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SOME RESULTS ON q-ANALOGUE TYPE OF FUBINI NUMBERS AND POLYNOMIALS

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ABSTRACT. In the present article, we define a new class of q-analogue type of Fubini numbers and polynomials and investigate some properties of these polynomials. We derive recurrence relation, derivative properties, integral representation, and summation formulas of these polynomials by summation techniques series. Furthermore, we consider some relationships for q-Fubini polynomials associated with q-Bernoulli polynomials, q-Euler polynomials, and q-Genocchi polynomials and q-Stirling numbers of the second kind.

1. Introduction

Throughout this presentation, we use the following standard notions $\mathbb{N} = \{1, 2, \cdots\}$, $\mathbb{N}_0 = \{0, 1, 2, \cdots\} = \mathrm{N} \cup \{0\}$, $\mathbb{Z}^- = \{-1, -2, \cdots\}$. Also as usual $\mathbb{Z}$ denotes the set of integers, $\mathbb{R}$ denotes the set of real numbers and $\mathbb{C}$ denotes the set of complex numbers.

Recently, many mathematicians like as (see [1, 8, 9, 10, 11, 12, 13, 14]) have been introduced to the subject of q-calculus. The applications of q-calculus in various fields of mathematics, physics, and engineering. The definitions and notations of q-calculus reviewed here are taken from (see [1]):

The q-analogue of the shifted factorial $(a)_n$ is given by

$$(a; q)_0 = 1, (a; q)_n = \prod_{m=0}^{n-1} (1 - q^m a), n \in \mathbb{N}.$$

The q-analogue of a complex number a and of the factorial function is given by

$$[a]_q = \frac{1 - q^a}{1 - q}, q \in \mathbb{C} - \{1\}; a \in \mathbb{C},$$

$$[n]_q! = \prod_{m=1}^{n} [m]_q = [1]_q [2]_q \cdots [n]_q = \frac{(q; q)_n}{(1 - q)^n}, q \neq 1; n \in \mathbb{N},$$

$$[0]_q! = 1, q \in \mathbb{C}; 0 <| q |< 1.$$

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The Gauss q-binomial coefficient $\left(\begin{array}{c} n \\ k \end{array}\right)_q$ is given by

$$\left(\begin{array}{c} n \\ k \end{array}\right)_q = \frac{[n]_q!}{[k]_q![n-k]_q!} = \frac{(q;q)_n}{(q;q)_k(q;q)_{n-k}}, k=0,1,\cdots,n.$$

The q-analogue of the function $(x+y)_q^n$ is given by

$$(x+y)_q^n = \sum_{k=0}^{n} \left(\begin{array}{c} n \\ k \end{array}\right)_q q^{k(k-1)/2} x^{n-k} y^k, n \in \mathbb{N}_0. \tag{1.1}$$

The q-analogue of exponential functions are given by

$$e_q(x) = \sum_{n=0}^{\infty} \frac{x^n}{[n]_q!} = \frac{1}{((1-q)x;q)_\infty}, 0 <| q |< 1; | x |<| 1-q |^{-1}, \tag{1.2}$$

and

$$E_q(x) = \sum_{n=0}^{\infty} q^{n(n-1)/2} \frac{x^n}{[n]_q!} = (-(1-q)x;q)_\infty, 0 <| q |< 1; x \in \mathbb{C}. \tag{1.3}$$

Moreover, the functions $e_q(x)$ and $E_q(x)$ satisfy the following properties:

$$D_q e_q(x) = e_q(x), D_q E_q(x) = E_q(qx), \tag{1.4}$$

where the q-derivative $D_q f$ of a function f at a point $0 \neq z \in \mathbb{C}$ is defined as follows:

$$D_q f(z) = \frac{f(qz) - f(z)}{qz - z}, 0 <| q |< 1.$$

For any two arbitrary functions $f(z)$ and $g(z)$, the q-derivative operator D_q satisfies the following product and quotient relations:

$$D_{q,z}(f(z)g(z)) = f(z)D_{q,z}g(z) + g(qz)D_{q,z}f(z), \tag{1.5}$$

$$D_{q,z}\left(\frac{f(z)}{g(z)}\right) = \frac{g(qz)D_{q,z}f(z) - f(qz)D_{q,z}g(z)}{g(z)g(qz)}.$$

The q-Bernoulli polynomials $B_{n,q}^{(\alpha)}(x,y)$ of order α, the q-Euler polynomials $E_{n,q}^{(\alpha)}(x,y)$ of order α and the q-Genocchi polynomials $G_{n,q}^{(\alpha)}(x,y)$ of order α are defined by means of the following generating function (see [1, 11, 12, 13]):

$$\left(\frac{t}{e_q(t)-1}\right)^\alpha e_q(xt)E_q(yt) = \sum_{n=0}^{\infty} B_{n,q}^{(\alpha)}(x,y)\frac{t^n}{[n]_q!}, | t |< 2\pi, 1^\alpha = 1, \tag{1.6}$$

$$\left(\frac{2}{e_q(t)+1}\right)^\alpha e_q(xt)E_q(yt) = \sum_{n=0}^{\infty} E_{n,q}^{(\alpha)}(x,y;\lambda)\frac{t^n}{[n]_q!}, | t |< \pi, 1^\alpha = 1, \tag{1.7}$$

$$\left(\frac{2t}{e_q(t)+1}\right)^\alpha e_q(xt)E_q(yt) = \sum_{n=0}^{\infty} G_{n,q}^{(\alpha)}(x,y)\frac{t^n}{[n]_q!}, | t |< \pi, 1^\alpha = 1. \tag{1.8}$$

Clearly, we have

$$B_{n,q}^{(\alpha)} = B_{n,q}^{(\alpha)}(0,0), E_{n,q}^{(\alpha)} = E_{n,q}^{(\alpha)}, G_{n,q}^{(\alpha)} = G_{n,q}^{(\alpha)}.$$

Geometric polynomials (also known as Fubini polynomials) are defined as follows (see [2]):

$$F_n(x) = \sum_{k=0}^{n} \left\{ \begin{array}{c} n \\ k \end{array} \right\} k! x^k, \tag{1.9}$$

where $\left\{ \begin{array}{c} n \\ k \end{array} \right\}$ is the Stirling number of the second kind (see [5]).
For $x = 1$ in (1.9), we get n^{th} Fubini number (ordered Bell number or geometric number) F_n [2, 3, 4, 5, 6, 7, 16] is defined by

$$F_n(1) = F_n = \sum_{k=0}^{n} \left\{ \begin{array}{c} n \\ k \end{array} \right\} k!. \tag{1.10}$$

The exponential generating functions of geometric polynomials is given by (see [2]):

$$\frac{1}{1 - x(e^t - 1)} = \sum_{n=0}^{\infty} F_n(x) \frac{t^n}{n!}, \tag{1.11}$$

and related to the geometric series (see [3]):

$$\left(x \frac{d}{dx} \right)^m \frac{1}{1-x} = \sum_{k=0}^{\infty} k^m x^k = \frac{1}{1-x} F_m\left(\frac{x}{1-x}\right), \mid x \mid < 1.$$

Let us give a short list of these polynomials and numbers as follows:

$F_0(x) = 1, F_1(x) = x, F_2(x) = x+2x^2, F_3(x) = x+6x^2+6x^3, F_4(x) = x+14x^2+36x^3+24x^4,$

and

$$F_0 = 1, F_1 = 1, F_2 = 3, F_3 = 13, F_4 = 75.$$

Geometric and exponential polynomials are connected by the relation (see [2]):

$$F_n(x) = \int_0^{\infty} \phi_n(x) e^{-\lambda} d\lambda. \tag{1.12}$$

The goal of this paper is as follows. In section 2, we define generating functions for q-type Fubini numbers and polynomials and give some properties of these numbers and polynomials. In section 3, we derive summation formulas of q-type-Fubini numbers and polynomials and some relationships between q-Bernoulli polynomials, q-Euler polynomials, and q-Genocchi polynomials and Stirling numbers of the second kind.

2. q-analogue type of Fubini numbers and polynomials

In this section, we introduce q-type Fubini polynomials $F_{n,q}(x; y)$ and investigate some basic properties of these polynomials. We begin the following definition as follows.

Definition 2.1. Let $q \in \mathbb{C}$ with $0 <\mid q \mid< 1$, the q-type Fubini polynomials $F_{n,q}(x; y)$ of two variables are defined by means of the following generating function:

$$\frac{1}{1-y(e_q(t)-1)}e_q(xt)=\sum_{n=0}^{\infty}F_{n,q}(x;y)\frac{t^n}{[n]_q!}. \tag{2.1}$$

From definition (2.1), we have

$$F_{n,q}(x;1)=F_{n,q}(x), F_{n,q}(0;1)=F_{n,q},$$

where $F_{n,q}$ are called the q-type Fubini numbers.

Remark 2.1. On setting $q\longrightarrow 1^-$ in (2.1), the result reduces to the known result of Kargin [6] as follows:

$$\frac{1}{1-y(e^t-1)}e^{xt}=\sum_{n=0}^{\infty}F_n(x;y)\frac{t^n}{n!}. \tag{2.2}$$

Theorem 2.1. The following series representation for the q-type Fubini polynomials $F_{n,q}(x;y)$ holds true:

$$F_{n,q}(x;y)=\sum_{m=0}^{n}\left(\begin{array}{c}n\\k\end{array}\right)_q F_{k,q}(y)x^{n-k}. \tag{2.3}$$

Proof. Using equation (1.2) and (2.1), we have

$$\sum_{n=0}^{\infty}F_{n,q}(x;y)\frac{t^n}{[n]_q!}=\frac{1}{1-y(e_q(t)-1)}e_q(xt)$$

$$=\left(\sum_{k=0}^{\infty}F_{k,q}(y)\frac{t^k}{[k]_q!}\right)\left(\sum_{n=0}^{\infty}x^n\frac{t^n}{[n]_q!}\right).$$

Applying the Cauchy product rule and equating the coefficients of same powers of t in both sides of resultant equation, we get representation (2.3). □

Theorem 2.2. For $n\geq 0$, the following formula for q-type Fubini polynomials holds true:

$$x^n=F_{n,q}(x;y)-yF_n(x+1;y)+yF_{n,q}(x;y). \tag{2.4}$$

Proof. We begin with the definition (2.1) and write

$$e_q(xt)=\frac{1-y(e_q(t)-1)}{1-y(e_q(t)-1)}e_q(xt)$$

$$=\frac{e_q(xt)}{1-y(e_q(t)-1)}-\frac{y(e_q(t)-1)}{1-y(e_q(t)-1)}e_q(xt).$$

From (1.4) and (2.1), we have

$$\sum_{n=0}^{\infty}x^n\frac{t^n}{[n]_q!}=\sum_{n=0}^{\infty}\left[F_{n,q}(x;y)-yF_{n,q}(x+1;y)+yF_{n,q}(x;y)\right]\frac{t^n}{[n]_q!}.$$

Finally, comparing the coefficients of $\frac{t^n}{n!}$, we get (2.4). □

Theorem 2.3. The following formula for q-type Fubini polynomials holds true:

$$yF_{n,q}(x+1;y) = (1+y)F_{n,q}(x;y) - x^n. \tag{2.5}$$

Proof. From (2.1), we have

$$\begin{aligned}\sum_{n=0}^{\infty}\left[F_{n,q}(x+1;y) - F_{n,q}(x;y)\right]\frac{t^n}{[n]_q!} &= \frac{e_p(xt)}{1-y(e_q(t)-1)}(e_q(t)-1)\\ &= \frac{1}{y}\left[\frac{e_q(xt)}{1-y(e_q(t)-1)} - e_q(xt)\right]\\ &= \frac{1}{y}\sum_{n=0}^{\infty}\left[F_{n,q}(x;y) - x^n\right]\frac{t^n}{[n]_q!}.\end{aligned}$$

Comparing the coefficients of $\frac{t^n}{[n]_q!}$ on both sides, we obtain (2.5). ☐

Theorem 2.4. The following recursive formula for the q-type Fubini polynomials $F_{n,q}(x;y)$ holds true:

$$D_{p,q;x}F_{n,q}(x;y) = [n]_q F_{n-1,q}(x;y). \tag{2.6}$$

Proof. Differentiating generating function (2.1) with respect to x and y with the help of equation (1.5), we have

$$\begin{aligned}\sum_{n=0}^{\infty} D_{q;x}F_{n,q}(x;y)\frac{t^n}{[n]_q!} &= D_{q;x}\sum_{n=0}^{\infty}F_{n,q}(x;y)\frac{t^n}{[n]_q!}\\ &= \frac{1}{1-y(e_q(t)-1)}D_{q;x}e_q(xt)\\ &= \frac{t}{1-y(e_q(t)-1)}e_q(xt)\\ &= \sum_{n=0}^{\infty}F_{n,q}(x;y)\frac{t^{n+1}}{[n]_q!},\end{aligned}$$

and then simplifying with the help of the Cauchy product rule formulas (2.6) are obtained. ☐

Theorem 2.5. The following definite q-integral is valid

$$\int_a^b F_{n,q}(x;y)d_qx = \frac{F_{n+1}(x,\frac{b}{q};y) - F_{n+1,q}(x,\frac{a}{q};y)}{[n+1]_q}. \tag{2.7}$$

Proof. Since

$$\int_a^b \frac{\delta}{\delta_q x}F_{n,q}(x;y)d_qx = f(b) - f(a), (\text{see [7]}),$$

in terms of equation (2.7) and equations (1.4) and (1.5), we arrive at the asserted result

$$\begin{aligned}\int_a^b \frac{\delta}{\delta_q x}F_{n,q}(x;y)d_qx &= \frac{1}{[n+1]_{p,q}}\int_a^b D_qF_{n+1,q}(x;y)\,d_qx\\ &= \frac{F_{n+1}(x,\frac{b}{q};y) - F_{n+1,q}(x,\frac{a}{q};y)}{[n+1]_q}.\end{aligned}$$

The other can be shown using similar method. Therefore, the complete the proof of this theorem. □

Theorem 2.6. The following relationship holds true:

$$F_{n+1,q}(x;y) = xF_{n,q}\left(\frac{x}{q};y\right)q^n + y\sum_{k=0}^{n}\left(\begin{array}{c} n \\ k \end{array}\right)_q F_{n-k,q}(x;y)F_{k,q}(q^{-1};y)q^k. \quad (2.8)$$

Proof. By (1.4), (1.5) and (2.1), we get

$$\sum_{n=0}^{\infty} F_{n+1,q}(x;y)\frac{t^n}{[n]_q!} = D_{q;t}\sum_{n=0}^{\infty} F_{n,q}(x;y)\frac{t^n}{[n]_q!}$$

$$= D_{q;t}\left(\frac{1}{1-y(e_q(t)-1)}e_q(xt)\right)$$

$$= \frac{xe_q(xt)}{1-y(e_q(qt)-1)} + \frac{ye_q(xt)e_q(t)}{(1-y(e_q(t)-1))(1-y(e_q(qt)-1))}$$

$$= x\sum_{n=0}^{\infty} F_{n,q}\left(\frac{x}{q};y\right)q^n\frac{t^n}{[n]_q!} + y\left(\sum_{n=0}^{\infty} F_{n,q}(x;y)\frac{t^n}{[n]_q!}\right)\left(\sum_{k=0}^{\infty} F_{k,q}(q^{-1};y)q^k\frac{t^k}{[k]_q!}\right).$$

Using the Cauchy product and comparing the coefficients of $\frac{t^n}{[n]_q!}$ in both sides, which yields to the desired result. □

Theorem 2.7. The following relation for the q-type Fubini polynomials $F_{n,q}(x;y)$ holds true:

$$(1+y)F_{n,q}(x;y) = y\sum_{k=0}^{n}\left(\begin{array}{c} n \\ k \end{array}\right)_q F_{n-k,q}(x;y) + x^n. \quad (2.9)$$

Proof. Consider the following identity

$$\frac{1+y}{(1-y(e_q(t)-1))ye_q(t)} = \frac{1}{1-y(e_q(t)-1)} + \frac{1}{ye_q(t)}.$$

Evaluating the following fraction using above identity, we find

$$\frac{(1+y)e_q(xt)}{(1-y(e_q(t)-1))ye_q(t)} = \frac{e_q(xt)}{1-y(e_q(t)-1)} + \frac{e_q(xt)}{ye_q(t)}$$

$$(1+y)\sum_{n=0}^{\infty} F_{n,q}(x;y)\frac{t^n}{[n]_q!}$$

$$= y\sum_{n=0}^{\infty} F_{n,q}(x;y)\frac{t^n}{[n]_q!}\sum_{k=0}^{\infty}\frac{t^k}{[k]_q!} + \sum_{n=0}^{\infty} x^n\frac{t^n}{[n]_q!}.$$

Applying the Cauchy product rule in the above equation and then equating the coefficients of like powers of t in both sides of the resultant equation, assertion (2.9) follows. □

3. Main results

First, we prove the following result involving the q-type Fubini polynomials $F_{n,q}(x;y)$ by using series rearrangement techniques and considered its special case:

Theorem 3.1. The following summation formula for q-type Fubini polynomials $F_{n,q}(x;y)$ holds true:

$$F_{k+l,q}(w;y) = \sum_{n,s=0}^{k,l} \binom{k}{n}_q \binom{l}{s}_q (w-x)^{n+s} F_{k+l-n-s,q}(x;y). \tag{3.1}$$

Proof. Replacing t by $t + u$ in (2.1) and then using the formula [15]:

$$\sum_{N=0}^{\infty} f(N)\frac{(x+y)^N}{N!} = \sum_{n,m=0}^{\infty} f(n+m)\frac{x^n}{n!}\frac{y^m}{m!}, \tag{3.2}$$

in the resultant equation, we find the following generating function for the q-type Fubini polynomials $F_{n,q}(x;y)$:

$$\frac{1}{1-y(e_q(t+u)-1)} = e_q(-x(t+u))\sum_{k,l=0}^{\infty} F_{k+l,q}(x;y)\frac{t^k}{[k]_q!}\frac{u^l}{[l]_q!}, \text{(see [11])}. \tag{3.3}$$

Replacing x by w in the above equation and equating the resultant equation to the above equation, we find

$$e_q((w-x)(t+u))\sum_{k,l=0}^{\infty} F_{k+l,q}(x;y)\frac{t^k}{[k]_q!}\frac{u^l}{[l]_q!} = \sum_{k,l=0}^{\infty} F_{k+l,q}(w;y)\frac{t^k}{[k]_q!}\frac{u^l}{[l]_q!}. \tag{3.4}$$

On expanding exponential function (3.4) gives

$$\sum_{N=0}^{\infty} \frac{[(w-x)(t+u)]^N}{[N]_q!}\sum_{k,l=0}^{\infty} F_{k+l,q}(x;y)\frac{t^k}{[k]_q!}\frac{u^l}{[l]_q!} = \sum_{k,l=0}^{\infty} F_{k+l,q}(w;y)\frac{t^k}{[k]_q!}\frac{u^l}{[l]_q!}, \tag{3.5}$$

which on using formula (3.2) in the first summation on the left hand side becomes

$$\sum_{n,s=0}^{\infty} \frac{(w-x)^{n+s}t^n u^s}{[n]_q![s]_q!}\sum_{k,l=0}^{\infty} F_{k+l,q}(x;y)\frac{t^k}{[k]_q!}\frac{u^l}{[l]_q!} = \sum_{k,l=0}^{\infty} F_{k+l,q}(w;y)\frac{t^k}{[k]_q!}\frac{u^l}{[l]_q!}. \tag{3.6}$$

Now replacing k by $k-n$, l by $l-s$ and using the lemma (see [13]):

$$\sum_{k=0}^{\infty}\sum_{n=0}^{\infty}A(n,k)=\sum_{k=0}^{\infty}\sum_{n=0}^{k}A(n,k-n), \tag{3.7}$$

in the l.h.s. of (3.6), we find

$$\sum_{k,l=0}^{\infty}\sum_{n,s=0}^{k,l}\frac{(w-x)^{n+s}}{[n]_q![s]_q!}F_{k+l-n-s,q}(x;y)\frac{t^k}{(k-n)_q!}\frac{u^l}{(l-s)_q!}$$
$$=\sum_{k,l=0}^{\infty}F_{k+l,q}(w;y)\frac{t^k}{[k]_q!}\frac{u^l}{[l]_q!}. \tag{3.8}$$

Finally, on equating the coefficients of the like powers of t and u in the above equation, we get the assertion (3.1) of Theorem 3.1. □

Remark 3.1. Taking $l=0$ in assertion (3.1) of Theorem 3.1, we deduce the following consequence of Theorem 3.1.

Corollary 3.1. The following summation formula for q-type Fubini polynomials $F_{n,q}(x;y)$ holds true:

$$F_{k,q}(w;y) = \sum_{n=0}^{k}\left(\begin{array}{c}k\\ n\end{array}\right)_q (w-x)^n F_{k-n,q}(x;y). \tag{3.9}$$

Remark 3.2. Replacing w by $w+x$ in (3.9), we obtain

$$F_{k,q}(w+x;y) = \sum_{n=0}^{k}\left(\begin{array}{c}k\\ n\end{array}\right)_q w^n F_{k-n,q}(x;y). \tag{3.10}$$

Theorem 3.2. The following summation formula for q-type Fubini polynomials $F_{n,q}(x;y)$ holds true:

$$F_{n,q}(w;y)F_{m,q}(W;Y)=\sum_{r,k=0}^{n,m}\left(\begin{array}{c}n\\ r\end{array}\right)_q\left(\begin{array}{c}m\\ k\end{array}\right)_q (w-x)_q^r$$
$$\times F_{n-r,q}(x;y)(W-X)_q^k F_{m-k,q}(X;Y). \tag{3.11}$$

Proof. Consider the product of the q-type Fubini polynomials, we can be written as generating function (2.1) in the following form:

$$\frac{1}{1-y(e_q(t)-1)}e_q(xt)\frac{1}{1-Y(e_q(T)-1)}e_q(XT)$$
$$=\sum_{n=0}^{\infty}F_{n,q}(x;y)\frac{t^n}{[n]_q!}\sum_{m=0}^{\infty}F_{m,q}(X;Y)\frac{T^m}{[m]_q!}. \tag{3.12}$$

Replacing x by w, X by W in (3.12) and equating the resultant to itself,

$$\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}F_{n,q}(w;y)F_{m,q}(W;Y)\frac{t^n}{[n]_q!}\frac{T^m}{[m]_q!}$$
$$=e_q\left((w-x)t\right)e_q\left((W-X)T\right)$$

$$\times \sum_{n=0}^{\infty}\sum_{m=0}^{\infty} F_{n,q}(x;y)F_{m,q}(X;Y)\frac{t^n}{[n]_q!}\frac{T^m}{[m]_q!},$$

which on using the generating function (3.7) in the exponential on the r.h.s., becomes

$$\sum_{n=0}^{\infty}\sum_{m=0}^{\infty} F_{n,q}(w;y)F_{m,q}(W;Y)\frac{t^n}{[n]_q!}\frac{T^m}{[m]_q!}$$
$$= \sum_{n,r=0}^{\infty} (w-x)_q^r F_{n,q}(x;y)\frac{t^{n+r}}{[n]_q![r]_q!}$$
$$\times \sum_{m,k=0}^{\infty} (W-X)_q^k F_{m,q}(X;Y)\frac{T^{m+k}}{[m]_q![k]_q!}. \tag{3.13}$$

Finally, replacing n by $n-r$ and m by $m-k$ and using equation (3.7) in the r.h.s. of the above equation and then equating the coefficients of like powers of t and T, we get assertion (3.11) of Theorem 3.2. □

Theorem 3.3. The following summation formula for q-type Fubini polynomials $F_{n,q}(x;y)$ holds true:

$$F_{n,q}(x+1;y) = \sum_{r=0}^{n} \binom{n}{r}_q F_{n-r,q}(x;y). \tag{3.14}$$

Proof. Using the generating function (2.1), we have

$$\sum_{n=0}^{\infty} F_{n,q}(x+1;y)\frac{t^n}{[n]_q!} - \sum_{n=0}^{\infty} F_{n,q}(x;y)\frac{t^n}{[n]_q!}$$
$$= \left(\frac{1}{1-y(e_q(t)-1)}\right)(e_q(t)-1)e_q(xt)$$
$$= \sum_{n=0}^{\infty} F_{n,q}(x;y)\frac{t^n}{[n]_q!}\left(\sum_{r=0}^{\infty}\frac{t^r}{[r]_q!}-1\right)$$
$$= \sum_{n=0}^{\infty} F_{n,q}(x;y)\frac{t^n}{[n]_q!}\sum_{r=0}^{\infty}\frac{t^r}{[r]_q!} - \sum_{n=0}^{\infty} F_{n,q}(x;y)\frac{t^n}{[n]_q!}$$
$$= \sum_{n=0}^{\infty}\sum_{r=0}^{n}\binom{n}{r}_q F_{n-r,q}(x;y)\frac{t^n}{[n]_q!} - \sum_{n=0}^{\infty} F_{n,q}(x;y)\frac{t^n}{[n]_q!}.$$

Finally, equating the coefficients of the like powers of t on both sides, we get (3.14). □

Theorem 3.4. For $n \geq 0$ and $y_1 \neq y_2$, the following formula for q-type Fubini polynomials holds true:

$$\sum_{k=0}^{n}\binom{n}{k}_q F_{n-k,q}(x_1;y_1)F_{k,q}(x_2;y_2)$$
$$= \frac{y_2F_{n,q}(x_1+x_2;y_1) - y_1F_{n,q}(x_1+x_2;y_2)}{y_2-y_1}. \tag{3.15}$$

Proof. The products of (2.1) can be written as

$$\sum_{n=0}^{\infty}\sum_{k=0}^{\infty} F_{n,q}(x_1;y_1)F_{k,q}(x_2;y_2)\frac{t^n}{[n]_q!}\frac{t^k}{[k]_q!}$$

$$=\frac{e_q(x_1t)}{1-y_1(e_q(t)-1)}\frac{e_q(x_2t)}{1-y_2(e_q(t)-1)}$$

$$\sum_{n=0}^{\infty}\left(\sum_{k=0}^{n}\left(\begin{array}{c} n \\ k \end{array}\right)_q F_{n-k,q}(x_1;y_1)F_{k,q}(x_2;y_2)\right)\frac{t^n}{[n]_q!}$$

$$=\frac{y_2}{y_2-y_1}\frac{e_q[(x_1+x_2)t]}{1-y_1(e_q(t)-1)}-\frac{y_1}{y_2-y_1}\frac{e_q[(x_1+x_2)t]}{1-y_2(e_q(t)-1)}$$

$$=\left(\frac{y_2F_{n,q}(x_1+x_2;y_1)-y_1F_{n,q}(x_1+x_2;y_2)}{y_2-y_1}\right)\frac{t^n}{[n]_q!}.$$

By equating the coefficients of $\frac{t^n}{[n]_q!}$ on both sides, we get (3.15).

4. Applications

In this section, we derive some relationships for q-type Fubini polynomials related to q-Bernoulli polynomials, q-Euler polynomials and q-Genocchi polynomials and Stirling numbers of the second kind. We start a following theorem.

Theorem 4.1. Each of the following relationships holds true:

$$F_{n,q}(x;y)$$
$$=\sum_{s=0}^{n+1}\left(\begin{array}{c} n+1 \\ s \end{array}\right)_q\left[\sum_{k=0}^{s}\left(\begin{array}{c} s \\ k \end{array}\right)_q B_{s-k,q}(x)-B_{s,q}(x)\right]\frac{F_{n+1-s,q}(y)}{[n+1]_q}, \quad (4.1)$$

where $B_{n,q}(x)$ is q-Bernoulli polynomials.

Proof. By using definition (2.1), we have

$$\left(\frac{1}{1-y(e_q(t)-1)}\right)e_q(xt)$$
$$=\left(\frac{1}{1-y(e_q(t)-1)}\right)\frac{t}{e_q(t)-1}\frac{e_q(t)-1}{t}e_q(xt)$$
$$=\frac{1}{t}\sum_{n=0}^{\infty}\left(\sum_{k=0}^{s}\left(\begin{array}{c} s \\ k \end{array}\right)_q B_{s-k,q}(x)\right)\frac{t^s}{[s]_q!}\sum_{n=0}^{\infty}F_{n,q}(0,y)\frac{t^n}{[n]_q!}$$
$$-\frac{1}{t}\sum_{s=0}^{\infty}B_{s,q}(x)\frac{t^s}{[s]_q!}\sum_{n=0}^{\infty}F_{n,q}(y)\frac{t^n}{[n]_q!}$$
$$=\frac{1}{t}\sum_{n=0}^{\infty}\left[\sum_{s=0}^{n}\left(\begin{array}{c} n \\ s \end{array}\right)_q\sum_{k=0}^{s}\left(\begin{array}{c} s \\ k \end{array}\right)_q B_{s-k,q}(x)\right]F_{n-s,q}(y)\frac{t^n}{[n]_q!}$$
$$-\frac{1}{t}\sum_{n=0}^{\infty}\left[\sum_{s=0}^{n}\left(\begin{array}{c} n \\ s \end{array}\right)_q B_{s,q}(x)\right]F_{n-s,q}(y)\frac{t^n}{[n]_q!}.$$

By using Cauchy product and comparing the coefficients of $\frac{t^n}{[n]_q!}$, we arrive at the required result (4.1).

Theorem 4.2. Each of the following relationships holds true:

$$F_{n,q}(x;y) = \sum_{s=0}^{n} \left(\begin{array}{c} n \\ s \end{array} \right)_q \left[\sum_{k=0}^{s} \left(\begin{array}{c} s \\ k \end{array} \right)_q E_{s-k,q}(x) + E_{s,q}(x) \right] \frac{F_{n-s,q}(y)}{[2]_q}, \tag{4.2}$$

where $E_{n,q}(x)$ is q-Euler polynomials.

Proof. By using definition (2.1), we have

$$\begin{aligned}
&\left(\frac{1}{1 - y(e_q(t) - 1)} \right) e_q(xt) \\
&= \left(\frac{1}{1 - y(e_q(t) - 1)} \right) \frac{[2]_q}{e_q(t) + 1} \frac{e_q(t) + 1}{[2]_q} e_q(xt) \\
&= \frac{1}{[2]_q} \left[\sum_{n=0}^{\infty} \left(\sum_{k=0}^{n} \left(\begin{array}{c} n \\ k \end{array} \right)_q E_{n-k,q}(x) \right) \frac{t^n}{[n]_q!} + \sum_{n=0}^{\infty} E_{n,q}(x) \frac{t^n}{[n]_q!} \right] \\
&\times \sum_{n=0}^{\infty} F_{n,q}(y) \frac{t^n}{[n]_q!} \\
&= \frac{1}{[2]_q} \sum_{n=0}^{\infty} \left[\sum_{s=0}^{n} \left(\begin{array}{c} n \\ s \end{array} \right)_q \sum_{k=0}^{s} \left(\begin{array}{c} s \\ k \end{array} \right)_q E_{s-k,q}(x) + \sum_{s=0}^{n} \left(\begin{array}{c} n \\ s \end{array} \right)_q E_{s,q}(x) \right] \\
&\times F_{n-s,q}(y) \frac{t^n}{[n]_q!}.
\end{aligned}$$

Comparing the coefficients of $\frac{t^n}{[n]_q!}$, we arrive at the desired result (4.2).

Theorem 4.3 . Each of the following relationships holds true:

$$F_{n,q}(x;y) = \sum_{s=0}^{n} \left(\begin{array}{c} n+1 \\ s \end{array} \right)_q \left[\sum_{k=0}^{s} \left(\begin{array}{c} s \\ k \end{array} \right)_q G_{s-k,q}(x) + G_{s,q}(x) \right] \frac{F_{n+1-s,q}(y)}{[2]_q [n+1]_q}, \tag{4.3}$$

where $G_{n,q}(x)$ is q-Genocchi polynomials.

Proof. By using definition (2.1), we have

$$\left(\frac{1}{1-y(e_q(t)-1)}\right)e_q(xt)$$

$$=\left(\frac{1}{1-y(e_q(t)-1)}\right)e_q(xt)\frac{[2]_qt}{e_q(t)+1}\frac{e_q(t)+1}{[2]_qt}e_q(xt)$$

$$=\frac{1}{[2]_qt}\left[\sum_{n=0}^{\infty}\left(\sum_{k=0}^{n}\left(\begin{array}{c}n\\k\end{array}\right)_q G_{n-k,q}(x)\right)\frac{t^n}{[n]_q!}+\sum_{n=0}^{\infty}G_{n,q}(x)\frac{t^n}{[n]_q!}\right]$$

$$\times\sum_{n=0}^{\infty}F_{n,q}(y)\frac{t^n}{[n]_q!}$$

$$=\frac{1}{[2]_q}\sum_{n=0}^{\infty}\left[\sum_{s=0}^{n}\left(\begin{array}{c}n\\s\end{array}\right)_q\sum_{k=0}^{s}\left(\begin{array}{c}s\\k\end{array}\right)_q G_{s-k,q}(x)+\sum_{s=0}^{n}\left(\begin{array}{c}n\\s\end{array}\right)_q G_{s,q}(x)\right]$$

$$\times F_{n+1-s,q}(y)\frac{t^n}{[n+1]_q!}.$$

Comparing the coefficients of $\frac{t^n}{[n]_q!}$, then we have the asserted result (4.3).

Theorem 4.4. For $n\geq 0$, the following formula for q-type Fubini polynomials holds true:

$$F_{n,q}(x;y)=\sum_{l=0}^{n}\left(\begin{array}{c}n\\l\end{array}\right)_q x^{n-l}\sum_{k=0}^{l}y^k k!S_{2,q}(l,k). \tag{4.4}$$

Proof. From (2.1), we have

$$\sum_{n=0}^{\infty}F_{n,q}(x;y)\frac{t^n}{[n]_q!}=\frac{1}{1-y(e_q(t)-1)}e_q(xt)$$

$$=e_q(xt)\sum_{k=0}^{\infty}y^k(e_q(t)-1)^k$$

$$=e_q(xt)\sum_{k=0}^{\infty}y^k\sum_{l=k}^{\infty}k!S_{2,q}(l,k)\frac{t^l}{[l]_q!}$$

$$=\sum_{n=0}^{\infty}x^n\frac{t^n}{[n]_q!}\sum_{l=0}^{\infty}y^k\sum_{k=0}^{l}k!S_{2,q}(l,k)\frac{t^l}{[l]_q!}.$$

Replacing n by $n-l$ in above equation, we get

$$\sum_{n=0}^{\infty}F_{n,q}(x;y)\frac{t^n}{[n]_q!}$$

$$=\sum_{n=0}^{\infty}\left(\sum_{l=0}^{n}\left(\begin{array}{c}n\\l\end{array}\right)_q x^{n-l}\sum_{k=0}^{l}y^k k!S_{2,q}(l,k)\right)\frac{t^n}{[n]_q!}.$$

Comparing the coefficients of $\frac{t^n}{[n]_q!}$ in both sides, we get (4.4).

Theorem 4.5. For $n \geq 0$, the following formula for q-type Fubini polynomials holds true:

$$F_{n,q}(x+r;y) = \sum_{l=0}^{n} \left(\begin{array}{c} n \\ l \end{array} \right)_q x^{n-l} \sum_{k=0}^{l} y^k k! S_{2,q}(l+r,k+r). \tag{4.5}$$

Proof. Replacing x by $x+r$ in (2.1), we have

$$\begin{aligned}\sum_{n=0}^{\infty} F_{n,q}(x+r;y)\frac{t^n}{[n]_q!} &= \frac{1}{1-y(e_q(t)-1)}e_q((x+r)t) \\ &= e_q(xt)e_q(rt)\sum_{k=0}^{\infty} y^k (e_q(t)-1)^k \\ &= e_q(xt)e_q(rt)\sum_{k=0}^{\infty} y^k \sum_{l=k}^{\infty} k! S_{2,q}(l,k)\frac{t^l}{[l]_q!} \\ &= \sum_{n=0}^{\infty} x^n \frac{t^n}{[n]_q!}\sum_{l=0}^{\infty} y^k \sum_{k=0}^{l} k! S_{2,q}(l+r,k+r)\frac{t^l}{[l]_q!}.\end{aligned}$$

Replacing n by $n-l$ in above equation, we get

$$\sum_{n=0}^{\infty} F_{n,q}(x+r;y)\frac{t^n}{[n]_q!}$$

$$= \sum_{n=0}^{\infty}\left(\sum_{l=0}^{n} \left(\begin{array}{c} n \\ l \end{array} \right)_q x^{n-l} \sum_{k=0}^{l} y^k k! S_{2,q}(l+r,k+r)\right)\frac{t^n}{[n]_q!}.$$

Comparing the coefficients of $\frac{t^n}{[n]_q!}$ in both sides, we get (4.5).

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