbb{R}^m)$ and the corresponding solution $z(t)=z(t,\eta,u)$ of problem (4.11). Then, we consider a number $d>0$ small enough such that $d<\min\{r,\tau-r,\tau-t_p,\epsilon/MN\}$, where

$$
M=\sup_{0\leq\theta\leq\tau}\{\|S(\tau,\theta)\|\|A_0(\theta)\|,\|S(\tau,\theta)\|\}, \quad N=\max_{\theta\in[0,\tau]}\{\rho_{-1}\left(\|z(\theta-r)\|\right)+\rho_1\left(\|z(\theta-r)\|\right)\}.
$$

Define the following control

$$
u^d(t)=\begin{cases}
u(t), & \text{if } t\in[0,\tau-d],\\
u_d(t), & \text{if } t\in(\tau-d,\tau],
\end{cases}
$$

where

$$u_d(t) = B^*(t)(t)S(\tau, t)\mathfrak{Y}^{-1}_{\tau-d}\left(z - S(\tau, \tau - d)z_0\right)$$

with $\mathfrak{Y}^{-1}_{\tau-d}$ defined by (3.10), and z_0 is to be defined later. Let $z^d(t) = z(t, \eta, u^d)$ be the corresponding solution of (4.11) for the control u^d defined above. Then,

$$\begin{aligned} z^d(\tau) =& S(\tau, 0)\Big[\eta(0) - h(z^d_{\tau_1}, z^d_{\tau_2}, \dots, z^d_{\tau_q})(0) - f_{-1}(0, \eta - h(z^d_{\tau_1}, z^d_{\tau_2}, \dots, z^d_{\tau_q}))\Big] \\ &+ f_{-1}(\tau, z^d_\tau) + \int_0^\tau S(\tau, \theta)A_0(\theta)f_{-1}(\theta, z^d_\theta)d\theta + \int_0^\tau S(\tau, \theta)f_1(\theta, z^d_\theta, u(\theta))d\theta \\ &+ \int_0^\tau S(\tau, \theta)B(\theta)u(\theta)d\theta + \sum_{0<t_k<\tau-d} S(\tau - d, t_k)J_k(t_k, z^d(t_k), u^d(t_k)). \end{aligned}$$

Therefore,

$$\begin{aligned} &z^d(\tau) = f_{-1}(\tau, z^d_\tau) \\ &+ S(\tau, \tau - d)\Big\{S(\tau - d, 0)\Big[\eta(0) - h(z^d_{\tau_1}, z^d_{\tau_2}, \dots, z^d_{\tau_q})(0) - f_{-1}(0, \eta - h(z^d_{\tau_1}, z^d_{\tau_2}, \dots, z^d_{\tau_q}))\Big] \\ &+ \int_0^{\tau-d} S(\tau - d, \theta)\left[A_0(\theta)f_{-1}(\theta, z^d_\theta) + f_1(\theta, z^d_\theta, u^d(\theta))\right]d\theta \\ &+ \int_0^{\tau-d} S(\tau - d, \theta)B(\theta)u^d(\theta)d\theta + \sum_{0<t_k<\tau-d} S(\tau - d, t_k)J_k(t_k, z(t_k), u(t_k))\Big\} \\ &+ \int_{\tau-d}^{\tau} S(\tau, \theta)\left[A_0(\theta)f_{-1}(\theta, z^d_\theta) + f_1(\theta, z^d_\theta, u^d(\theta))\right]d\theta + \int_{\tau-d}^{\tau} S(\tau, \theta)B(\theta)u(\theta)d\theta. \end{aligned}$$

Thus

$$\begin{aligned} z^d(\tau) =& f_{-1}(\tau, z_\tau) - S(\tau, \tau - d)f_{-1}(\tau - d, z_{\tau-d}) + S(\tau, \tau - s)z(\tau - d) \\ &+ \int_{\tau-d}^{\tau} S(\tau, \theta)\left[A_0(\theta)f_{-1}(\theta, z^d_\theta) + f_1(\theta, z_\theta, u_d(\theta))\right]d\theta \\ &+ \int_{\tau-d}^{\tau} S(\tau, \theta)B(\theta)u_d(\theta)d\theta. \end{aligned}$$

The solution of (4.12) for $t_0 = \tau - d$ at $t \in (\tau - d, \tau]$ is given by

$$y_d(t) = S(t, \tau - d)z_0 + \int_{\tau-d}^{t} S(t, \theta)B(\theta)u_d(\theta)d\theta.$$

So,

$$y_d(\tau) = S(\tau, \tau - d)z_0 + \int_{\tau-d}^{\tau} S(\tau, \theta)B(\theta)u_d(\theta)d\theta.$$

Taking

$$z_0 = S(\tau - d, \tau)f_{-1}(\tau, z_\tau) - f_{-1}(\tau - d, z_{\tau-d}) + z(\tau - d),$$

we get that

$$y_d(\tau) = f_{-1}(\tau, z_\tau) - S(\tau, \tau - d)f_{-1}(\tau - d, z_{\tau-d}) + S(\tau, \tau - d)z(\tau - d) + \int_{\tau-d}^{\tau} S(\tau, \theta)B(\theta)u_d(\theta)d\theta.$$

Hence

$$\begin{aligned}\left\|z^d(\tau)-y_d(\tau)\right\| &\leq \int_{\tau-d}^{\tau}\|S(\tau,\theta)\|\left\|A_0(\theta)f_{-1}(\theta,z_\theta^d)+f_1(\theta,z_\theta^d,u_d(\theta))\right\|d\theta\\ &\leq \int_{\tau-d}^{\tau}\|S(\tau,\theta)\|\|A_0(\theta)\|\rho_{-1}\left(\left\|z^d(\theta-r)\right\|\right)d\theta\\ &\quad+\int_{\tau-d}^{\tau}\|S(\tau,\theta)\|\rho_1\left(\left\|z^d(\theta-r)\right\|\right)d\theta.\end{aligned}$$

Since $0<d<r$ and $\tau-\theta\leq\theta\leq\tau$, we have that $\theta-r\leq\tau-r\leq\tau-d$. Then $z^d(\theta-r)=z(\theta-r)$. This implies that

$$\left\|z^d(\tau)-z_1\right\|\leq\int_{\tau-d}^{\tau}M\left(\rho_{-1}\left(\|z(\theta-r)\|\right)+\rho_1\left(\|z(\theta-r)\|\right)\right)d\theta.$$

Then

$$\left\|z^d(\tau)-z_1\right\|\leq dMN<\epsilon.$$

□

4.2. **Exact Controllability of** (4.11)**.** In this subsection, we will study the exact controllability of the (4.11) system using the ideas from Sections 2 and 3 where we turn the exact controllability problem into a fixed point problem. In other words, to prove the exact controllability, we will impose some conditions in such a way that an operator associated with the system has a fixed point. For which, we will assume the following hypotheses:

(*H*3) There exists constants $d_k, L_g>0, k=1,2,\dots,p$ such that

$$\|J_k(t,y)-J_k(t,z)\|_{\mathbb{R}^n}\leq d_k\|y-z\|_{\mathbb{R}^n},\quad y,z\in\mathbb{R}^n, t\in[0,\tau],$$

$$\|h(y)(t)-h(v)(t)\|_{\mathbb{R}^n}\leq L_g\sum_{i=1}^{q}\|y_i(t)-v_i(t)\|_{\mathbb{R}^n},\quad y,v\in\mathcal{PW}_{qp}.$$

(*H*4) The function f_{-1} satisfies

$$\|f_{-1}(t,\eta_1)-f_{-1}(t,\eta_2)\|_{\mathbb{R}^n}\leq L_{-1}\|\eta_1-\eta_2\|_r,\quad \eta_1,\eta_2\in\mathcal{PW}_r,$$

and f_1 satisfies

$$\|f_1(t,\eta_1)-f_1(t,\eta_2)\|_{\mathbb{R}^n}\leq L_1\|\eta_1-\eta_2\|_r,\quad \eta_1,\eta_2\in\mathcal{PW}_r.$$

Also we shall consider the following notation:

$$M=\sup_{t,\theta\in[0,\tau]}\|S(t,\theta)\|,\quad \|B\|_\infty=\sup_{\theta\in[0,\tau]}\|B(\theta)\|,\quad \|\Gamma\|=\sup_{\theta\in[0,\tau]}\|B^*(\theta)S^*(\tau,\theta)\mathfrak{Y}^{-1}\|,$$

$$M_1=M\sup_{\theta\in[0,\tau]}\|A_0(\theta)\|,\quad T=\sum_{k=1}^{q}d_k,\quad M_2=L_{-1}+L_gMq+M_1L_{-1}\tau+ML_1\tau+MT.$$

THEOREM 4.2. *Suppose that* **(*H*3)** *and* **(*H*4)** *hold, the linear system of ordinary differential equations* (4.12) *is exactly controllable on* $[0,\tau]$ *and the following condition holds.*

$$L_{-1}+ML_gq+ML_{-1}L_gq+M_1\tau+ML_1\tau+M\|B\|_\infty\|\Gamma\|M_2+MT\tau<1 \tag{4.13}$$

Then, the system (4.11) *is exactly controllable* $[0,\tau]$.

Proof. From [4, 5, 9–11] it is well known that system (4.12) is exactly controllable if, and only if, the Gramian matrix

$$\mathfrak{Y}=\int_0^{\tau}S(\tau,\theta)B(\theta)B^*(\theta)S(\tau,\theta)^*d\theta$$

is invertible, and a control steering system (4.12) from initial state z_0 to the final state z_1 is given by

$$u(t) = B^*(t)S^*(\tau, t)\mathfrak{Y}^{-1}(z_1 - S(\tau, 0)z_0),$$

and the steering operator $\Gamma : \mathbb{R}^n \longrightarrow L^2(0, \tau; \mathbb{R}^m)$, defined by $\Gamma\xi = B^*(\cdot)S^*(\tau, \cdot)\mathfrak{Y}^{-1}\xi$, is a right inverse of the controllability operator $\mathfrak{C} : L^2(0, \tau; \mathbb{R}^m) \longrightarrow \mathbb{R}^n$ define by

$$\mathfrak{C}u = \int_0^\tau S(\tau, \theta)B(\theta)u(\theta)d\theta.$$

i.e.,

$$\mathfrak{C}\Gamma = I_{\mathbb{R}^n} \quad \text{and} \quad u = \Gamma(z_1 - S(\tau, 0)z_0).$$

Again, the controllability of system (4.11) will be equivalent to find a fixed points for the following operator
$K : C(-r, \tau; \mathbb{R}^n) \longrightarrow C(-r, \tau; \mathbb{R}^n)$ defined by

$$\begin{aligned}(Ky)(t) =& f_{-1}(t, z_t) + S(t, 0)\left[\eta(0) - h(z_{\tau_1}, z_{\tau_2}, \ldots, z_{\tau_q})(0) - f_{-1}(0, \eta - h(z_{\tau_1}, z_{\tau_2}, \ldots, z_{\tau_q}))\right]\\ &+ \int_0^t S(t, \theta)A_0(\theta)f_{-1}(\theta, z_\theta)d\theta + \int_0^t S(t, \theta)f_1(\theta, z_\theta)d\theta\\ &+ \int_0^t S(t, \theta)B(\theta)\Gamma\mathscr{L}(\theta)d\theta + \sum_{0<t_k<t} S(t, t_k)J_k(t_k, z(t_k)),\end{aligned}$$

where $\mathscr{L} : C(-r, \tau; \mathbb{R}^n) \longrightarrow \mathbb{R}^n$ is the operator defined by

$$\begin{aligned}&\mathscr{L}(z) = z_1 - f_{-1}(\tau, z_\tau)\\ &- S(\tau, 0)\left[\eta(0) - h(z_{\tau_1}, z_{\tau_2}, \ldots, z_{\tau_q})(0) - f_{-1}(0, \eta - h(z_{\tau_1}, z_{\tau_2}, \ldots, z_{\tau_q}))\right]\\ &- \int_0^\tau S(\tau, \theta)A_0(\theta)f_{-1}(\theta, z_\theta)d\theta + \int_0^\tau S(\tau, \theta)f_1(\theta, z_\theta, u(\theta))d\theta\\ &- \sum_{0<t_k<\tau} S(\tau, t_k)J_k(t_k, z(t_k)).\end{aligned}$$

To apply Banach contraction mapping we need to prove that K is a contraction mapping. In deed, consider $z, y \in C(-r, \tau; \mathbb{R}^n)$ and

$$\begin{aligned}&\|(Ky)(t) - (Kz)(t)\| \le \|f_{-1}(0, \eta - h(z_{\tau_1}, z_{\tau_2}, \ldots, z_{\tau_q})) - f_{-1}(0, \eta - h(y_{\tau_1}, y_{\tau_2}, \ldots, y_{\tau_q}))\|\\ &\|f_{-1}(t, z_t) - f_{-1}(t, y_t)\| + \|h(z_{\tau_1}, z_{\tau_2}, \ldots, z_{\tau_q})(0) - h(y_{\tau_1}, y_{\tau_2}, \ldots, y_{\tau_q})(0)\|\\ &+ \int_0^t \|S(t, \theta)A_0(\theta)(f_{-1}(\theta, z_\theta) - f_{-1}(\theta, y_\theta))\|d\theta + \int_0^t \|S(t, \theta)(f_1(\theta, z_\theta) - f_1(\theta, y_\theta)\|d\theta\\ &+ \int_0^t \|S(t, \theta)B(\theta)\Gamma(\mathscr{L}(\theta) - \mathscr{L}(\theta))\|d\theta + \sum_{0<t_k<t} \|S(t, t_k)\|\|J_k(t_k, z(t_k)) - J_k(t_k, z(t_k))\|\end{aligned}$$

Then, using the above notation, we get that

$$\begin{aligned}&\|(Ky)(t) - (Kz)(t)\| \le L_{-1}\|z - y\| + ML_gq\|z - y\| + ML_{-1}L_gq\|z - y\| + \tau M_1\|z - y\|\\ &+ M\tau L_1\|z - y\| + \tau M\|B\|_\infty\|\Gamma\|\|\mathscr{L}(z) - \mathscr{L}(y)\| + MT\|z - y\|.\end{aligned}$$

On the other hand, we have the following estimate

$$\|\mathscr{L}(z) - \mathscr{L}(y)\| \le M_2\|z - y\|.$$

Thus

$$\|K(y)-K(z)\| \leq$$
$$\{L_{-1}+ML_gq+ML_{-1}L_gq+\tau M_1+M\tau L_1+\tau M\|B\|_\infty\|\Gamma\|M_2+MT\}\,\|z-y\|.$$

Since $L_{-1}+ML_gq+ML_{-1}L_gq+\tau M_1+M\tau L_1+\tau M\|B\|_\infty\|\Gamma\|M_2+MT<1$, then K is a contraction mapping, and consequently has a fixed point. That is to say,

$$z=K(z).$$

Since $u=\Gamma\mathscr{L}(z)$, then

$$\begin{aligned}\mathfrak{C}u=&\mathscr{L}(z)=z_1-f_{-1}(\tau,z_\tau)\\&-S(\tau,0)\left[\phi(0)-h(z_{\tau_1},z_{\tau_2},\dots,z_{\tau_q})(0)-f_{-1}(0,\phi-h(z_{\tau_1},z_{\tau_2},\dots,z_{\tau_q}))\right]\\&-\int_0^\tau S(\tau,\theta)A_0(\theta)f_{-1}(\theta,z_\theta)d\theta+\int_0^\tau S(\tau,\theta)f_1(\theta,z_\theta,u(\theta))d\theta\\&-\sum_{0<t_k<\tau}S(\tau,t_k)J_k(t_k,z(t_k)).\end{aligned}$$

5. Final Remark

In this work we study the approximate controllability and the exact controllability for semi-linear neutral differential equations with non-local conditions and impulses, this is done assuming that the associated system of linear ordinary differential equations is controllable, plus some additional conditions imposed on the non-linear terms that could be seen as perturbations of the linear system of ODEs; in fact, what we have proven is that from the point of view of controllability, semi-linear neutral control systems can be seen as perturbations of the corresponding linear system of ordinary differential equations, which may represent the novelty of this work. Of course, once we have studied in depth the finite-dimensional case, we will study the case of control systems modeled by semi-linear neutral equations in infinite-dimensional spaces.

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