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# APPROXIMATE CONTROLLABILITY OF TIME-DEPENDENT IMPULSIVE SEMILINEAR RETARDED DIFFERENTIAL EQUATIONS WITH INFINITE DELAY AND NONLOCAL CONDITIONS

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Abstract. Here we study the approximate controllability of time dependent impulsive retarded semilinear differential equations with infinite delay and nonlocal conditions, where some ideas are taking from a previous works for this kind of systems with impulses, nonlocal conditions and finite delay, this is done using a techniques evading fixed point theorems used by A.E. Bashirov et al. In this case we have to impose some conditions on the nonlinear term depending of the last time impulse $t_p$, so that we can prove the approximate controllability of this system by living the impulses behind on a fixed solution curve in a small interval of time, and from this position, we are able to reach a ball of center the final state and radius $\epsilon > 0$ small enough, at time $\tau$, by assuming that the associated linear control system is exactly controllable on any interval $[t_0, \tau]$, $0 < t_0 < \tau$.

*Key words and phrases.* approximate controllability, time-dependent retarded differential equation with infinite delay, Hale and Kato axiomatic theory, impulses, nonlocal conditions, Bashirov et al. technique.

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## 1. Introduction.

In this paper, the approximate controllability of a time-dependent impulsive semilinear retarded differential equations with infinite delay and nonlocal is proved without using fixed point theorem technique, rather than this, we use a technique evading fixed point theorems used by A.E. Bashirov et al.[3, 4, 5]. In this case we have to impose some conditions on the nonlinear term depending of the last boost time $t_p$, so that we can prove the approximate controllability of this system by living the impulses behind on a fixed curve in a short time interval, and from this position, we are able to reach a neighborhood of the final state on time $\tau$, by assuming that the corresponding linear control system is exactly controllable on any interval $[t_0, \tau]$, $0 < t_0 < \tau$. Without further ado, this system is giving by the following

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impulsive retarded differential equation with infinite delay and nonlocal condition:

$$(1.1)\qquad \begin{cases} z'(t) = \mathcal{A}(t)z(t) + \mathcal{B}(t)u(t) + f(t, z_t, u(t)), \quad t \neq t_k, \quad t > 0 \\ z(s) + \mathcal{K}(z_{\tau_1}, z_{\tau_2}, \cdots, z_{\tau_q})(s) = \phi(s), \quad s \in \mathbb{R}_- = (-\infty, 0] \\ z(t_k^+) = z(t_k^-) + J_k(t_k, z(t_k)), \quad k = 1, 2, \cdots, p, \end{cases}$$

where $0 < t_1 \leq t_2 < ... < t_p < \tau$, $0 < \tau_1 < \tau_2 < \cdots < \tau_q < r < \tau$, are fixed real numbers $z(t) \in \mathbb{R}^n$, $u(t) \in \mathbb{R}^m$, $\mathcal{A}(t)$, $\mathcal{B}(t)$ are continuous matrices of dimension $n \times n$ and $n \times m$ respectively, the control function $u$ belongs to $L_2([0, \tau]; \mathbb{R}^m)$, $\phi \in \mathfrak{B}$, with $\mathfrak{B}$ being a particular phase space satisfying the axiomatic theory defined by Hale and Kato (which will be specified later), $z_t(s) = z(t + s)$, $z_t \in \mathfrak{B}$, and finally $f : (-\infty, \tau] \times \mathfrak{B} \times \mathbb{R}^m \to \mathbb{R}^n$, $J_k : (-\infty, \tau] \times \mathbb{R}^n \to \mathbb{R}^n$ and $\mathcal{K} : (\mathfrak{B})^q \to \mathfrak{B}$ are smooth enough functions. Under these conditions, using some ideas from [10, 12], in [2] it is proved that this control system with infinite delay, impulse, and non local conditions has solutions. Additionally, we assume the following conditions on the nonlinear term $f$

$$(1.2)\qquad |f(t, \varphi, u)| \leq \zeta(\|\varphi(-t_p)\|), \quad u \in \mathbb{R}^m, \varphi \in \mathfrak{B},$$

where $\zeta : \mathbb{R}_+ \to [0, \infty)$ is a continuous function. In particular, $\zeta(\xi) = a(\xi)^\beta + b$, with $\beta \geq 1$. Associated with the semilinear system (1.1), we consider also the linear system

$$(1.3)\qquad \begin{cases} z'(t) = \mathcal{A}(t)z(t) + \mathcal{B}(t)u(t), \quad t \in (t_0, \tau], \\ z(t_0) = z_0. \end{cases}$$

Also, we shall assume the following hypothesis:
H1) The linear control system (1.3) is exactly controllable on any interval $[\tau - \delta, \tau]$, for all $\delta$ with $0 < \delta < \tau$.
The hypothesis H1) is satisfied in the case that $\mathcal{A}(t) = A$ and $\mathcal{B}(t) = B$ are constant matrices since the algebraic Kalman's condition(see [11]) for exact controllability of linear autonomous systems does not depend on the time interval.

$$\text{Rank}[B|AB|\cdots|A^{n-1}B] = n.$$

Others examples of time-dependent systems satisfying the hypothesis H1) can be found(see [9]).
There are several papers on the existence of solutions of semilinear evolution equations with impulses, with impulses and bounded delay, with bounded delay and non-local condition, and with non-local conditions and impulses. To mention, one can see [15]. Recently, in [1], the existence of periodic mild solution of infinite delay evolution equations with non-instantaneous impulses have been study by using Koratowski's measure of non-compactness and Sadowski's fixed point theorem. In recently work [2], using some ideas from this paper and from [8], [13],[14], to define a particular phase space $\mathfrak{B}$ satisfying Hale-Kato axiomatic theory, the existence of solutions for this type of systems has been proved applying Karakosta's fixed

point theorem as in [10, 12]. But, as far as we know, the controllability of this system has not being studied before.

## 2. Preliminaries 2

In this section, we shall set some notation and define the phase space $\mathfrak{B}$ in which our initial state will take place. Let $\Phi$ be the fundamental matrix of the linear system

$$z'(t) = \mathcal{A}(t)z(t), \quad t \in \mathbb{R}. \tag{2.4}$$

Then, the evolution operator $\mathcal{U}(t,s)$ is define by $\mathcal{U}(t,s) = \Phi(t)\Phi^{-1}(s), t, s \in \mathbb{R}$. For $\tau > 0$, we consider the following bound for the evolution operator

$$M = \sup_{t,s \in (0,\tau]} \|\mathcal{U}(t,s)\|.$$

Now, we shall define the function space $\mathcal{PW}_p = \mathcal{PW}_p((-\infty,0];\mathbb{R}^N)$, as it follows:

$$\mathcal{PW}_p = \big\{ z : (-\infty,0] \to \mathbb{R}^N : z \text{ is continuous except on } s_{kz} \in (-\infty,0], \\ k = 1,2,\dots,p, \text{ the side limits } z(s_{kz}^+), z(s_{kz}^-) \text{ exist and } z(s_{kz}^+) = z(s_{kz}) \big\}.$$

Using some ideas from [13, 8, 13, 14], we consider a function $h : \mathbb{R} \to \mathbb{R}_+$ in such a way that

(1) $h(0) = 1$,
(2) $h(-\infty) = +\infty$,
(3) $h$ is decreasing.

Remark 2.1. *A particular function $h$ is $h(s) = \exp(-as)$, with $a > 0$.*

Now, we define the following functions space

$$C_{hp} = \left\{ z \in \mathcal{PW}_p : \sup_{s \le 0} \frac{\|z(s)\|}{h(s)} < \infty \right\}.$$

Lemma 2.1 (See [2]). *The space $C^{hp}$ equipped with the norm*

$$\|z\|_{hp} = \sup_{s \le 0} \frac{\|z(s)\|}{h(s)}, \quad z \in C_{hp}$$

*is a completed metric space.*

Now, we shall consider the following larger space $\mathcal{PW}_{h\tau} := \mathcal{PW}_{h\tau}((-\infty,\tau];\mathbb{R}^N)$

$$\mathcal{PW}_{h\tau} = \left\{ z : (-\infty,\tau] \to \mathbb{R}^N : z\Big|_{\mathbb{R}_-} \in \mathfrak{B} \text{ and } z\Big|_{(0,\tau]} \text{ is a continuous except at } t_k, \right. \\ \left. k = 1,2,...,p, \text{ where side limits } z(t_k^+), z(t_k^-) \text{ exist and } z(t_k^+) = z(t_k) \right\}.$$

From Lemma 2.1, the following Lemma is obtained

LEMMA 2.2. *$\mathcal{PW}_{h\tau}$ is a completed metric space equipped with the norm*

$$\|z\| = \left\|z|_{\mathbb{R}_-}\right\|_{\mathfrak{B}} + \|z|_I\|_\infty ,$$

*where* $\|z|_I\|_\infty = \sup_{t\in I=(0,\tau]} \|z(t)\|$.

Our phase space will be

$$\mathfrak{B} := C_{hp},$$

equipped with the norm

$$\|z\|_{hp} = \|z\|_{\mathfrak{B}}.$$

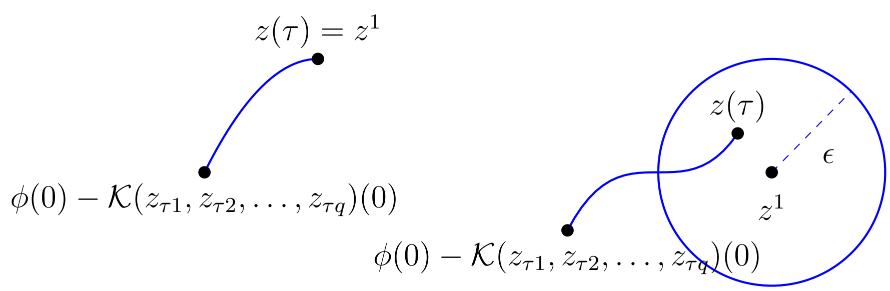
It is not hard to verify that the phase space $\mathfrak{B}$ satisfies the Hale and Kato axiomatic theory for the phase space of retarded differential equations with infinite delay: For more details about it, one can see [8, 13, 14].
We assume that the reader is familiar with the concept of exact controllability and approximate controllability, for more detail abut it, one can see [6, 7].

DEFINITION 2.1. **(Exact Controllability)** *The system (1.1) is said to be exactly controllable on $[0,\tau]$ if for every $\phi \in \mathfrak{B}$, $z_1 \in \mathbb{R}^n$, there exists $u \in L_2([0,\tau];\mathbb{R}^m)$ such that the solution $z(t)$ of (1.1) corresponding to $u$ verifies:*

$$z(0) + \mathcal{K}(z_{\tau_1}, z_{\tau_2}, \dots, z_{\tau_q})(0) = \phi(0) \quad \textit{and} \quad z(\tau) = z_1.$$

DEFINITION 2.2. **(Approximate Controllability)** *The system (1.1) is said to be approximately controllable on $[0,\tau]$ if for every $\phi \in \mathfrak{B}$, $z_1 \in \mathbb{R}^n$ and $\epsilon > 0$, there exists $u \in L_2([0,\tau];\mathbb{R}^m)$ such that the solution $z(t)$ of (1.1) corresponding to $u$ verifies:*

$$z(0) + \mathcal{K}(z_{\tau_1}, \dots, z_{\tau_q})(0) = \phi(0), \quad \textit{and} \quad \left\|z(\tau) - z^1\right\|_{\mathbb{R}^n} < \epsilon.$$




## 3. CONTROLLABILITY OF LINEAR SYSTEM

In this section, we shall present some known characterization of the controllability of the linear system (1.3) without impulses, delays and nonlocal conditions. To this end, we note that for all $z_0 \in \mathbb{R}^n$ and $u \in L_2(0,\tau;\mathbb{R}^m)$ the initial value problem

$$\begin{cases} y' = \mathcal{A}(t)y(t) + \mathcal{B}(t)u(t), \quad y \in \mathbb{R}^n, \quad t \in [\tau-\delta,\tau], \\ y(\tau-\delta) = z_0, \end{cases} \tag{3.5}$$

admits only one solution given by

$$y(t) = \mathcal{U}(t,\tau-\delta)z_0 + \int_{\tau-\delta}^{t} \mathcal{U}(t,s)\mathcal{B}(s)u(s)ds, \;\; t \in [\tau-\delta,\tau], \tag{3.6}$$

Definition 3.1. *Corresponding with* (3.5)*, we define the following matrix: The Gramian controllability matrix by:*

$$\mathcal{W}_{\tau\delta} = \int_{\tau-\delta}^{\tau} \mathcal{U}(\tau,s)\mathcal{B}(s)\mathcal{B}^*(s)\mathcal{U}^*(\tau,s)ds. \tag{3.7}$$

Proposition 3.1. *(See* [6, 7]*) The system (3.5) is controllable on* $[\tau-\delta,\tau]$ *if, and only if, the matrix* $\mathcal{W}_{\tau\delta}$ *is invertible.*
*Moreover, a control steering the system (3.5) from initial state* $z_0$ *to a final state* $z^1$ *on the interval* $[\tau-\delta,\tau]$ *is given by*

$$v^\delta(t) = B^*(t)\mathcal{U}^*(\tau,t)\mathcal{W}_{\tau\delta}^{-1}(z_1 - \mathcal{U}(\tau,\tau-\delta)z_0), \quad t \in [\tau-\delta,\tau]. \tag{3.8}$$

*i.e.,*
*The corresponding solution* $y^\delta(t)$ *of the linear system (3.5) satisfies the boundary condition:*

$$y^\delta(\tau-\delta) = z_0 \quad \textit{and} \quad y^\delta(\tau) = z^1.$$

## 4. Main Result

In this section, we shall prove the main results of this work, the approximate controllability of the semilinear retarded system (1.1) with infinite delay, impulses, and nonlocal conditions. In this regard, according to [2], for all $\phi \in \mathfrak{B}$ and $u \in L^2(0,\tau;\mathbb{R}^m)$ the problem (1.1) admits only one solution $z \in \mathcal{PW}_{h\tau}$ given by
(4.9)

$$\begin{aligned} z(t) &= \mathcal{U}(t,0)\phi(0) - \mathcal{U}(t,0)[(\mathcal{K}(z_{\tau_1},\dots,z_{\tau_q}))(0)] + \int_0^t \mathcal{U}(t,s)\mathcal{B}(s)u(s)ds \\ &+ \int_0^t \mathcal{U}(t,s)f(s,z(s-t_p),u(t))ds + \sum_{0<t_k<t} \mathcal{U}(t,t_k)I_k(t_k,z(t_k)u(t_k)), \quad t \in [0,\tau], \\ z(t) &+ (\mathcal{K}(z_{\tau_1},\dots,z_{\tau_q}))(t) = \phi(t), \quad t \in (-\infty,0]. \end{aligned}$$

Theorem 4.1. *If the functions* $f, I_k, h$ *are smooth enough, condition (1.2) holds and the linear system (3.5) is exact controllable on any interval* $[\tau-\delta,\tau]$*,* $0<\delta<\tau$*, then system (1.1) is approximately controllable on* $[0,\tau]$*.*

**Proof** . Given $\phi \in \mathfrak{B}$, a final state $z^1$ and $\varepsilon > 0$, we want to find a control $u^\delta \in L^2(0,\tau;\mathbb{R}^m)$ steering the system to a ball of center $z^1$ and radius $\varepsilon > 0$ on $[0,\tau]$. In indeed, we consider any fixed control $u \in L^2(0,\tau;\mathbb{R}^m)$ and the corresponding solution $z(t) = z(t,0,\phi,u)$ of the problem (1.1). For $0 < \delta < min\{\tau - t_p, t_p, \frac{\varepsilon}{MN}\}$, we define the control $u^\delta \in L^2(0,\tau;\mathbb{R}^m)$ as follows

$$u^\delta(t) = \begin{cases} u(t), & \text{if } \; 0 \leq t \leq \tau - \delta, \\ v^\delta(t), & \text{if } \; \tau - \delta < t \leq \tau. \end{cases}$$

where $N = \sup_{s\in[0,\tau]}\{\zeta(\|z(s)\|\}$ and

$$v^\delta(t) = B^*(t)\mathcal{U}^*(\tau,t)(\mathcal{W}_{\tau\delta})^{-1}(z^1 - \mathcal{U}(\tau,\tau-\delta)z(\tau-\delta)), \quad \tau - \delta < t \leq \tau.$$

Since $0 < \delta < \tau - t_p$, then $\tau - \delta > t_p$; and using the cocycle property $\mathcal{U}(t,l)\mathcal{U}(l,s) = \mathcal{U}(t,s)$, the associated solution $z^\delta(t) = z(t,0,\phi,u^\delta)$ of the time-dependent inpulsive semeilinear retarded differential equation with infinite delay and nonlocal (1.1), at time $\tau$, can be expressed as follows:

$$\begin{aligned} z^\delta(\tau) &= \mathcal{U}(\tau,0)\phi(0) - \mathcal{U}(\tau,0)[(\mathcal{K}(z_{\tau_1},\dots,z_{\tau_q})(0)] + \int_0^\tau \mathcal{U}(\tau,s)\mathcal{B}(s)u^\delta(s)ds \\ &+ \int_0^\tau \mathcal{U}(\tau,s)f(s,z_s^\delta,u^\delta(s))ds + \sum_{0<t_k<\tau} \mathcal{U}(\tau,t_k)I_k(t_k,z(t_k),u^\delta(t_k)). \end{aligned}$$

Therefore,

$$\begin{aligned} z^\delta(\tau) &= \mathcal{U}(\tau,\tau-\delta)\Big\{\mathcal{U}(\tau-\delta,0)\phi(0) - \mathcal{U}(\tau-\delta,0)[(\mathcal{K}(z_{\tau_1},\dots,z_{\tau_1}))(0)] \\ &+ \int_0^{\tau-\delta} \mathcal{U}(\tau-\delta,s)\mathcal{B}(s)(s)u^\delta(s)ds \\ &+ \int_0^{\tau-\delta} \mathcal{U}(\tau-\delta,s)f(s,z_s^\delta,u^\delta(s))ds \\ &+ \sum_{0<t_k<\tau-\delta} \mathcal{U}(\tau-\delta,t_k)I_k^e(t_k,z^\delta(t_k),u^\delta(t_k))\Big\} \\ &+ \int_{\tau-\delta}^\tau \mathcal{U}(\tau,s)\mathcal{B}(s)u^\delta(s)ds + \int_{\tau-\delta}^\tau \mathcal{U}(\tau,s)f(s,z_s^\delta,u^\delta(s))ds. \end{aligned}$$

Hence,

$$\begin{aligned} z^\delta(\tau) &= \mathcal{U}(\tau,\tau-\delta)z(\tau-\delta) + \int_{\tau-\delta}^\tau \mathcal{U}(\tau,s)\mathcal{B}(s)v^\delta(s))ds \\ &+ \int_{\tau-\delta}^\tau \mathcal{U}(\tau,s)f(s,z_s^\delta,v^\delta(s))ds. \end{aligned}$$

So,

$$\begin{aligned} z^{\delta}(\tau) &= \mathcal{U}(\tau,\tau-\delta)z(\tau-\delta) + \int_{\tau-\delta}^{\tau} \mathcal{U}(\tau,s)\mathcal{B}(s)v^{\delta}(s)ds \\ &+ \int_{\tau-\delta}^{\tau} \mathcal{U}(\tau,s)f(s,z_s^{\delta},v^{\delta}(s))ds. \end{aligned}$$

The corresponding solution $y^{\delta}(t) = y(t,\tau-\delta,z(\tau-\delta),v^{\delta})$ of the initial value problem (3.5) at time $\tau$, for the control $v^{\delta}$ and the initial condition $z_0 = z(\tau-\delta)$, is given by:

$$y^{\delta}(\tau) = \mathcal{U}(\tau,\tau-\delta)z(\tau-\delta) + \int_{\tau-\delta}^{\tau} \mathcal{U}(\tau,s)\mathcal{B}(s)v^{\delta}(s)ds,$$

and from Proposition 3.1, we get that

$$y^{\delta}(\tau) = z^1.$$

Therefore,

$$\left\| z^{\delta}(\tau) - z_1 \right\| \le \int_{\tau-\delta}^{\tau} \|\mathcal{U}(\tau,s)\| \left\| f(s,z_s^{\delta},u^{\delta}(s)) \right\| ds.$$

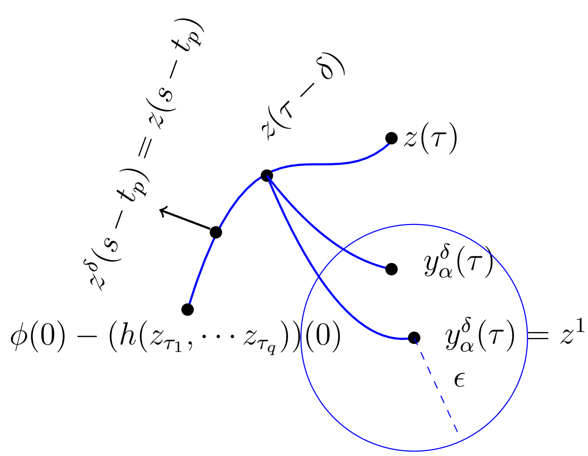
Now, since $0<\delta<t_p$ and $\tau-\delta \le s \le \tau$, then $s-t_p \le \tau-t_p < \tau-\delta$ and

$$z^{\delta}(s-t_p) = z(s-t_p).$$

Hence, since $\delta$ satisfies $0<\delta<\min\{t_p,\tau-t_p,\frac{\varepsilon}{MN}\}$ and

$$\begin{aligned} \left\| z^{\delta}(\tau) - z_1 \right\| &\le \int_{\tau-\delta}^{\tau} \|\mathcal{U}(\tau,s)\| \left\| f(s,z_s^{\delta},v^{\delta}(s)) \right\| ds \\ &\le \int_{\tau-\delta}^{\tau} \|\mathcal{U}(\tau,s)\| \, \zeta(\|z(s-t_p)\| ds < MN\delta < \varepsilon. \end{aligned}$$

A geometric interpretation of this theorem can be seen in the following picture:



This finishes the proof of our main Theorem.

□

## 5. Application

As an application of Theorem 4.1, we shall consider an example of a control system governed by a semilinear time-dependent retarded equation with infinite delay,impulses, and nonlocal conditions

$$\left\{\begin{array}{ll} z'(t) = \mathcal{A}(t)z(t) + \mathcal{B}(t)u(t) + f(t, z_t, u(t)), & t \in (0, \tau], t \neq t_k \\ z(s) + \mathcal{K}(z_{\tau_1}, z_{\tau_2}, \ldots, z_{\tau_q})(s) = \phi(s), & s \in [-r, 0], \\ z(t_k^+) = z(t_k^-) + I_k(z(t_k), u(t_k)), & k = 1, 2, 3, \ldots, p, \end{array}\right. \tag{5.10}$$

with $\mathcal{A}(t) = a(t)A$, $\mathcal{B}(t) = b(t)B$, $A$ and $B$ are $n \times n$ and $n \times m$ constant matrices, respectively.

$a \in L^1[0, \tau]$, $b \in C[0, \tau]$ satisfying the following conditions

$$\int_0^\tau a(s)ds \neq 0, \qquad b(t) \neq 0, \qquad t \in [0, \tau]$$

We suppose that the linear autonomous system $y'(t) = Ay(t) + Bu(t)$ is controllable, which is equivalent to the Kalman's rank conditions

$$Rank[B; AB; \cdots; A^{n-1}B] = n.$$

But, from [9], the Kalman's rank conditions holds if, and only if, the time-dependent linear system given by

$$z'(t) = \mathcal{A}(t)z(t) + \mathcal{B}(t)u(t), \qquad t \in [0, \tau]$$

($\mathcal{A}(t) = a(t)A$, $\mathcal{B}(t) = b(t)B$) is exactly controllable on $[t_0, \tau]$, for all $0 \leq t_0 < \tau$.
In this example we shall assume that the Kalman's rank conditions holds. The nonlinear terms and the impulsive functions are given as follows

$f : [0, \tau] \times \mathfrak{B} \times \mathbb{R}^m \to \mathbb{R}^n$,

$$f(t, \phi, u) = \begin{pmatrix} \sqrt[3]{\sin \|u\| + 1} & \cdot & \sqrt[3]{\phi_1(-t_p)} \\ \sqrt[3]{\sin \|u\| + 1} & \cdot & \sqrt[3]{\phi_2(-t_p)} \\ \vdots & \cdot & \vdots \\ \sqrt[3]{\sin \|u\| + 1} & \cdot & \sqrt[3]{\phi_n(-t_p)} \end{pmatrix},$$

$\mathcal{K} : \mathfrak{B}^q \to \mathfrak{B}$,
given by

$$\mathcal{K}(\phi_1, \phi_2, \cdots, \phi_1) = \sum_{i=1}^{q} \begin{pmatrix} sin(\phi_{i1}) \\ sin(\phi_{i2}) \\ \vdots \\ sin(\phi_{in}) \end{pmatrix},$$

$I_k : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n$, $k = 1, 2, \cdots, p$,
given by

$$I_k(z, u) = cos(\sqrt{\|u\| + 1}) \begin{pmatrix} sin(z_1^k) \\ sin(z_2^k) \\ \vdots \\ sin(z_n^k) \end{pmatrix},$$

Then

$\|f(t, \phi, u)\| \leq \sqrt{n}\|\phi(-t_P)\|^{2/3} + 2\sqrt{n}\sin\|u\|^{2/3} + 2\sqrt{n} \leq \sqrt{n}\|\phi(-t_P)\|^{2/3} + 3\sqrt{n} = \zeta(\|\phi(-t_P)\|)$, and since $g$ and $I_k$, $k = 1, 2, \cdots, p$ are smooth enough and conditions and condition (1.2) is satisfied. Hence, the system (1.1) is approximately controllable on $[0, \tau]$.

## 6. Final Remark

In this work, the approximate controllability of a control system governed by a retarded equation with infinite delay, impulses, and non-local conditions is proved. The problem is set in a phase space that satisfies the axiomatic theory introduced by Hale and Kato to study delayed equations with infinite delay. Now, the fact that we have nonlocal conditions and impulses forces us to consider a slightly more general space in the sense that the impulses that occur in the present are transferred to the historical past, as the present also preserves properties of the same historical past. In this sense, this problem is very interesting and has not been studied before, as far as we know. Another important issue is that the technique we apply evades the use of fixed point theorems, which was used first by Bashirov et al(See [3, 4, 5]). to study the controllability of equations without impulses, without delays, and without non-local conditions, and assuming that the non-linear term was bounded. But in this work, the condition imposed on the last impulse time helps us to go back to a previously chosen solution curve, from which we can steer the system to a neighborhood of the final state on time $\tau$, thus proving the approximate controllability of the system.

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