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EXISTENCE OF SOLUTIONS FOR RETARDED EQUATIONS WITH INFINITE DELAY, IMPULSES, AND NONLOCAL CONDITIONS

MARÍA J. AYALA BOLAGAY [1], HUGO LEIVA[1] AND DARWIN TALLANA CHIMARRO[1]

Abstract. In this work, we study the existence and uniqueness of solutions for retarded equations with infinite delay, impulses, and nonlocal conditions. We set the problem in a natural Banach phase space satisfying Hale-Kato axiomatic Theory about the phase space for retarded equations with unbounded delay. The result about the existence is obtained by applying Karakosta's Fixed Point Theorem.

1. Introduction

It is well known that J. Hale and J. Kato wrote a magnificent paper on the phase space for ordinary retarded differential equations with unbounded delay (see [3]). They made an Axiomatic Theory on the conditions that the phase space should satisfies for this type of equations, which allowed them to analyze the existence of solutions and the asymptotic behavior of this type of equations, of course, without impulses and nonlocal conditions. In this work, we will use a particular and natural phase space $\mathfrak{B}$ appropriated to our problem, since the initial function $\phi : (-\infty, 0] \to \mathbb{R}^N$ has a fixed number p of points of discontinuity for which the side limits exist and the function ϕ is continuous to the right at such points. In fact, this phase space will satisfy the axioms proposed by Hale and Kato. Without further ado, we will study the existence and uniqueness of solutions for the next non-autonomous semi-linear retarded equation, which has infinite delay, impulses, and nonlocal conditions,

$$\begin{cases} z'(t) = A(t)z(t) + f(t, z_t), \quad t \neq t_k, t > 0, \\ z(s) + g(z_{\tau_1}, z_{\tau_2}, \cdots, z_{\tau_q})(s) = \phi(s), \quad s \in \mathbb{R}_- = (-\infty, 0], \\ z(t_k^+) = z(t_k^-) + \mathcal{J}_k(t_k, z(t_k)), \quad k = 1, 2, \cdots, p. \end{cases} \tag{1.1}$$

Here $0 < t_1 < t_2 < \cdots < t_p$, $0 < \tau_1 < \tau_2 < \cdots < \tau_q$, the $n \times n$ matrix $A(t)$ is continuous, $\phi \in \mathfrak{B}$, with $\mathfrak{B}$ being the phase space (which will be specified later) satisfying the axiomatic theory proposed by Hale and Kato , $z_t(s) = z(t+s)$, $z_t \in \mathfrak{B}$, and finally $f : (-\infty, \tau] \times \mathfrak{B} \to \mathbb{R}^N$, $\mathcal{J}_k : (-\infty, \tau] \times \mathbb{R}^N \to \mathbb{R}^N$ and $g : (\mathfrak{B})^q \to \mathfrak{B}$ are smooth enough functions.

The function z_t illustrates the history of the state up to the time t, and also remembers much of the historical past of ϕ, caring part of the present to the past (see Fig. 1).

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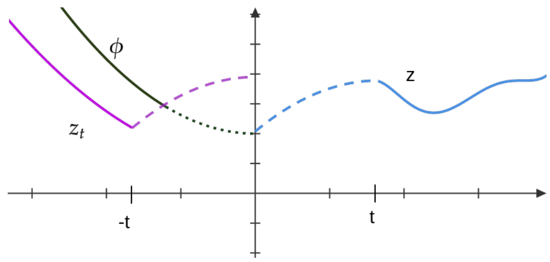

FIGURE 1. Graph of the functions z_t, ϕ, and z.

There are several papers on the existence of solutions of semilinear evolution equations with impulses or with impulses and bounded delay or with unbounded delay, non-local condition or with non-local conditions and impulses. To mention, one can see [1, 2, 4, 7, 8, 14, 13, 15]. Recently, in [1], the existence of periodic mild solution of infinite delay evolution equations with non-instantaneous impulses have been studied by using Koratowski's measure of non-compactness, and Sadovki's fixed point theorem. In our work, we will use some ideas from this paper and from [3, 9, 10] to define a particular phase space $\mathfrak{B}$ satisfying Hale-Kato axiomatic theory. Since our problem is under the influence of unbounded delay, impulses, and nonlocal condition simultaneously, it suggests us to use a new method to achieve the existence result by applying Karakosta's Fixed Point Theorem (see [7, 8]).

2. Preliminaries

In this section, we shall establish some useful notation and define the phase space $\mathfrak{B}$ in which our initial state will take place. Let's denote as Φ the fundamental matrix of the linear system

$$z'(t) = A(t)z(t), \quad t \in \mathbb{R}, \tag{2.2}$$

then the evolution operator $\mathcal{V}$ is defined by $\mathcal{V}(t,s) = \Phi(t)\Phi^{-1}(s)$, where $t, s \in \mathbb{R}$. For $\tau > 0$, we consider the following bound for the evolution operator

$$\mathcal{M} = \sup_{t,s\in(0,\tau]} \|\mathcal{V}(t,s)\|. \tag{2.3}$$

Now, we shall define the function space $\mathcal{PW}_p = \mathcal{PW}_p((-\infty,0];\mathbb{R}^N)$, as follows:

$$\mathcal{PW}_p = \big\{ z : (-\infty,0] \to \mathbb{R}^N : z \in C(J;\mathbb{R}^N),\, J = (-\infty,0]\backslash\{s_{1z},\dots,s_{pz}\}, s_{kz} \in (-\infty,0],$$
$$k = 1,2,\dots,p, \text{ where } z(s_{kz}^+), z(s_{kz}^-) \text{ exist and } z(s_{kz}^+) = z(s_{kz}) \big\}.$$

Using some ideas from [10], we consider a function $h : \mathbb{R} \to \mathbb{R}_+$ such that

a) $h(0) = 1$,
b) $h(-\infty) = +\infty$,
c) h is decreasing.

Remark 2.1. A particular function h is $h(s) = \exp(-as)$, with $a > 0$. See Fig. 2.

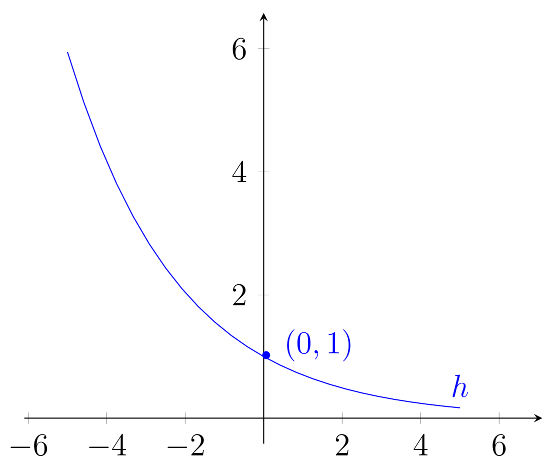

FIGURE 2. Example of h function described above.

Now, we define the following functions space

$$C_{hp} = \left\{ z \in \mathcal{PW}_p : \sup_{s\le 0} \frac{\|z(s)\|}{h(s)} < \infty \right\}.$$

In [1, 10, 9, 3] and other references it is mentioned that this space is a Banach space, but neither of them gives proof if, so we decided to give proof of it for a better understanding of the reader.

Lemma 2.2. *The space C_{hp} equipped with the norm*

$$\|z\|_{hp} = \sup_{s\le 0} \frac{\|z(s)\|}{h(s)}, \quad z \in C_{hp},$$

is a Banach space.

Proof. Let $\{\phi_n\}$ be a Cauchy sequence in C_{hp} . It is easy to see that the sequences $\{\phi_n\}$ converges pointwise to a function $\phi : (-\infty, 0] \to \mathbb{R}^N$. Also, note that,

$$\lim_{n\to+\infty} \|\phi_n - \phi\|_{hp} = 0$$

So, it is enough to show that $\phi \in C_{hp}$. Indeed, let us put

$$\varphi_n(s) = \frac{\phi_n(s)}{h(s)} \quad \text{for all} \quad s \in \mathbb{R}_-, \quad n \in \mathbb{N}.$$

Then

$$\varphi(s) = \frac{\phi(s)}{h(s)} \quad \text{and} \quad \lim_{n\to+\infty} \|\varphi_n - \varphi\| = 0, \quad \text{uniformly.}$$

Since $\{\varphi_n\}$ is a Cauchy sequence in the uniform convergence, there exists $N \in \mathbb{N}$ such that for all $n \geq N$, $s_{k\phi_n} = s_k \in [-N, 0]$, $\quad k = 1, 2, \ldots, p$, and φ_n is continuous in $[-N, 0]$ except in $\{s_k\}_{k=1}^p$, where the side limits $\varphi_n(s_k^-), \varphi_n(s_k^+) = \varphi_n(s_k)$ exist. Next, we want to show that

$$\lim_{s\to s_k^-} \varphi(s) = \varphi(s_k^-) \quad \text{and} \quad \lim_{s\to s_k^+} \varphi(s) = \varphi(s_k^+) = \varphi(s_k).$$

Indeed, let $\{s_i\} \subset [-N, 0]$ be a sequence such that $s_i > s_k$ and $\lim_{i\to\infty} s_i = s_k$.
Then, by Cantor's diagonalization process the sequence $\{\varphi_i(s_i)\}$ converges uniformly. Now, we want to prove that $\lim_{i\to\infty} \varphi(s_i) = L \in \mathbb{R}^N$ exists.
Let us prove that $\{\varphi(s_i)\}$ is a Cauchy sequence. Indeed, we consider the following estimate:

$$\|\varphi(s_j) - \varphi(s_i)\| \leq \|\varphi_j(s_j) - \varphi(s_j)\| + \|\varphi_i(s_i) - \varphi(s_i)\| + \|\varphi_j(s_j) - \varphi_i(s_i)\|,$$

From the uniform convergence and the fact that $\{\varphi_i(s_i)\}$ is a Cauchy sequence, we obtain that $\{\varphi(s_i)\}$ is a Cauchy sequence. Hence,

$$\lim_{i\to\infty} \varphi(s_i) = L \in \mathbb{R}^N. \tag{2.4}$$

Let us prove that this limit doesn't depend on the sequence $\{s_i\} \subset [-N, 0]$ such that $s_i < s_k$ and $\lim_{i\to\infty} s_i = s_k$. In fact, consider another sequence $\{\tau_i\} \subset [-N, 0]$ such that $\tau_i < s_k$ and $\lim_{i\to\infty} \tau_i = s_k$. Then, applying Cantor's diagonalization process again, we get that $\{\varphi_i(\tau_i)\}$ converges and

$$\lim_{i\to\infty} \varphi(\tau_i) = l \in \mathbb{R}^N. \tag{2.5}$$

Also, from Cantor's diagonalization process, we get that

$$\lim_{i\to\infty} \varphi_i(s_i) = \lim_{i\to\infty} \varphi_i(\tau_i). \tag{2.6}$$

Next, we consider the following estimate

$$\begin{aligned} \|L - l\| \;\leq\; & \|\varphi(s_i) - L\| + \|\varphi(\tau_i) - l\| + \|\varphi_i(s_i) - \varphi(s_i)\| \\ + \; & \|\varphi_i(\tau_i) - \varphi(\tau_i)\| + \|\varphi_i(s_i) - \varphi_i(\tau_i)\|. \end{aligned}$$

From (2.4), (2.5), (2.6) and the uniform convergence, we have that $L = l$. Therefore $\lim_{s\to s_k^+}\varphi(s) = \varphi(s_k^+)$ exists. Analogously, $\lim_{s\to s_k^-}\varphi(s) = \varphi(s_k^-)$ exists. i.e.,

$$\lim_{s\to s_k^-}\varphi(s) = \varphi(s_k^-) \quad \text{and} \quad \lim_{s\to s_k^+}\varphi(s) = \varphi(s_k^+).$$

Since, $\varphi_n(s_k^+) = \varphi_n(s_k)$, for all $n > N$, we get that $\varphi(s_k^+) = \varphi(s_k)$. On the other hand

$$\varphi(s) = \frac{\phi(s)}{h(s)} \quad \text{for all} \quad s \in \mathbb{R}_-,$$

Hence, the following limits exist

$$\lim_{s\to s_k^-}\phi(s) = \phi(s_k^-) \quad \text{and} \quad \lim_{s\to s_k^+}\phi(s) = \phi(s_k^+) = \phi(s_k),$$

and the proof of the Lemma is completed. $\square$

Our phase space will be

$$\mathfrak{B} := C_{hp},$$

endowed with the norm

$$\|z\|_{hp} = \|z\|_{\mathfrak{B}}.$$

Now, we shall consider the following larger space $\mathcal{PW}_{h\tau} := \mathcal{PW}_{h\tau}((-\infty,\tau];\mathbb{R}^N)$

$$\mathcal{PW}_{h\tau} = \Big\{ z : (-\infty,\tau] \to \mathbb{R}^N : z\Big|_{\mathbb{R}_-} \in \mathfrak{B} \text{ and } z\Big|_{(0,\tau]} \in C(J';\mathbb{R}^N),\, J' = (0,\tau]\backslash\{t_1,\dots,t_p\},$$
$$t_k \in (0,\tau], k = 1,2,...,p, \text{ where } z(t_k^+), z(t_k^-) \text{ exist and } z(t_k^+) = z(t_k) \Big\}.$$

From Lemma 2.2, it follows that,

Lemma 2.3. *$\mathcal{PW}_{h\tau}$ endowed with the norm*

$$\|z\| = \left\|z|_{\mathbb{R}_-}\right\|_{\mathfrak{B}} + \|z|_I\|_\infty,$$

where $\|z|_I\|_\infty = \sup_{t\in I=(0,\tau]} \|z(t)\|$, is a Banach space.

Remark 2.4. It is not hard to verify that our phase space $\mathfrak{B}$ satisfies the Hale and Kato axiomatic theory (stated in e.g. [1]) for the phase space of retarded differential equations with infinite delay. Thus, $\mathfrak{B}$ will be a linear space of functions mapping $(-\infty,0]$ into $\mathbb{R}^N$ endowed with a norm $\|\cdot\|_{\mathfrak{B}}$. For more details about this axiomatization, one can see [1, 10, 9, 3].

The following Lemma is the key to prove our existence theorem, and its proof is due to the fact that the function h is defined on the entire real line, this result is stronger than axiom A1)-ii) from Hale and Kato axiomatic theory for the phase space presented in [1].

Lemma 2.5. *For all function $z \in \mathcal{PW}_{h\tau}$ the following estimate holds for all $s \in [0, \tau]$:*

$$\|z_s\|_{\mathfrak{B}} \leq \|z\|_{\mathcal{PW}_{h\tau}}.$$

Proof.

$$\begin{aligned}
\|z_s\|_{\mathfrak{B}} &= \sup_{\theta \in \mathbb{R}_-} \frac{\|z_s(\theta)\|}{h(\theta)} \\
&= \sup_{\theta \in \mathbb{R}_-} \frac{\|z(s+\theta)\|}{h(\theta)} \\
&= \sup_{\theta \in \mathbb{R}_-} \frac{\|z(s+\theta)\|}{h(s+\theta)} \cdot \frac{h(s+\theta)}{h(\theta)} \\
&\leq \sup_{\theta \in \mathbb{R}_-} \frac{\|z(s+\theta)\|}{h(s+\theta)} \\
&= \sup_{l \in (-\infty, s]} \frac{\|z(l)\|}{h(l)} \leq \sup_{l \in (-\infty, 0]} \frac{\|z(l)\|}{h(l)} + \sup_{l \in (0, s]} \|z(l)\| \\
&\leq \|z\Big|_{\mathfrak{B}}\| + \|z\Big|_{I}\|_\infty = \|z\|_{\mathcal{PW}_{h\tau}} = \|z\|.
\end{aligned}$$

□

3. Main Results

In this section we shall show that, by assuming some conditions on f, $\mathcal{J}_k$ and g, the non-autonomous differential equation (1.1) has a solution on $(-\infty, \tau]$, for $\tau > 0$.

In the following proposition, we present the characterization of the solutions of Problem (1.1). Its proof is base in the variation constant formula for non-homogeneous ordinary differential equations(See [11]).

Proposition 3.1. *Problem (1.1) permits a solution $z(\cdot)$ on $(-\infty, \tau]$ if, and only if, $z(\cdot)$ satisfies the following expression,*

$$\begin{aligned}
z(t) &= \mathcal{V}(t,0)[\phi(0) - g(z_{\tau_1} \ldots z_{\tau_q})(0)] + \int_0^t \mathcal{V}(t,s) f(s, z_s) ds \\
&+ \sum_{0<t_k<t} \mathcal{V}(t,t_k) \mathcal{J}_k(t_k, z(t_k)), \quad t \in (0, \tau],
\end{aligned}$$

$$z(t) = \phi(t) - g\left(z_{\tau_1}, z_{\tau_2}, z_{\tau_3}, \ldots, z_{\tau_q}\right)(t), \quad t \in (-\infty, 0].$$

Now, let us denote by

$$(\mathbb{R}^N)^q = \mathbb{R}^N \times \mathbb{R}^N \times ... \times \mathbb{R}^N = \prod_{i=1}^{q} \mathbb{R}^N,$$

endowed with the norm

$$\|y\|_q = \sum_{i=1}^{q} \|y_i\|_{\mathbb{R}^N}, \quad y = (y_1, ..., y_q)^T \in (\mathbb{R}^N)^q$$

and the norm in the space $(\mathfrak{B})^q$ is given by

$$\|y\|_{q\mathfrak{B}} = \max_{i=1,2,\dots,q} \{\|y_i\|_{\mathfrak{B}}\}$$

or

$$\|y\|_{q\mathfrak{B}} = \sum_{i=1}^{q} \|y_i\|_{\mathfrak{B}}$$

3.1. **Existence Theorems.** In this subsection we shall assume the hypotheses that will allow us to prove the first existence theorem. The constant M is defined in (2.3).

(H1) Let $\mathcal{K} : \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}_+$ and $\tilde{\psi} : \mathbb{R}_+ \to \mathbb{R}_+$ be both continuous and increasing functions. The following conditions are satified for $f : [0, \tau] \times \mathfrak{B} \to \mathbb{R}^N$:
   i) $\|f(t, \eta_1) - f(t, \eta_2)\|_{\mathbb{R}^N} \le \mathcal{K}(\|\eta_1\|_{\mathfrak{B}}, \|\eta_2\|_{\mathfrak{B}}) \|\eta_1 - \eta_2\|_{\mathfrak{B}}, \quad \forall \eta_1, \eta_2 \in \mathfrak{B},$ $\forall t \in I = (0, \tau]$,
   ii) $\|f(t, \eta)\|_{\mathbb{R}^N} \le \tilde{\psi}(\|\eta\|_{\mathfrak{B}}), \quad \forall \eta \in \mathfrak{B}.$

(H2) There exist positive constants $L_q, r_k, k = 1, 2, ..., p$ such that $\forall y, z \in \mathbb{R}^N, t \in I$:
   i) $ML_q q < M \sum_{k=1}^{p} r_k < \frac{1}{4}$, and $\|\mathcal{J}_k(t, y) - \mathcal{J}_k(t, z)\|_{\mathbb{R}^N} \le r_k \|y - z\|_{\mathbb{R}^N}$.

   ii) $g(0) = 0$ and

$$\|g(\widetilde{y}) - g(\widetilde{z})\|_{\mathfrak{B}} \le L_q \sum_{i=1}^{q} \|\widetilde{z}_i - \widetilde{y}_i\|_{\mathfrak{B}}, \quad \forall \widetilde{y}, \widetilde{z} \in (\mathfrak{B})^q.$$

(H3) For such τ there exist positive constant α such that

$$\left(ML_q q + M \sum_{k=1}^{p} r_k \right) (\|\tilde{\phi}\| + \alpha) + \tau M \tilde{\psi}(\|\tilde{\phi}\| + \alpha) \le \frac{\alpha}{2},$$

where $\tilde{\phi} \in \mathcal{PW}_{h\tau}$ is such that

$$\tilde{\phi} = \begin{cases} \mathcal{V}(t, 0)\phi(0), & t \in (0, \tau], \\ \phi(t), & t \in \mathbb{R}_-. \end{cases} \tag{3.7}$$

(H4) For α as in (H3) we have the following inequality

$$\tau M\mathcal{K}(\|\tilde{\phi}\|+\alpha,\|\tilde{\phi}\|+\alpha)+M\sum_{k=1}^{p} r_k<\frac{1}{2}.$$

We shall use the following well-known result to prove the existence of solutions for (1.1),

Theorem 3.2. *(See [5]) (G.L. Karakostas Fixed Point Theorem) Let Z and Y be Banach spaces and D be a closed convex subset of Z, and let $\mathcal{C}: D \to Y$ be a continuous operator such that $\mathcal{C}(D)$ is a relatively compact subset of Y, and*

$$\mathcal{T}: D\times\overline{\mathcal{C}(D)}\to D$$

is a continuous operator such that the family $\{\mathcal{T}(\cdot,y): y\in\overline{\mathcal{C}(D)}\}$ is equicontractive. Then, the operator equation

$$\mathcal{T}(z,\mathcal{C}(z))=z$$

admits a solution on D.

Theorem 3.3. *The system 1.1 has at least one solution on $(-\infty,\tau]$ under the hypothesis (H1)-(H3).*

Proof. Let's consider the following operators:

$$\begin{aligned}\mathcal{T}:&\mathcal{PW}_{h\tau}\times\mathcal{PW}_{h\tau}\longrightarrow\mathcal{PW}_{h\tau},\\ \mathcal{C}:&\mathcal{PW}_{h\tau}\longrightarrow\mathcal{PW}_{h\tau},\end{aligned}$$

where

$$\mathcal{T}(z,y)(t)=\begin{cases} y(t)+\displaystyle\sum_{0<t_k<t}\mathcal{V}(t,t_k)\mathcal{J}_k(t_k,z(t_k)), & t\in(0,\tau],\\ \phi(t)-g(z_{\tau_1},\dots,z_{\tau_q})(t), & t\in\mathbb{R}_-,\end{cases}$$

and

$$\mathcal{C}(z)(t)=\begin{cases}\mathcal{V}(t,0)[\phi(0)-g(z_{\tau_1},\dots,z_{\tau_q})(0)]+\displaystyle\int_0^t\mathcal{V}(t,s)f(s,z_s)ds, & t\in(0,\tau],\\ \phi(t), & t\in\mathbb{R}_-.\end{cases}$$

Moreover, let $\tilde{\phi}$ be the function defined in (3.7) and α given as in (H3). We define the the following closed and convex set

$$D=D(\alpha,\tau,\phi)=\left\{y\in\mathcal{PW}_{h\tau}:\|y-\tilde{\phi}\|\le\alpha\right\}. \tag{3.8}$$

Note that finding a solution of problem (1.1) is equivalent to find solutions of the next operator equation

$$\mathcal{T}(z,\mathcal{C}(z))=z.$$

With the goal to get solutions of such equation, we shall use Karakostas Fixed Point Theorem. In the following lines, we are going to verify that operators $\mathcal{C}$ and $\mathcal{T}$ satisfy the assumptions presented in Theorem 3.2. First, we shall verify that $\mathcal{C}$ is continuous and that $\mathcal{C}(D)$ is a relatively compact set. Next, we shall see that $\{\mathcal{T}(\cdot, y) : y \in \overline{\mathcal{C}(D)}\}$ is equicontractive and finally we will check that $\mathcal{T}(\cdot, \mathcal{C}(\cdot))(D) \subseteq D$. We divide the proof in the following Affirmations:

Affirmation 1: $\mathcal{C}$ is continuous.
In order to prove this Affirmation, we shall use the hypothesis (H1)-i),(H2)-ii) of (H2) and Lemma 2.5. We have the following estimate for $z, y \in \mathcal{PW}_{h\tau}$.

Consider $t \in (-\infty, 0]$. Then,

$$\|\mathcal{C}(z)(t) - \mathcal{C}(y)(t)\|_{\mathbb{R}^N} = \|\phi(t) - \phi(t)\|_{\mathbb{R}^N} = 0,$$

that is, $\left\| \left.(\mathcal{C}(z) - \mathcal{C}(y))\right|_{\mathbb{R}_-} \right\|_{\mathfrak{B}} = 0$. Now, if $t \in (0, \tau]$, we have that,

$$\begin{aligned}
(3.9)\qquad & \|\mathcal{C}(z)(t) - \mathcal{C}(y)(t)\|_{\mathbb{R}^N} \\
& \leq ML_q \sum_{i=1}^{q} \|y_{\tau_i}(0) - z_{\tau_i}(0)\|_{\mathbb{R}^N} + \int_0^t \|\mathcal{V}(t,s)\| \, \|f(s, z_s) - f(s, y_s)\|_{\mathbb{R}^N} \, ds \\
& \leq ML_q \sum_{i=1}^{q} \sup_{t \in (0,\tau]} \|z(t) - y(t)\|_{\mathbb{R}^N} + \int_0^t M \, \|f(s, z_s) - f(s, y_s)\|_{\mathbb{R}^N} \, ds \\
& \leq ML_q q \| \left.(z - y)\right|_I \|_\infty + M \int_0^t \mathcal{K}\left(\|z_s\|_{\mathfrak{B}}, \|y_s\|_{\mathfrak{B}}\right) \|z_s - y_s\|_{\mathfrak{B}} \, ds \\
& \leq ML_q q \| \left.(z - y)\right|_I \|_\infty + M \int_0^t \mathcal{K}\left(\|z_s\|, \|y_s\|\right) \|z_s - y_s\| \, ds \\
& \leq ML_q q \| \left.(z - y)\right|_I \|_\infty + M\tau\mathcal{K}(\|z\|, \|y\|)\|z - y\| \\
(3.10)\qquad & \leq ML_q q \|z - y\| + M\tau\mathcal{K}(\|z\|, \|y\|)\|z - y\|.
\end{aligned}$$

Hence, by taking the supremum on I, by (3.9), we get that

$$\begin{aligned}
\|\mathcal{C}(z) - \mathcal{C}(y)\| &= \| \left.(C(z) - C(y))\right|_{\mathbb{R}_-} \|_{\mathfrak{B}} + \| \left.(C(z) - C(y))\right|_I \|_\infty \\
&= \| \left.(\mathcal{C}(z) - \mathcal{C}(y))\right|_I \|_\infty \\
&\leq (ML_q + M\tau\mathcal{K}(\|z\|, \|y\|)) \|z - y\|.
\end{aligned}$$

Hence, we conclude that $\mathcal{C}$ is locally Lipschitz, which implies that it is continuous.

Affirmation 2: $\mathcal{C}$ maps bounded sets of $\mathcal{PW}_{h\tau}$ into bounded sets $\mathcal{PW}_{h\tau}$
It is enough to prove that for any $R > 0$, $\exists r > 0$ s.t. for each $y \in B_R = \{z \in \mathcal{PW}_{h\tau} : \|z\| \leq R\}$

we have that $\|\mathcal{C}(y)\| \leq r$. Indeed, taking an arbitrary $y \in B_R$ and baring in mind (H1)-ii), (H2)-ii), and Lemma 2.5, the following estimates hold

$$\|\mathcal{C}(y)(t)\|_{\mathbb{R}^N} = \|\phi(t)\|_{\mathbb{R}^N}, \quad \forall t \in (-\infty, 0],$$

from which follows that,

$$\left\| (C(y))|_{\mathbb{R}_-} \right\|_{\mathfrak{B}} = \sup_{t \leq 0} \frac{\|C(y)(t)\|_{\mathbb{R}^N}}{h(t)} = \sup_{t \leq 0} \frac{\|\phi(t)\|_{\mathbb{R}^N}}{h(t)} = \|\phi\|_{\mathfrak{B}}, \tag{3.11}$$

and for $t \in (0, \tau]$,

$$\begin{aligned} \|\mathcal{C}(y)(t)\|_{\mathbb{R}^N} &\leq \left\|\mathcal{V}(t,0)\left\{\phi(0) - g\left(y_{\tau_1}, y_{\tau_2}, \ldots, y_{\tau_q}\right)(0)\right\}\right\| \\ &+ \int_0^t \|\mathcal{V}(t,s) f(s, y_s)\|_{\mathbb{R}^N} ds \\ &\leq M\left\{\|\phi(0)\|_{\mathbb{R}^N} + L_q q \|y\|\right\} + \tau M \tilde{\psi}(\|y\|_{\mathfrak{B}}) \\ &\leq M\left\{\|\phi(0)\|_{\mathbb{R}^N} + L_q q R\right\} + \tau M \tilde{\psi}(R) = l. \end{aligned}$$

Taking supremum on $t \in [0, \tau]$ and putting $r = \|\phi\|_{\mathfrak{B}} + l$, we have that

$$\|\mathcal{C}(y)\| = \| \left(C(y)\right)|_{\mathbb{R}_-} \|_{\mathfrak{B}} + \| \left(C(y)\right)|_I \|_\infty \leq r.$$

Hence, Affirmation 2 holds.
Affirmation 3: $\mathcal{C}$ maps bounded sets of $\mathcal{PW}_{h\tau}$ into equicontinuous sets of $PW_{h\tau}$.
Let's consider B_R as in Affirmation 2. We shall prove that $\mathcal{C}(B_R)$, on the interval is $(-\infty, \tau]$ is equicontinuous. Clearly, it is enough to show this on $(0, \tau]$, by definition of $\mathcal{C}$.
Let's take $y \in B_R$. Baring in mind (H1)-ii), (H2)-ii) and Lemma 2.5, we have the following

estimates,

$$
\begin{aligned}
&\|\mathcal{C}(y)\,(t_2)-\mathcal{C}(y)\,(t_1)\|_{\mathbb{R}^N}\\
\leq &\Big\|\mathcal{V}\,(t_2,0)\left\{\phi(0)-g\left(y_{\tau_1},y_{\tau_2},\ldots,y_{\tau_q}\right)(0)\right\}+\int_0^{t_2}\mathcal{V}\,(t_2,s)\,f\,(s,y_s)\,ds\\
&-\mathcal{V}\,(t_1,0)\left\{\phi(0)-g\left(y_{\tau_1},y_{\tau_2},\ldots,y_{\tau_q}\right)(0)\right\}-\int_0^{t_1}\mathcal{V}\,(t_1,s)\,f\,(s,y_s)\,ds\Big\|_{\mathbb{R}^N}\\
\leq &\|[\mathcal{V}(t_2,0)-\mathcal{V}(t_1,0)]\left\{\phi(0)-g\left(y_{\tau_1},y_{\tau_2},\ldots,y_{\tau_q}\right)(0)\right\}\|_{\mathbb{R}^N}\\
&+\Big\|\int_0^{t_1}\mathcal{V}\,(t_2,s)\,f\,(s,y_s)\,ds+\int_{t_1}^{t_2}\mathcal{V}\,(t_2,s)\,f\,(s,y_s)\,ds\\
&-\int_0^{t_1}\mathcal{V}\,(t_1,s)\,f\,(s,y_s)\,ds\Big\|_{\mathbb{R}^N}\\
\leq &\|\mathcal{V}\,(t_2,0)-\mathcal{V}\,(t_1,0)\|\left(\left\|\phi(0)-g\left(y_{\tau_1},y_{\tau_2},\ldots,y_{\tau_q}\right)(0)\right\|_{\mathbb{R}^N}\right)\\
&+\int_0^{t_1}\|[\mathcal{V}\,(t_2,s)-\mathcal{V}\,(t_1,s)]\,f\,(s,y_s)\|_{\mathbb{R}^N}\,ds+\int_{t_1}^{t_2}\|\mathcal{V}\,(t_2,s)\,f\,(s,y_s)\|_{\mathbb{R}^N}\,ds\\
\leq &\|\mathcal{V}\,(t_2,0)-\mathcal{V}\,(t_1,0)\|\left(\|\phi(0)\|_{\mathbb{R}^N}+L_q\sum_{i=1}^{q}\|y_i(t)\|_{\mathbb{R}^N}\right)\\
&+\tilde{\psi}(\|y\|_{\mathfrak{B}})\int_0^{t_1}\|\mathcal{V}\,(t_2,s)-\mathcal{V}\,(t_1,s)\|\,ds+M\tilde{\psi}(\|y\|_{\mathfrak{B}})\int_{t_1}^{t_2}ds
\end{aligned}
$$

$$
\begin{aligned}
\leq &\|\mathcal{V}\,(t_2,0)-\mathcal{V}\,(t_1,0)\|\left(\|\phi(0)\|_{\mathbb{R}^N}+L_q q\|y\|_{\mathbb{R}^N}\right)\\
&+\tilde{\psi}(R)\int_0^{t_1}\|\mathcal{V}\,(t_2,s)-\mathcal{V}\,(t_1,s)\|\,ds+M\tilde{\psi}(R)\,(t_2-t_1)\\
\leq &\|\mathcal{V}\,(t_2,0)-\mathcal{V}\,(t_1,0)\|\left(\|\phi(0)\|_{\mathbb{R}^N}+L_q qR\right)\\
&+\tilde{\psi}(R)\int_0^{t_1}\|(\mathcal{V}\,(t_2,s)-\mathcal{V}\,(t_1,s))\|\,ds+M\tilde{\psi}(R)\,(t_2-t_1)\,.
\end{aligned}
$$

Because of continuity of $\mathcal{V}(t,s)$, we have that

$$\|\mathcal{C}(y)\,(t_2)-\mathcal{C}(y)\,(t_1)\|_{\mathbb{R}^N}\to 0\quad\text{ as }\quad t_2\to t_1,$$

independently on $y\in B_R$.

Affirmation 4: The subset $\mathcal{C}(D)$ is relatively compact in $\mathcal{PW}_{h\tau}$.

Let us prove Affirmation 4. Let D be the bounded subset of $\mathcal{PW}_{h\tau}$ defined in (3.8). By Affirmations 2 and 3, $\mathcal{C}(D)$ is bounded and equicontinuous in $\mathcal{PW}_{h\tau}$. Let $\{y_n\}_{n\in\mathbb{N}}\subseteq\mathcal{C}(D)$;

then

$$y_n\Big|_{\mathbb{R}_-} = \phi, \quad \forall n \in \mathbb{N}.$$

Hence, $y_n\Big|_{\mathbb{R}_-}$ converges uniformly on $\mathbb{R}_-$.

Now, putting $\varphi_n = y_n\Big|_{[0,\tau]}$, we get that $\{\varphi_n\}_{n\in\mathbb{N}} \subseteq \mathcal{PW}_{t_1..t_p}$, where

$$\begin{aligned}\mathcal{PW}_{t_1..t_p}([-0,\tau];\mathbb{R}^N) \quad = \quad & \{z:[-r,\tau]\to\mathbb{R}^N : z \quad \text{is continuous except in the points} \quad t_k,\\ & \text{and limits exist} \quad z(t_k^-), \quad z(t_k)=z(t_k^+)\},\end{aligned}$$

endowed with the sumpremum norm. Let us put $t_0 = 0$ and $t_{p+1} = \tau$. Then, applying Arzela-Ascoli Theorem, the sequence $\{\varphi_n\}_{n\in\mathbb{N}}$ contains a subsequence $\{\varphi_n^1\}_{n\in\mathbb{N}}$ that converges in the interval $[t_0, t_1]$. Now, applying again Arzela-Ascoli Theorem, the sequence $\{\varphi_n^1\}_{n\in\mathbb{N}}$ contains a subsequence $\{\varphi_n^2\}_{n\in\mathbb{N}}$ that converges in the interval $[t_1, t_2]$. Continuing with this process we find a subsequence $\{\varphi_n^{p+1}\}_{n\in\mathbb{N}}$ of $\{\varphi_n\}_{n\in\mathbb{N}}$ that converges in each interval $[t_k, t_{k+1}]$, with $k = 0, 1, 2, \ldots, p$. Therefore,

$$\varphi_n^{p+1} = y_n^{p+1}\Big|_{[0,\tau]}, \quad \text{converges on} \quad [0,\tau].$$

Consequently, $\{\varphi_n^{p+1}\}_{n\in\mathbb{N}} = \{y_n^{p+1}\}_{n\in\mathbb{N}}$ converges uniformly on $(-\infty, \tau]$. Thus, $\mathcal{C}(D)$ is relatively compact, and the proof of Affirmation 4 is completed.
Affirmation 5: The family $\{\mathcal{T}(\cdot, y) : y \in \overline{\mathcal{C}(D)}\}$ is equicontractive.
Let's take $z, x \in PW_{h\tau}$, $y \in \overline{\mathcal{C}(D)}$, and $t \in (-\infty, 0]$. By using ii) of (H_2), we obtain

$$\begin{aligned}\frac{1}{h(t)}\|\mathcal{T}(z,C(y))(t)-\mathcal{T}(x,C(y))(t)\|_{\mathbb{R}^N} &\leq \frac{1}{h(t)}\left\|g\left(z_{\tau_1},z_{\tau_2},\ldots,z_{\tau_q}\right)(t)-g\left(x_{\tau_1},x_{\tau_2},\ldots,x_{\tau_q}\right)(t)\right\|_{\mathbb{R}^N}\\ &\leq \left\|g\left(z_{\tau_1},z_{\tau_2},\ldots,z_{\tau_q}\right)-g\left(x_{\tau_1},x_{\tau_2},\ldots,x_{\tau_q}\right)\right\|_{\mathfrak{B}}\\ &\leq L_q q\left\|(z-x)|_{\mathbb{R}_-}\right\|_{\mathfrak{B}}\\ &\leq ML_q q\|z-x\|.\end{aligned}$$

By taking the supremum in t, we have that,

$$\left\|(\mathcal{T}(z,C(y))-\mathcal{T}(x,C(y)))|_{\mathbb{R}_-}\right\|_{\mathfrak{B}} \leq ML_q q\|z-x\|. \tag{3.12}$$

Let $t \in (0, \tau]$. From the hypothesis (H2)-i), we get:

$$\begin{aligned}
\|\mathcal{T}(z, C(y))(t) - \mathcal{T}(x, C(y))(t)\|_{\mathbb{R}^N} &\leq \left\| \sum_{0<t_k<t} U(t, t_k)\, \mathcal{J}_k(t_k, z(t_k)) - \sum_{0<t_k<t} U(t, t_k)\, \mathcal{J}_k(t_k, x(t_k)) \right\|_{\mathbb{R}^N} \\
&\leq \sum_{0<t_k<t} \|U(t, t_k)\, (\mathcal{J}_k(t_k, z(t_k)) - \mathcal{J}_k(t_k, x(t_k)))\|_{\mathbb{R}^N} \\
&\leq M \sum_{k=1}^{p} \|(\mathcal{J}_k(t_k, z(t_k)) - \mathcal{J}_k(t_k, x(t_k)))\|_{\mathbb{R}^N} \\
&\leq M \sum_{k=1}^{p} r_k \, \|z(t_k) - x(t_k)\|_{\mathbb{R}^N} \\
&\leq \left(M \sum_{k=1}^{p} r_k \right) \| \, (z - x)|_I \, \|_\infty \\
&\leq \left(M \sum_{k=1}^{p} r_k \right) \|z - x\|.
\end{aligned}$$

Hence,

$$\|\mathcal{T}(z, C(y)) - \mathcal{T}(x, C(y))\| \leq \left(M \sum_{k=1}^{p} r_k \right) \|z - x\|. \tag{3.13}$$

Therefore, from (3.12) and (3.13), we obtain that

$$\|\mathcal{T}(z, C(y)) - \mathcal{T}(x, C(y))\|, \leq \frac{1}{2}\|z - x\|.$$

which is a contraction independently of $y \in \overline{C(D)}$. So , the family $\{\mathcal{T}(\cdot, y) : y \in \overline{C(D)}\}$ is equicontractive.

Affirmation 6: Finally, we shall prove that

$$\mathcal{T}(\cdot, C(\cdot))(D(\alpha, \tau, \phi)) \subseteq D(\alpha, \tau, \phi)$$

Let us take $z \in D(\alpha, \tau, \phi)$ and consider $t \in (-\infty, 0]$. Then, for $t \in (-\infty, 0]$, from (H2)-ii) and (H3)-ii), we obtain

$$\begin{aligned}
\frac{1}{h(t)}\|\mathcal{T}(z, C(z))(t) - \tilde{\phi}(t)\|_{\mathbb{R}^N} &\leq \|g\left(z_{\tau_1}, z_{\tau_2}, \ldots, z_{\tau_q}\right)(t)\|_{\mathbb{R}^N} \\
&\leq L_q q\|z\| \leq M L_q q\|z\| \leq M L_q q(\|\tilde{\phi}\| + \alpha) \leq \alpha/2.
\end{aligned}$$

Moreover, for $t \in (0,\tau]$, and baring in mind (H1)-ii),(H2)-ii), (H3) and Lemma 2.5, we have that

$$\begin{aligned}
\|\mathcal{T}(z,C(z))(t)-\tilde{\phi}(t)\|_{\mathbb{R}^N} \leq & M\left\|g\left(z_{\tau_1},z_{\tau_2},\dots,z_{\tau_q}\right)(0)\right\|_{\mathbb{R}^N}+\int_0^t\|\mathcal{V}(t,s)f\left(s,z_s\right)\|_{\mathbb{R}^N}\,ds \\
& +\sum_{0<t_k<t}\|\mathcal{V}\left(t,t_k\right)\mathcal{J}_k(t_k,z(t_k))\|_{\mathbb{R}^N} \\
& \leq ML_q q\|z\|+M\tau\tilde{\psi}(\|z\|+M\sum_{k=1}^{p}r_k\|z\| \\
& \leq ML_q q(\|\tilde{\phi}\|+\alpha)+M\tau\tilde{\psi}(\|\tilde{\phi}\|+\alpha)+\left(M\sum_{k=1}^{p}r_k\right)(\|\tilde{\phi}\|+\alpha) \\
& =\left(ML_q q+M\sum_{k=1}^{p}r_k\right)(\|\tilde{\phi}\|+\alpha)+M\tau\tilde{\psi}(\|\tilde{\phi}\|+\alpha) \\
& \leq \alpha/2.
\end{aligned}$$

Hence, $\mathcal{T}(\cdot,B(\cdot))D(\alpha,\tau,\phi)\subseteq D(\alpha,\tau,\phi)$. Since Affirmation 1, Affirmation 4 and Affirmation 5 hold, the conditions of Karakostas Fixed Point Theorem are satisfied for the closed and convex set given in (3.8), and the proof of Theorem 3.3 immediately follows by applying Theorem 3.2. □

Theorem 3.4. *With the conditions of Theorem 3.3and now we suppose that (H4) holds. Then the problem (1.1) has only one solution on the interval* $(-\infty,\tau]$.

Proof. . Let z^1 and z^2 are two solutions for problem (1.1). Then, we consider the following estimate for $t\in[0,\tau]$:

$$\begin{aligned}
\|z_1(t)-z_2(t)\|_{\mathbb{R}^N} \leq & \|(g(z^1_{\tau_1},z^1_{\tau_2},\dots,z^1_{\tau_q}))(0)-(g(z^2_{\tau_1},z^2_{\tau_2},\dots,z^2_{\tau_q}))(0)\|_{\mathbb{R}^N}+ \\
& \int_0^t\|\mathcal{V}(t,s)(f(s,z^1_s)-f(s,z^2_s))\|ds \\
& +\sum_{0<t_k<t}\|\mathcal{V}(t,t_k)(\mathcal{J}_k(t_k,z^1(t_k))-\mathcal{J}_k(t_k,z^2(t_k)))\| \\
& \leq \left(L_q q+M\tau\mathcal{K}(\|\tilde{\phi}\|+\alpha),\|\tilde{\phi}\|+\alpha)+M\sum_{k=1}^{p}r_k\right)\|z^1-z^2\|.
\end{aligned}$$

From the hypotheses (H1)-(H4), we know that:

$$L_q q + \tau M \mathcal{K}(\|\phi\| + \alpha), \|\phi\| + \alpha) + M \sum_{k=1}^{p} r_k < 1,$$

which implies that $z^1\Big|_I = z^2\Big|_I$.
In the same way, we can prove that

$$\|(z^1 - z^2)|_{\mathbb{R}_-}\|_{\mathfrak{B}} \le L_q q \|(z^1 - z^2)|_{\mathbb{R}_-}\|_{\mathfrak{B}},$$

Thus,

$$z^1\Big|_{\mathbb{R}_-} = z^2\Big|_{\mathbb{R}_-}.$$

Hence, $z^1 = z^2$.

$\square$

Now, we have to consider the following subset $\tilde{D}$ of $\mathbb{R}^N$:

$$\tilde{D} = \{y \in \mathbb{R}^N : \|y\|_{\mathbb{R}^N} \le R\}, \quad \text{with} \quad R = \|\tilde{\phi}\| + \alpha. \tag{3.14}$$

Therefore, for all $z \in D$ we have $z(t) \in \tilde{D}$ for t $(-\infty, \tau]$.

Definition 3.5. We shall say that $(-\infty, s_1)$ is a maximal interval of existence of the solution $z(\cdot)$ of problem (1.1) if there is not solution of the (1.1) on $(-\infty, s_2)$ with $s_2 > s_1$.

THEOREM 3.1. Suppose that the conditions of Theorem 3.4 hold. If z is a solution of problem (1.1) on $(-\infty, s_1)$ and s_1 is maximal, then either $s_1 = +\infty$ or there exists a sequence $\tau_n \to s_1$ as $n \to \infty$ s.t. $z(\tau_n) \to \partial\tilde{D}$.

Proof. Suppose, for the purpose of contradiction, that there exist a neighborhood N of $\partial\tilde{D}$ such that $z(t)$ does not enter in it, for $0 < s_2 \le t < s_1$. We can take $N = \tilde{D}\backslash B$, where B is a closed subset of $\tilde{D}$, then $z(t) \in B$ for $0 < t_p < s_2 \le t < s_1$. We need to prove that

$\lim_{t\to z_1^+} z(t) = z_1 \in B$. Indeed, if we consider $0 < t_p < s_2 \leq \ell < t < s_1$, then:

$$\begin{aligned}
\|z(t)-z(\ell)\|_{\mathbb{R}^N} &\leq \|\mathcal{V}(t,0)-\mathcal{V}(\ell,0)\|\|\phi(0)\|_{\mathbb{R}^N} + \|\mathcal{V}(t,0)-\mathcal{V}(\ell,0)\|\|g(z_{\tau_1}, z_{\tau_2}, \dots, z_{\tau_q})(0)\|_{\mathbb{R}^N} \\
&\quad + \int_0^\ell \|\mathcal{V}(t,s)-\mathcal{V}(\ell,s)\|\|f(s,z_s)\|ds + \int_\ell^t \|(\mathcal{V}(t,s)\|\|f(s,z_s)\|ds \\
&\quad + \|\sum_{0<t_k<t} \mathcal{V}(t,t_k)\mathcal{J}_k(t_k, z(t_k)) - \sum_{0<t_k<t} \mathcal{V}(\ell,t_k)\mathcal{J}_k(t_k, z(t_k))\| \\
&\leq (\|\mathcal{V}(t,0)-\mathcal{V}(\ell,0)\|)\left(\|\phi(0)\|_{\mathbb{R}^N} + L_q q\right) \\
&\quad + \left(\int_0^\ell \|\mathcal{V}(t,s)-\mathcal{V}(\ell,s)\|ds + \int_\ell^t \|(\mathcal{V}(t,s)\|ds\right)\Psi(R) \\
&\quad + \|\mathcal{V}(t,\ell) - I_{\mathbb{R}^N}\| \sum_{k=1}^q \|\mathcal{V}(\ell,t_k)\|\|\mathcal{J}_k(t_k, z(t_k))\| \\
&\leq (\|\mathcal{V}(t,0)-\mathcal{V}(\ell,0)\|)\left(\|\phi(0)\|_{\mathbb{R}^N} + L_q q\right) \\
&\quad + \left(\int_0^\ell \|\mathcal{V}(t,s)-\mathcal{V}(\ell,s)\|ds + \int_\ell^t \|(\mathcal{V}(t,s)\|ds\right)\Psi(R) \\
&\quad + \|\mathcal{V}(t,\ell) - I_{\mathbb{R}^N}\| MR \sum_{k=1}^q r_k
\end{aligned}$$

Since $\mathcal{V}(t,s)$ is uniformly continuous for $t \geq 0$, then $\|z(t)-z(l)\|_{\mathbb{R}^N}$ goes to zero as $l < \ell \to s_1$. Therefore, $\lim_{t\to s_1} z(t) = z_1$ exists in $\mathbb{R}^N$, and since B is closed, z_1 belongs to B. This completes the proof.

$\square$

COROLLARY 3.1. In the conditions of Theorem 3.4, if the second part of hypothesis (H1) has changed to

$$\|f(t,\phi)\| \leq \mu(t)(1 + \|\phi(0)\|_{\mathbb{R}^N}), \quad \phi \in \mathfrak{B}, \quad t \in \mathbb{R},$$

where $\mu(\cdot)$ is a continuous function on $(-\infty,\infty)$, then the problem (1.1) have a unique solution on $(-\infty,\infty)$.

Proof.

$$\begin{aligned}
\|z(t)\|_{\mathbb{R}^N} &\leq M\left(\|z(0)\|_{\mathbb{R}^N}\right) + M\int_0^t \|f(s, z_s)\|ds \\
&\quad + M \sum_{0<t_k<t} \|\mathcal{J}_k(t_k, z(t_k))\| \\
&\leq M\|z(0)\|_{\mathbb{R}^N} + \int_0^\tau M\mu(s)(1 + \|z(s)\|_{\mathbb{R}^N})ds \\
&\quad + M\sum_{k=1}^p r_k \|z(t_k))\|_{\mathbb{R}} \\
&\leq M\left(\|z(0)\|_{\mathbb{R}^N} + \int_0^\tau \mu(s)ds\right) + \int_0^t M\mu(s)\|z(s)\|_{\mathbb{R}^N}ds \\
&\quad + \sum_{k=1}^p Mr_k\|z(t_k))\|_{\mathbb{R}^N}.
\end{aligned}$$

Then, applying Gronwall Inequality for impulsive differential equations (see [6, 12, 13, 14]), we obtain that

$$\|z(t)\|_{\mathbb{R}^N} \leq M\left(\|z(0)\|_{\mathbb{R}^N} + \int_0^\tau \mu(s)ds\right) \prod_{t_0<t_k<t} (1 + Mr_k)e^{\int_0^\tau M\mu(s)ds},$$

This implies that $\|z(t)\|_{\mathbb{R}^N}$ stays bounded as $t \to s_1$ and we apply the Theorem 3.1 we get the result.

□

4. Conclusion and Final Remark

In this work we prove the existence and uniqueness of solutions of retarded equations with infinite delay, impulses, and nonlocal conditions; showing that the phase space that we choose satisfies the axioms proposed by Hale and Kato to study retarded equations with unbounded delay, but in this case, our phase space is a subspace of the piecewise continuous functions due to impulses and non-local conditions. Once we have proven the existence of solutions for this type of equations, we shall study the existence of bounded solutions, working in the phase space of bounded and continuous functions except for a fixed number of points p, for which the lateral limits exist and the functions of this space are continuous on the right of these points; this space is continuously embedded in our phase space $\mathfrak{B}$. Then, we shall study the controllability of control systems governed by such equations, proving approximate controllability on the one hand, and exact controllability on the other, depending on the conditions imposed on the non-linear terms and assuming that the associated linear system is controllable. Our future research will focus on studying the same results with noninstantaneous impulses, infinite dimensional case, and the stability of such equations, as well as other aspects of dynamical systems.

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[1]Yachay Tech, School of Mathematical and Computational Sciences, Department of Mathematics, San Miguel de Urcuqui-100119., Imbabura - Ecuador e-mail: hleiva@yachaytech.edu.ec