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CERTAIN INTEGRAL INEQUALITIES OF HERMITE-HADAMARD TYPE FOR H-CONVEX FUNCTIONS

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ABSTRACT. In this paper, we obtain new inequalities of the Hermite-Hadamard type, in the class of h-convex functions. It is shown that several results reported in the literature are obtained as our particular cases.

1. Introduction

Perhaps one of the most productive mathematical ideas lately, due to its variety of uses and interrelationships with different applications, is that of the convex function.

Definition 1.1. A function $\varphi : I \to \mathbb{R}$ is said to be **convex** on interval $I \subset \mathbb{R}$, if the inequality $\varphi(\tau u + (1-\tau)v) \leq \tau\varphi(u) + (1-\tau)\varphi(v)$, for $u, v \in I$ is fulfilled with $\tau \in [0, 1]$.

We say that φ is concave if $-\varphi$ is convex.

The area of Integral inequalities has become one of the most dynamic of Mathematical Sciences, both pure and applied, which translates into a constant increase in the number of researchers and the results obtained in recent years. Within these, there is an inequality that is considered fundamental and that provides simple bounds for the integral mean value of a particular class of functions: convex functions, and it is the so-called Hermite-Hadamard inequality (see, e.g., [5, 10, 11]).

Let $\varphi : I \subseteq \mathbb{R} \to \mathbb{R}$ be a convex function defined on the interval I of real numbers and $a_1, a_2 \in I$ with $a_1 < a_2$. The following inequality

$$\varphi\Big(\frac{a_1 + a_2}{2}\Big) \leq \frac{1}{a_2 - a_1}\int_{a_1}^{a_2} \varphi(u)du \leq \frac{\varphi(a_1) + \varphi(a_2)}{2} \tag{1.1}$$

holds. Since its discovery, this inequality has received considerable attention, some extensions and generalizations of this inequality, with different fractional and generalized operators and using different convexity operators, can be consulted in [1, 2, 7, **?**, 9, 14, 15, 16, 18, 19].

The consequent extensions of concept of the convex functions, which have appeared lately, have transformed it into an extremely complex concept. To reflect on this, we suggest that the user read the work [18], where a fairly complete classification of most of the known definitions is made.

Toader in [24] defined m-convexity in the following way:

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Definition 1.2. The function $\varphi : [0,b] \to \mathbb{R}$, $b > 0$, is said to be m-convex, where $m \in [0,1]$, if

$$\varphi(\tau u + (1-\tau)v) \leq \tau\varphi(u) + m(1-\tau)\varphi(v) \tag{1.2}$$

holds for all $u, v \in [0,b]$ and $\tau \in [0,1]$.

If the above inequality holds in reverse, then we say that the function φ is m-concave.

Definition 1.3. [6] $\varphi : [0,b] \to [0,+\infty)$ with $b > 0$ belongs to the class $P(I)$ if it is nonnegative and, for all $u, v \in I$, $I \subset [0,b]$ and satisfies the following inequality

$$\varphi(\tau u + m(1-\tau)v) \leq \varphi(u) + \varphi(v), \tag{1.3}$$

with $\tau \in (0,1)$.

Definition 1.4. [3, 12] Let $s \in (0,1]$ be a real number. A function $\varphi : [0,b] \to [0,+\infty)$ with $b > 0$ is said to be s-convex (in the first sense) if

$$\varphi(\tau u + m(1-\tau)v) \leq \tau^s\varphi(u) + (1-\tau^s)\varphi(v), \tag{1.4}$$

for all $u, v \in [0,b]$ and $\tau \in (0,1)$.

Definition 1.5. [3, 12] Let $s \in (0,1]$ be a real number. A function $\varphi : [0,b] \to [0,+\infty)$ with $b > 0$ is said to be s-convex (in the second sense) if

$$\varphi(\tau u + m(1-\tau)v) \leq \tau^s\varphi(u) + (1-\tau)^s\varphi(v) \tag{1.5}$$

for all $u, v \in [0,b]$ and $\tau \in (0,1)$.

Definition 1.6. [27] Let $h : J \subseteq \mathbb{R} \to \mathbb{R}$ be a positive function . We say that the real function $\varphi : I \subseteq \mathbb{R} \to \mathbb{R}$ is called h-convex function, or that φ belongs to the class $SX(h,I)$, if φ is nonnegative and for all $u, v \in I$ and $w \in (0,1)$ we have

$$\varphi(wu + (1-w)v) \leq h(w)\varphi(u) + h(1-w)\varphi(v) \tag{1.6}$$

It is known that fractional calculus, that is, calculus with derivatives and integrals of non-integer order, despite being as old as classical calculus, has been gaining attention in the last 40 years and new operators have been defined, which have proven its usefulness in different applications. In particular, new integral operators have appeared that are natural generalizations of the classical fractional Riemann-Liouville integral. In a previous work (see [8]) the authors define a generalized operator that contains, as particular cases, several of the known fractional integral operators.

Definition 1.7. The k-generalized fractional Riemann-Liouville integral of order α with $\alpha \in \mathbb{R}$, and $s \neq -1$ of an integrable function $\varphi(u)$ on $[0,\infty)$, are given as follows (right and left, respectively):

$$ {}^{s}J_{F,a_1^+}^{\frac{\alpha}{k}}\varphi(u) = \frac{1}{k\Gamma_k(\alpha)}\int_{a_1}^{u}\frac{F(\tau,s)\varphi(\tau)d\tau}{[\mathbb{F}(u,\tau)]^{1-\frac{\alpha}{k}}}, \tag{1.7}$$

$$ {}^{s}J^{\frac{\alpha}{k}}_{F,a_2^-}\varphi(u)=\frac{1}{k\Gamma_k(\alpha)}\int_u^{a_2}\frac{F(\tau,s)\varphi(\tau)d\tau}{[\mathbb{F}(\tau,u)]^{1-\frac{\alpha}{k}}}, \tag{1.8} $$

with $F(\tau,0)=1$, $\mathbb{F}(u,\tau)=\int_\tau^u F(\theta,s)d\theta$ and $\mathbb{F}(\tau,u)=\int_u^\tau F(\theta,s)d\theta$.

With the functions Γ (see [21, 22, 23, 25, 26]) and Γ_k defined by (cf. [5]):

$$ \Gamma(z)=\int_0^\infty \tau^{z-1}e^{-\tau}\,\mathrm{d}\tau, \quad \Re(z)>0, \tag{1.9} $$

$$ \Gamma_k(z)=\int_0^\infty \tau^{z-1}e^{-\tau^k/k}\,\mathrm{d}\tau, k>0. \tag{1.10} $$

It is clear that if $k\to 1$ we have $\Gamma_k(z)\to\Gamma(z)$, $\Gamma_k(z)=(k)^{\frac{z}{k}-1}\Gamma\left(\frac{z}{k}\right)$ and $\Gamma_k(z+k)=z\Gamma_k(z)$. As well, we define the k-beta function as follows

$$ B_k(u,v)=\frac{1}{k}\int_0^1 \tau^{\frac{u}{k}-1}(1-\tau)^{\frac{v}{k}-1}d\tau, $$

notice that $B_k(u,v)=\frac{1}{k}B(\frac{u}{k},\frac{v}{k})$ and $B_k(u,v)=\frac{\Gamma_k(u)\Gamma_k(v)}{\Gamma_k(u+v)}$.

The next result will be crucial in the future, your proof is in [28].

Lemma 1.8. *Let $\varphi\in SX(h,I)$, $a_1,a_2\in I$ with $0<a_1<a_2<\infty$. Then for any $w\in[a_1,a_2]$,*

$$ \varphi(a_1+a_2-w)\le (h(\tau)+h(1-\tau))[\varphi(a_1)+\varphi(a_2)]-\varphi(w); \tag{1.11} $$

with $\tau\in[a_1,a_2]$ depends on w.

The main purpose of this paper, using the generalized fractional integral operator of the Riemann- Liouville type, of Definition 1.7, is to establish several integral inequalities of Hermite-Hadamard type for h-convex functions, which contain as particular cases, several of those reported in the literature.

2. Main Results

In the sequel of the paper, I and J are intervals in $\mathbb{R}$, $[0,1]\subset J$ and functions h and φ are real nonnegative functions defined on J and I , respectively. Let $a_1,a_2\in I$ with $0<a_1<a_2<\infty$. We assume that $\varphi\in L_1[a_1,a_2]$ such that ${}^{s}J^{\frac{\alpha}{k}}_{F,a_1^+}\varphi(u)$ and ${}^{s}J^{\frac{\alpha}{k}}_{F,a_2^-}\varphi(u)$ are well defined. We define

$$ \tilde{\varphi}(u):=\varphi(a_1+a_2-u),\ u\in[a_1,a_2] $$

and

$$ G_\varphi(u):=\varphi(u)+\tilde{\varphi}(u),\ u\in[a_1,a_2]. $$

Notice that by using the change of variables $w=\frac{\tau-a_1}{u-a_1}$, we have that (1.7) becomes in

$$ {}^{s}J^{\frac{\alpha}{k}}_{F,a_1^+}\varphi(u)=\frac{(u-a_1)}{k\Gamma_k(\alpha)}\int_0^1\frac{F(wu+a_1(1-w),s)\varphi\big(wu+a_1(1-w)\big)dw}{\big[\mathbb{F}\big(u,wu+a_1(1-w)\big)\big]^{1-\frac{\alpha}{k}}}, \tag{2.1} $$

where $u>a_1$.

Theorem 2.1. *Let $\varphi \in SX(h,I)$, $a_1, a_2 \in I$, with $0 < a_1 < a_2 < \infty$ and $\varphi \in L_1[a_1,a_2]$. Then*

$$\begin{aligned} &\frac{\varphi\left(\frac{a_2+a_1}{2}\right)\left[\mathbb{F}\big(a_2,a_1\big)\right]^{\frac{\alpha}{k}}}{\alpha h(\frac{1}{2})\Gamma_k(\alpha)} \leq \frac{1}{a_2-a_1}\ {}^{s}J^{\frac{\alpha}{k}}_{F,a_1^+}G_{\varphi}(a_2) \\ &\leq \frac{2[\varphi(a_1)+\varphi(a_2)]}{k\Gamma_k(\alpha)}\int_0^1 \frac{F(wa_2+(1-w)a_1,s)}{\left[\mathbb{F}\big(u,wa_2+(1-w)a_1\big)\right]^{1-\frac{\alpha}{k}}}[h(w)+h(1-w)]dw \end{aligned} \tag{2.2}$$

Proof. Since $\varphi \in SX(h,I)$, with $\tau = 1/2$, $u = wa_2+(1-w)a_1$, $v = (1-w)a_2+wa_2$ y $w \in [0,1]$, we have

$$\varphi(\tfrac{a+b}{2}) \leq h\left(\tfrac{1}{2}\right)\varphi(wa_2+(1-w)a_1) + h\left(\tfrac{1}{2}\right)\varphi((1-w)a_2+wa_1) \tag{2.3}$$

Multiplying both sides of (2.3) by

$$\frac{(a_2-a_1)}{k\Gamma_k(\alpha)}\frac{F(wa_2+(1-w)a_1,s)}{[\mathbb{F}\big(a_2,wa_2+(1-w)a_1\big)]^{1-\frac{\alpha}{k}}}$$

and integrating over $(0,1)$ with respect to w, and using the identity $\tilde{\varphi}\big((1-w)a_1+a_2w\big) = \varphi(a_1w+(1-w)a_2)$, we get

$$\begin{aligned} &\frac{\varphi(\frac{a+b}{2})}{h(\frac{1}{2})}\frac{(a_2-a_1)}{k\Gamma_k(\alpha)}\left[\mathbb{F}\big(a_2,a_1\big)\right]^{\frac{\alpha}{k}} \leq \\ &\frac{(a_2-a_1)}{k\Gamma_k(\alpha)}\int_0^1 \frac{F(wa_2+(1-w)a_1,s)\varphi\big(wa_2+(1-w)a_1\big)dw}{[\mathbb{F}\big(a_2,wa_2+a_1(1-w)\big)]^{1-\frac{\alpha}{k}}} \\ &+\frac{(a_2-a_1)}{k\Gamma_k(\alpha)}\int_0^1 \frac{F(wa_2+(1-w)a_1,s)\varphi\big(wa_1+(1-w)a_2\big)dw}{[\mathbb{F}\big(a_2,wa_2+a_1(1-w)\big)]^{1-\frac{\alpha}{k}}} \\ &=\frac{(a_2-a_1)}{k\Gamma_k(\alpha)}\int_0^1 \frac{F(wa_2+(1-w)a_1,s)\varphi\big(wa_2+(1-w)a_1\big)dw}{[\mathbb{F}\big(a_2,wa_2+a_1(1-w)\big)]^{1-\frac{\alpha}{k}}} \\ &+\frac{(a_2-a_1)}{k\Gamma_k(\alpha)}\int_0^1 \frac{F(wa_2+(1-w)a_1,s)\widetilde{\varphi}\big(wa_2+(1-w)a_1\big)dw}{[\mathbb{F}\big(a_2,wa_2+a_1(1-w)\big)]^{1-\frac{\alpha}{k}}} \end{aligned} \tag{2.4}$$

We note that

$$\int_0^1 \frac{F(wa_2+(1-w)a_1,s)dw}{[\mathbb{F}\big(a_2,wa_2+a_1(1-w)\big)]^{1-\frac{\alpha}{k}}} = \frac{k[\mathbb{F}(a_2,a_1)]^{\frac{\alpha}{k}}}{\alpha(a_2-a_1)}.$$

Now, from (2.1) we get

$$\frac{(a_2-a_1)}{k\Gamma_k(\alpha)}\int_0^1 \frac{F(wa_2+(1-w)a_1,s)\varphi(a_1w+(1-w)a_2)dw}{[\mathbb{F}\big(a_2,wa_2+a_1(1-w)\big)]^{1-\frac{\alpha}{k}}} = {}^{s}J^{\frac{\alpha}{k}}_{F,a_1^+}\tilde{\varphi}(a_2)$$

and

$$\frac{(a_2-a_1)}{k\Gamma_k(\alpha)}\int_0^1 \frac{F(wa_2+(1-w)a_1,s)\varphi((1-w)a_1+a_2w)dw}{[\mathbb{F}\big(a_2,wa_2+a_1(1-w)\big)]^{1-\frac{\alpha}{k}}} = {}^{s}J^{\frac{\alpha}{k}}_{F,a_1^+}\varphi(a_2).$$

Therefore, the equation (2.4) becomes in

$$\frac{\varphi\left(\frac{a+b}{2}\right)\left[\mathbb{F}\big(a_2,a_1\big)\right]^{\frac{\alpha}{k}}}{\alpha h(\frac{1}{2})\Gamma_k(\alpha)} \leq \frac{1}{a_2-a_1}\left\{ {}^{s}J^{\frac{\alpha}{k}}_{F,a_1^+}\varphi(a_2) + {}^{s}J^{\frac{\alpha}{k}}_{F,a_1^+}\widetilde{\varphi}(a_2)\right\} = \frac{1}{a_2-a_1}\ {}^{s}J^{\frac{\alpha}{k}}_{F,a_1^+}G_\varphi(a_2),$$

and the first inequality is proved. The proof of the second inequality follows by using (1.6) with $u = a_2$ and $v = a_1$ That is,

$$\varphi(wa_2 + (1-w)a_1) \leq h(w)\varphi(a_2) + h(1-w)\varphi(a_1) \tag{2.5}$$

Multiplying both sides of (2.5) by

$$\frac{(a_2-a_1)}{k\Gamma_k(\alpha)}\frac{F(wa_2+(1-w)a_1,s)}{[\mathbb{F}\big(a_2,wa_2+(1-w)a_1\big)]^{1-\frac{\alpha}{k}}}$$

and integrating over $(0,1)$ with respect to w, we get

$$\begin{aligned}
&\frac{(a_2-a_1)}{k\Gamma_k(\alpha)}\int_0^1 \frac{F(wa_2+(1-w)a_1,s)\varphi\big(wa_2+(1-w)a_1\big)dw}{[\mathbb{F}\big(a_2,wa_2+a_1(1-w)\big)]^{1-\frac{\alpha}{k}}}\\
&\leq \frac{(a_2-a_1)\varphi(a_2)}{k\Gamma_k(\alpha)}\int_0^1 \frac{F(wa_2+(1-w)a_1,s)h\big(w\big)dw}{[\mathbb{F}\big(a_2,wa_2+a_1(1-w)\big)]^{1-\frac{\alpha}{k}}}\\
&+ \frac{(a_2-a_1)\varphi(a_1)}{k\Gamma_k(\alpha)}\int_0^1 \frac{F(wa_2+(1-w)a_1,s)h\big(1-w\big)dw}{[\mathbb{F}\big(a_2,wa_2+a_1(1-w)\big)]^{1-\frac{\alpha}{k}}}\\
&\leq \frac{(a_2-a_1)[\varphi(a_2)+\varphi(a_1)]}{k\Gamma_k(\alpha)}\int_0^1 \frac{F(wa_2+(1-w)a_1,s)}{[\mathbb{F}\big(a_2,wa_2+a_1(1-w)\big)]^{1-\frac{\alpha}{k}}}\big[h(w)+h(1-w)\big]dw,
\end{aligned} \tag{2.6}$$

then

$$\begin{aligned}
&{}^{s}J^{\frac{\alpha}{k}}_{F,a_1^+}\varphi(a_2) \leq \\
&\frac{(a_2-a_1)[\varphi(a_2)+\varphi(a_1)]}{k\Gamma_k(\alpha)}\int_0^1 \frac{F(wa_2+(1-w)a_1,s)}{[\mathbb{F}\big(a_2,wa_2+a_1(1-w)\big)]^{1-\frac{\alpha}{k}}}\big[h(w)+h(1-w)\big]dw,
\end{aligned} \tag{2.7}$$

Similarly,

$$\begin{aligned}
&{}^{s}J^{\frac{\alpha}{k}}_{F,a_1^+}\widetilde{\varphi}(a_2) \leq \\
&\frac{(a_2-a_1)[\varphi(a_2)+\varphi(a_1)]}{k\Gamma_k(\alpha)}\int_0^1 \frac{F(wa_2+(1-w)a_1,s)}{[\mathbb{F}\big(a_2,wa_2+a_1(1-w)\big)]^{1-\frac{\alpha}{k}}}\big[h(w)+h(1-w)\big]dw,
\end{aligned} \tag{2.8}$$

Thus, adding (2.7) and (2.8) we obtain the second part of the inequality (2.2). □

Remark 2.2. If in the previous result, we consider $F \equiv 1$, $\alpha = k$, and φ convex, that is, h the identity function, then we obtain the classic Hermite-Hadamard inequality (1.1). In the case that h is of class $P(I)$, then we obtain Theorem 3.1

of [6] and if $h(u) = u^s$, we obtain Theorem 2.1 of [4]. This Theorem contains as a particular case Theorem 6 of [27], if we take $F \equiv 1$ and $\alpha = k$.

Theorem 2.3. *Let $\varphi, \psi \in SX(h, I)$, $a_1, a_2 \in I$, with $0 < a_1 < a_2 < \infty$ and $\varphi, \psi, h_1, h_2 \in L_1([a_1, a_2])$. Then*

$$\begin{aligned}
&\frac{1}{b-a}\, {}^{s}J^{\frac{\alpha}{k}}_{F,a_1^+}\varphi(a_2)\psi(a_2) \leq \\
&\frac{M(a,b)}{k\Gamma_k(\alpha)}\int_0^1\left[\frac{F(\tau(w),s)}{[\mathbb{F}\big(a_2,\tau(w)\big)]^{1-\frac{\alpha}{k}}}+\frac{F(\theta(w),s)}{[\mathbb{F}\big(a_2,\theta(w)\big)]^{1-\frac{\alpha}{k}}}\right]h_1(w)h_2(w)dw \\
&+\frac{N(a,b)}{k\Gamma_k(\alpha)}\int_0^1\left[\frac{F(\tau(w),s)}{[\mathbb{F}\big(a_2,\tau(w)\big)]^{1-\frac{\alpha}{k}}}+\frac{F(\theta(w),s)}{[\mathbb{F}\big(a_2,\theta(w)\big)]^{1-\frac{\alpha}{k}}}\right]h_1(w)h_2(1-w)dw,
\end{aligned}$$

where $M(a,b) = \varphi(a_2)\psi(a_2)+\varphi(a_1)\psi(a_1)$, $N(a,b) = \varphi(a_1)\psi(a_2)+\varphi(a_2)\psi(a_1)$ and $\tau(w) = wa_2 + (1-w)a_1$ and $\theta(w) = wa_1 + a_2(1-w)$.

Proof. Since $\varphi, \psi \in SX(h, I)$, we have

$$\begin{aligned}
\varphi(a_2 w + (1-w)a_1) &\leq h_1(w)\varphi(a_2) + h_1(1-w)\varphi(a_1) \\
\psi(a_2 w + (1-w)a_1) &\leq h_2(w)\psi(a_2) + h_2(1-w)\psi(a_1),
\end{aligned}$$

for all $w \in [0,1]$. Since φ and ψ are nonnegative, so

$$\begin{aligned}
&\varphi(a_2 w + (1-w)a_1)\psi(a_2 w + (1-w)a_1) \leq \\
&h_1(w)h_2(w)\varphi(a_2)\psi(a_2) + h_1(w)h_2(1-w)\varphi(a_2)\psi(a_1) \\
&+ h_1(1-w)h_2(w)\varphi(a_1)\psi(a_2) + h_1(1-w)h_2(1-w)\varphi(a_1)\psi(a_1).
\end{aligned} \tag{2.9}$$

Multiplying both sides of (2.9) by

$$\frac{1}{k\Gamma_k(\alpha)}\frac{F(wa_2 + (1-w)a_1, s)}{[\mathbb{F}\big(a_2, wa_2 + (1-w)a_1\big)]^{1-\frac{\alpha}{k}}}$$

and integrating over $(0,1)$ with respect to w, we obtain

$$\begin{aligned}
&\frac{1}{k\Gamma_k(\alpha)}\int_0^1 \frac{F(wa_2+(1-w)a_1,s)\varphi\big(wa_2+(1-w)a_1\big)\psi\big(wa_2+(1-w)a_1\big)dw}{[\mathbb{F}\big(a_2,wa_2+a_1(1-w)\big)]^{1-\frac{\alpha}{k}}} \leq \\
&\varphi(a_2)\psi(a_2)\frac{(a_2-a_1)}{k\Gamma_k(\alpha)}\int_0^1 \frac{F(wa_2+(1-w)a_1,s)}{[\mathbb{F}\big(a_2,wa_2+a_1(1-w)\big)]^{1-\frac{\alpha}{k}}}h_1(w)h_2(w)dw \\
&+\varphi(a_2)\psi(a_1)\frac{(a_2-a_1)}{k\Gamma_k(\alpha)}\int_0^1 \frac{F(wa_2+(1-w)a_1,s)}{[\mathbb{F}\big(a_2,wa_2+a_1(1-w)\big)]^{1-\frac{\alpha}{k}}}h_1(w)h_2(1-w)dw \\
&+\varphi(a_1)\psi(a_2)\frac{(a_2-a_1)}{k\Gamma_k(\alpha)}\int_0^1 \frac{F(wa_2+(1-w)a_1,s)}{[\mathbb{F}\big(a_2,wa_2+a_1(1-w)\big)]^{1-\frac{\alpha}{k}}}h_1(1-w)h_2(w)dw \\
&+\varphi(a_1)\psi(a_1)\frac{(a_2-a_1)}{k\Gamma_k(\alpha)}\int_0^1 \frac{F(wa_2+(1-w)a_1,s)}{[\mathbb{F}\big(a_2,wa_2+a_1(1-w)\big)]^{1-\frac{\alpha}{k}}}h_1(1-w)h_2(1-w)dw \\
&\leq \frac{M(a,b)}{k\Gamma_k(\alpha)}\int_0^1 \left[\frac{F(\tau(w),s)}{[\mathbb{F}\big(a_2,\tau(w)\big)]^{1-\frac{\alpha}{k}}}+\frac{F(\theta(w),s)}{[\mathbb{F}\big(a_2,\theta(w)\big)]^{1-\frac{\alpha}{k}}}\right]h_1(w)h_2(w)dw \\
&+\frac{N(a,b)}{k\Gamma_k(\alpha)}\int_0^1 \left[\frac{F(\tau(w),s)}{[\mathbb{F}\big(a_2,\tau(w)\big)]^{1-\frac{\alpha}{k}}}+\frac{F(\theta(w),s)}{[\mathbb{F}\big(a_2,\theta(w)\big)]^{1-\frac{\alpha}{k}}}\right]h_1(w)h_2(1-w)dw,
\end{aligned} \tag{2.10}$$

with $\tau(w)=wa_2+(1-w)a_1$ and $\theta(w)=wa_1+a_2(1-w)$ and where $M(a,b)=\varphi(a_2)\psi(a_2)+\varphi(a_1)\psi(a_1)$ and $N(a,b)=\varphi(a_1)\psi(a_2)+\varphi(a_2)\psi(a_1)$.

Then

$$\begin{aligned}
&\frac{1}{b-a}\,{}^{s}J^{\frac{\alpha}{k}}_{F,a_1^+}\varphi(a_2)\psi(a_2) \leq \\
&\frac{M(a,b)}{k\Gamma_k(\alpha)}\int_0^1 \left[\frac{F(\tau(w),s)}{[\mathbb{F}\big(a_2,\tau(w)\big)]^{1-\frac{\alpha}{k}}}+\frac{F(\theta(w),s)}{[\mathbb{F}\big(a_2,\theta(w)\big)]^{1-\frac{\alpha}{k}}}\right]h_1(w)h_2(w)dw \\
&+\frac{N(a,b)}{k\Gamma_k(\alpha)}\int_0^1 \left[\frac{F(\tau(w),s)}{[\mathbb{F}\big(a_2,\tau(w)\big)]^{1-\frac{\alpha}{k}}}+\frac{F(\theta(w),s)}{[\mathbb{F}\big(a_2,\theta(w)\big)]^{1-\frac{\alpha}{k}}}\right]h_1(w)h_2(1-w)dw.
\end{aligned}$$

Therefore the result is obtained. □

Remark 2.4. If in the previous result we do $F\equiv\frac{1}{2}$ and $\alpha=k$, then it reduces to Theorem 7 of [27]. If, additionally, φ and ψ are convex functions, this result contains as a particular case the Theorem 1, 1) of [20].

Theorem 2.5. *Let* $\varphi, \psi \in SX(h, I)$*,* $a_1, a_2 \in I$*, with* $0 < a_1 < a_2 < \infty$ *and* $\varphi, \psi, h_1, h_2 \in L_1([a_1, a_2])$*. Then*

$$\frac{\varphi\left(\frac{a_2+a_1}{2}\right)\phi\left(\frac{a_2+a_1}{2}\right)\left[\mathbb{F}\big(a_2,a_1\big)\right]^{\frac{\alpha}{k}}}{\alpha(a_2-a_1)h_1(\frac{1}{2})h_2(\frac{1}{2})\Gamma_k(\alpha)} - \frac{1}{a_2-a_1}\ {}^sJ^{\frac{\alpha}{k}}_{F,a_1^+}G_\varphi(a_2) \leq$$
$$\frac{1}{k\Gamma_k(\alpha)}\Bigg\{M(a,b)\int_0^1\left[\frac{F(\tau(w),s)}{[\mathbb{F}\big(a_2,\tau(w)\big)]^{1-\frac{\alpha}{k}}} + \frac{F(\theta(w),s)}{[\mathbb{F}\big(a_2,\theta(w)\big)]^{1-\frac{\alpha}{k}}}\right]h_1(w)h_2(w)dw$$
$$+ N(a,b)\int_0^1\left[\frac{F(\tau(w),s)}{[\mathbb{F}\big(a_2\tau(w)\big)]^{1-\frac{\alpha}{k}}} + \frac{F(\theta(w),s)}{[\mathbb{F}\big(a_2,\theta(w)\big)]^{1-\frac{\alpha}{k}}}\right]h_1(w)h_2(1-w)\Bigg\}dw,$$

where, $\tau(w) = wa_2 + (1-w)a_1$ *and* $\theta(w) = wa_1 + a_2(1-w)$*.*

Proof. We can write $\dfrac{a_2+a_1}{2} = \dfrac{a_2w+(1-w)a_1}{2} + \dfrac{(1-w)a_2+wa_1}{2}$, so

$$\varphi\left(\frac{a_2+a_1}{2}\right)\psi\left(\frac{a_2+a_1}{2}\right)$$
$$= \varphi\left(\frac{a_2w+(1-w)a_1}{2} + \frac{(1-w)a_2+wa_1}{2}\right)\times\psi\left(\frac{a_2w+(1-w)a_1}{2} + \frac{(1-w)a_2+wa_1}{2}\right)$$
$$\leq h_1(\tfrac{1}{2})h_2(\tfrac{1}{2})\Big[\big[\varphi(a_2w+(1-w)a_1)+\varphi((1-w)a_2+wa_1)\big]$$
$$\times\big[\psi(a_2w+(1-w)a_1)+\psi((1-w)a_2+wa_1)\big]\Big]$$
$$\leq h_1(\tfrac{1}{2})h_2(\tfrac{1}{2})\Big[\varphi(a_2w+(1-w)a_1)\psi(a_2w+(1-w)a_1)$$
$$+\varphi(wa_2+(1-w)a_1)\psi((1-w)a_2+wa_1)$$
$$+\varphi((1-w)a_2+wa_1)\psi(wa_2+(1-w)a_1)$$
$$+\varphi((1-w)a_2+wa_1)\psi((1-w)a_2+wa_1)\Big]$$
$$\leq h_1(\tfrac{1}{2})h_2(\tfrac{1}{2})\Big\{\varphi(a_2w+(1-w)a_1)\psi(a_2w+(1-w)a_1)$$
$$+\varphi((1-w)a_2+wa_1)\psi((1-w)a_2+wa_1)\Big\}$$
$$+h_1(\tfrac{1}{2})h_2(\tfrac{1}{2})\Big\{\big[h_1(w)\varphi(a_2)+h_1(1-w)\varphi(a_1)\big]\big[h_2(1-w)\psi(a_2)+h_2(w)\psi(a_1)\big]$$
$$+\big[h_1(1-w)\varphi(a_2)+h_1(w)\varphi(a_1)\big]\big[h_2(w)\psi(a_2)+h_2(1-w)\psi(a_1)\big]\Big\}$$

for all $w \in [0, 1]$. Thus we get

$$\begin{aligned}
&\varphi\left(\frac{a_2+a_1}{2}\right)\psi\left(\frac{a_2+a_1}{2}\right)\\
&\le h_1(\tfrac{1}{2})h_2(\tfrac{1}{2})\Big\{\varphi(a_2w+(1-w)a_1)\psi(a_2w+(1-w)a_1)\\
&+\varphi((1-w)a_2+wa_1)\psi((1-w)a_2+wa_1)\Big\} \qquad (2.11)\\
&+h_1(\tfrac{1}{2})h_2(\tfrac{1}{2})\Big\{\big[h_1(w)h_2(1-w)+h_1(1-w)h_2(w)\big]N(a,b)\\
&+\big[h_1(w)h_2(w)+h_1(1-w)h_2)(1-w)\big]M(a,b)\Big\}
\end{aligned}$$

Multiplying both sides of (2.11) by

$$\frac{1}{k\Gamma_k(\alpha)}\frac{F(wa_2+(1-w)a_1,s)}{[\mathbb{F}\big(a_2,wa_2+(1-w)a_1\big)]^{1-\frac{\alpha}{k}}}$$

and integrating over $(0,1)$ with respect to w, we obtain

$$\begin{aligned}
&\frac{\varphi\left(\frac{a_2+a_1}{2}\right)\phi\left(\frac{a_2+a_1}{2}\right)\left[\mathbb{F}\big(a_2,a_1\big)\right]^{\frac{\alpha}{k}}}{\alpha(a_2-a_1)h(\frac{1}{2})\Gamma_k(\alpha)}-\frac{h_1(\frac{1}{2})h_2(\frac{1}{2})}{a_2-a_1}\,{}^{s}J^{\frac{\alpha}{k}}_{F,a_1^+}G_{\varphi}(a_2)\le\\
&\frac{h_1(\frac{1}{2})h_2(\frac{1}{2})}{k\Gamma_k(\alpha)}\left\{\frac{M(a,b)}{k\Gamma_k(\alpha)}\int_0^1\left[\frac{F(\tau(w),s)}{[\mathbb{F}\big(a_2,\tau(w)\big)]^{1-\frac{\alpha}{k}}}+\frac{F(\theta(w),s)}{[\mathbb{F}\big(a_2,\theta(w)\big)]^{1-\frac{\alpha}{k}}}\right]h_1(w)h_2(w)dw\right.\\
&\left.+\frac{N(a,b)}{k\Gamma_k(\alpha)}\int_0^1\left[\frac{F(\tau(w),s)}{[\mathbb{F}\big(a_2,\tau(w)\big)]^{1-\frac{\alpha}{k}}}+\frac{F(\theta(w),s)}{[\mathbb{F}\big(a_2,\theta(w)\big)]^{1-\frac{\alpha}{k}}}\right]h_1(w)h_2(1-w)\right\}dw
\end{aligned}$$

with $\tau(w)=wa_2+(1-w)a_1$ and $\theta(w)=wa_1+a_2(1-w)$. Consequently the theorem is proved. □

Remark 2.6. If in the previous result we do $F\equiv\frac{1}{2}$ and $\alpha=k$, then it reduces to Theorem 8 of [27]. If, in addition, φ and ψ are convex functions, this result contains as a particular case the Theorem 1, 2) of [20]. In the case that they are s-convex functions in the second sense, then the previous result reduces to Theorem 7 of [13].

Theorem 2.7. *Let $\varphi\in SX(h,I)$, $a_1,a_2\in I$ with $0<a_1<a_2<\infty$ and $\varphi\in L_1[a_1,a_2]$ and $\psi:[a_1,a_2]\to\mathbb{R}$ is nonnegative, integrable and symmetric about $(a_1+a_2)/2$. Then*

$$\begin{aligned}
&\frac{1}{h\big(\frac{1}{2}\big)}\varphi\left(\tfrac{a_2+a_1}{2}\right)\,{}^{s}J^{\frac{\alpha}{k}}_{F,a_1^+}\psi(a_2)\le\,{}^{s}J^{\frac{\alpha}{k}}_{F,a_1^+}G_{\varphi\psi}(a_2)\\
&\le\frac{\varphi(a_2)+\varphi(a_1)}{k\Gamma_k(\alpha)}\int_{a_1}^{a_2}\frac{F(w,s)\psi(w)}{[\mathbb{F}(a_2,w]^{1-\frac{\alpha}{k}}}\left\{h\left(\frac{a_2-w}{a_2-a_1}\right)+h\left(\frac{a_1-w}{a_2-a_1}\right)\right\}dw.
\end{aligned} \qquad (2.12)$$

Proof. Since $\varphi\in SX(h;I)$ and $\psi:[a_1,a_2]\to\mathbb{R}$ is nonnegative, integrable and symmetric about $(a_1+a_2)/2$ and using the Lemma 1.8; we have

$$\frac{1}{h\left(\frac{1}{2}\right)}\varphi\left(\tfrac{a_2+a_1}{2}\right)\ {}^{s}J^{\frac{\alpha}{k}}_{F,a_1^+}\psi(a_2)=\frac{1}{h\left(\frac{1}{2}\right)k\Gamma_k(\alpha)}\int_{a_1}^{a_2}\varphi\left(\frac{a_1+a_2}{2}\right)\frac{F(w,s)\psi(w)}{[\mathbb{F}(a_2,w]^{1-\frac{\alpha}{k}}}dw$$
$$=\frac{1}{h\left(\frac{1}{2}\right)k\Gamma_k(\alpha)}\int_{a_1}^{a_2}\varphi\left(\frac{a_1+a_2-w+w}{2}\right)\frac{F(w,s)\psi(w)}{[\mathbb{F}(a_2,w]^{1-\frac{\alpha}{k}}}dw$$
$$\leq\frac{1}{h\left(\frac{1}{2}\right)k\Gamma_k(\alpha)}\int_{a_1}^{a_2}\frac{F(w,s)\psi(w)}{[\mathbb{F}(a_2,w]^{1-\frac{\alpha}{k}}}h\left(\tfrac{1}{2}\right)\Big\{\varphi\left(a_1+a_2-w\right)+\varphi(w)\Big\}dw$$
$$=\frac{1}{k\Gamma_k(\alpha)}\int_{a_1}^{a_2}\frac{F(w,s)\psi(w)}{[\mathbb{F}(a_2,w]^{1-\frac{\alpha}{k}}}\varphi\left(a_1+a_2-w\right)\psi(a_1+a_2-w)dw$$
$$+\ \frac{1}{k\Gamma_k(\alpha)}\int_{a_1}^{a_2}\frac{F(w,s)}{[\mathbb{F}(a_2,w]^{1-\frac{\alpha}{k}}}\varphi(w)\psi(w)dw$$
$$=\frac{1}{k\Gamma_k(\alpha)}\int_{a_1}^{a_2}\frac{F(w,s)\widetilde{\varphi}(w)\,\widetilde{\psi}(w)}{[\mathbb{F}(a_2,w]^{1-\frac{\alpha}{k}}}dw\ +\ \frac{1}{k\Gamma_k(\alpha)}\int_{a_1}^{a_2}\frac{F(w,s)\varphi(w)\psi(w)}{[\mathbb{F}(a_2,w]^{1-\frac{\alpha}{k}}}dw$$
$$=\frac{1}{k\Gamma_k(\alpha)}\int_{a_1}^{a_2}\frac{F(w,s)G_{\varphi\psi}(w)}{[\mathbb{F}(a_2,w]^{1-\frac{\alpha}{k}}}dw$$
$$=\ {}^{s}J^{\frac{\alpha}{k}}_{F,a_1^+}G_{\varphi\psi}(a_2),$$

and the first inequality is proved.

Now reasoning similarly, we get

$${}^{s}J^{\frac{\alpha}{k}}_{F,a_1^+}G_{\varphi\psi}(a_2)=\frac{1}{k\Gamma_k(\alpha)}\int_{a_1}^{a_2}\frac{F(w,s)G_{\varphi\psi}(w)}{[\mathbb{F}(a_2,w]^{1-\frac{\alpha}{k}}}dw$$
$$=\frac{1}{k\Gamma_k(\alpha)}\int_{a_1}^{a_2}\frac{F(w,s)\widetilde{\varphi}(w)\,\widetilde{\psi}(w)}{[\mathbb{F}(a_2,w]^{1-\frac{\alpha}{k}}}dw\ +\ \frac{1}{k\Gamma_k(\alpha)}\int_{a_1}^{a_2}\frac{F(w,s)\varphi(w)\psi(w)}{[\mathbb{F}(a_2,w]^{1-\frac{\alpha}{k}}}dw$$
$$=\frac{1}{k\Gamma_k(\alpha)}\int_{a_1}^{a_2}\frac{F(w,s)\psi(w)\varphi\left(a_1+a_2-w\right)\psi(a_1+a_2-w)}{[\mathbb{F}(a_2,w]^{1-\frac{\alpha}{k}}}dw$$
$$+\ \frac{1}{k\Gamma_k(\alpha)}\int_{a_1}^{a_2}\frac{F(w,s)}{[\mathbb{F}(a_2,w]^{1-\frac{\alpha}{k}}}\varphi(w)\psi(w)dw$$
$$=\frac{1}{k\Gamma_k(\alpha)}\int_{a_1}^{a_2}\frac{F(w,s)\psi(w)}{[\mathbb{F}(a_2,w]^{1-\frac{\alpha}{k}}}\Big\{\varphi\left(a_1+a_2-w\right)+\varphi(w)\Big\}dw$$
$$\leq\frac{\varphi(a_2)+\varphi(a_1)}{k\Gamma_k(\alpha)}\int_{a_1}^{a_2}\frac{F(w,s)\psi(w)}{[\mathbb{F}(a_2,w]^{1-\frac{\alpha}{k}}}\left\{h\left(\frac{a_2-w}{a_2-a_1}\right)+h\left(\frac{a_1-w}{a_2-a_1}\right)\right\}dw.$$

Thus, we obtain the inequality (2.12). □

Remark 2.8. If in the previous result, we consider $F\equiv 1$, $\alpha=k$, then it reduces to Theorem 5 and Theorem 6 of [28].

3. Concluding Remarks

This paper gives new inequalities of the Hermite-Hadamard type, in the class of h-convex functions, some related inequalities (fractional or not) are also obtained as particular cases of our results.

From the results obtained, we can point out two open problems:

1) Using the operators of the Definition 1.7, we can generalize different results already reported in the literature, selecting different kernels and even new operators can be used, for example [17].

2) If we use other notions of convexity, (h, m)-convexity, s-convexity and others, the results obtained can be extended.

4. Conflict of Interest

On behalf of all authors, the corresponding author states that there is no conflict of interest.

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