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EXISTENCE, UNIQUENESS AND CONTROLLABILITY ANALYSIS OF BENJAMIN-BONA-MAHONY EQUATION WITH NON INSTANTANEOUS IMPULSES, DELAY AND NON LOCAL CONDITIONS

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Abstract. The main purpose of this article is to prove existence and uniqueness of the solutions for a semilinear evolution equation with non-instantaneous impulses, delay and nonlocal conditions. As an application we consider the Benjamin-Bona-Mahony equation (**BBM**) . The results are obtained by using Karakostas fixed point theorem and nonlinear functional analysis. In the second part we establish the approximate controllability of the controlled **BBM** equation. This is done applying a technique that pull back the control solution to a fixed curve in a short time interval.

1. Introduction

Impulsive dynamic systems are a type of hybrid systems for which the trajectory admits discontinuities at certain instants due to sudden jumps of the state called pulses (see more in [3,21]). Dynamical behavior of many systems in real life can be characterized by abrupt changes that appear suddenly, such as heartbeats, drug flows, the value of stocks, impulse vaccination, and bonds on the stock market (see [1, 25]). In literature, there are two kind of impulses, one is instantaneous impulses where the duration of these sudden changes is very small in comparison with the total duration of the whole process. Another one is non-instantaneous impulses where the impulsive action starts abruptly at a certain moment of time and remains active on a finite time interval. In real life problems there is no impulse which happens instantaneously, howsoever the time of the action is little, it is always a considerable interval of time.

Control theory is one of the most fundamental and important issues for impulsive differential equations which consists of finding a controls steering the system from an arbitrary initial state to a final state in a finite interval of time. In the last few decades, many authors worked on existence, stability and controllability results for impulsive dynamic systems, see for instance [4,7,12,14–16,22,24] and the references therein.

In this paper, the interest is a semilinear evolution equation with non-instantaneous impulses, delay and nonlocal conditions. Our mathematical motivation is to extend the existence and uniqueness results proved by H. Leiva *Communications in Mathematical Analysis*(2018) [17] by adding non-instantaneous impulses. Also to extend the controllability result for **BBM** equation done by H. Leiva *Systems and Control Letters,* (2017) [18] by adding non-local conditions and non-instantaneous impulses. Motivated by the above facts, we study the existence and uniqueness of solutions for the following semilinear evolution equation with non-instantaneous impulses, delay and nonlocal conditions

$$\begin{cases} z' = -Az + F(t, z_t), & t \in \cup_{i=0}^{N} (s_i, t_{i+1}], \\ z(s) + \mathcal{H}(z_{\tau_1}, \dots, z_{\tau_q})(s) = \phi(s), & t \in [-r, 0], \\ z(t) = \mathcal{G}_i(t, z(t_i^-)), & t \in t \in \cup_{i=1}^{N} (t_i, s_i], \end{cases} \tag{1.1}$$

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where A is a sectorial operator such that $-A : \mathcal{D}(A) \subset \mathcal{X} \to \mathcal{X}$ is a generator of a strongly continuous analytic semigroup $(S(t))_{t\geq 0}$ on a Banach space $\mathcal{X}$, $0 = s_0 = t_0 < t_1 \leq s_1 < \ldots < t_N \leq s_N < t_{N+1} = T$, $0 < \tau_1 < \tau_2 < \ldots < \tau_q < r < T$ are fixed real numbers, the function $z_t \in C([-r,0];\mathcal{X}^\eta)$ is defined by $z_t(s) = z(t+s)$ with $s \in [-r,0]$ and $\mathcal{G}_i : (t_i, s_i] \times \mathcal{X}^\eta \longrightarrow \mathcal{X}^\eta$, $\mathcal{H} : \mathcal{PW}^\eta_{rq} \longrightarrow \mathcal{PW}^\eta_{[-r,0]}$, and $\mathcal{F} : [0,T] \times \mathcal{PW}^\eta_{[-r,0]} \longrightarrow \mathcal{X}$ are smooth functions, and the spaces $\mathcal{X}^\eta$, $\mathcal{PW}^\eta_{rq}$ and $\mathcal{PW}^\eta_{[-r,0]}$ are defined below.

Our results will be applied to prove existence and uniqueness of solution for the following **BBM** equation with non-instantaneous impulses, delay and non-local conditions

$$\begin{cases} z_t + a\tilde{A}z_t + b\tilde{A}z = -z(t-r,x)z_x(t-r,x) & (t,x) \in \cup_{i=0}^{N} (s_i, t_{i+1}] \times [0,\pi], \\ \qquad\qquad +f(t, z(t-r,x)), & \\ z(t,0) = z(t,\pi) = 0, & t \in [0,T], \\ z(s,x) + h(z(\tau_1+s,x), \ldots, z(\tau_q+s,x)) = \phi(s,x), & (t,x) \in [-r,0] \times [0,\pi], \\ z(t,x) = G_i(t, z(t_i^-, x)), & (t,x) \in \cup_{i=1}^{N} (t_i, s_i] \times [0,\pi], \end{cases} \tag{1.2}$$

where $\mathcal{X} = L^2[0,\pi]$, $a \geq 0$ and $b > 0$ are constants, $\tilde{A} : \mathcal{D}(\tilde{A}) \subset \mathcal{X} \to \mathcal{X}$ is the operator given by $\tilde{A}\psi = -\psi_{xx}$ whose domain is defined as

$$\mathcal{D}(\tilde{A}) := \{\psi \in \mathcal{X} : \psi, \psi_x \quad \text{are absolutely continouos}, \quad \psi_{xx} \in \mathcal{X},\ \psi(0) = \psi(\pi) = 0\},$$

and $(\mathcal{D}(-\tilde{A}))^{1/2} = \mathcal{X}^{1/2} = H_0^1$.

For the controllability of **BBM** equation, the following system is considered

$$\begin{cases} z_t + a\tilde{A}z_t + b\tilde{A}z = 1_\omega u(t,x) - z(t-r,x)z_x(t-r,x) & (t,x) \in \cup_{i=0}^{N} (s_i, t_{i+1}] \times [0,\pi], \\ \qquad\qquad +f(t, z(t-r,x), u), & \\ z(t,0) = z(t,\pi) = 0, & t \in [0,T], \\ z(s,x) + h(z(\tau_1+s,x), \ldots, z(\tau_q+s,x)) = \phi(s,x), & (t,x) \in [-r,0] \times [0,\pi], \\ z(t,x) = G_i(t_i^-, z(t,x), u(t,x)), & (t,x) \in \cup_{i=1}^{N} (t_i, s_i] \times [0,\pi], \end{cases} \tag{1.3}$$

with the same details described in (1.2) including that ω is an open nonempty subset of $[0,\pi]$. $\mathbf{1}_\omega$ denotes the characteristic function of the set ω, and the distributed control function u belongs to $L^2([0,T];\mathcal{X})$. In order to establish the approximate controllability of the system (1.3), we use a technique that consists of considering a solution curve and with a specified control move the corresponding solution to such a fixed curve. Then use the linear controllability system to get closer to the final state.

The manuscript is structured as follows. **section 2** introduces preliminary facts and some notations. In **section 3**, we discuss the existence and uniqueness for system (1.1). In **section 4**, we shall apply our previous results to system (1.2). Finally **section 5** is devoted to establish the controllability result for the purposed system (1.3).

2. Preliminaries

We suppose that the operator A is a sectorial operator in $\mathcal{X}$, and therefore $-A$ is the generator of a strongly continuous compact semigroup $\{S(t)\}_{t\geq 0}$ on $\mathcal{X}$ with $0 \in \rho(A)$. Thus, the fractional operator A^η with $\eta \in (0,1]$ is well defined. Moreover, A^η is a closed operator whose domain is a Banach space endowed with the following norm

$$\|z\|_\eta = \|A^\eta z\|, \qquad \forall z \in \mathcal{D}(A^\eta) = \mathcal{X}^\eta.$$

We know that $\mathcal{X}^\eta$ is dense in $\mathcal{X}$. Furthermore, if the resolvent of A is compact. Then for all $\eta \in (\beta, 1]$ with $\beta > 0$ the embedding $\mathcal{X}^\eta \to \mathcal{X}^\beta$ is compact.
Next, we consider the following properties for the strongly continuous semigroup $\{S(t)\}_{t\geq 0}$, generated by $-A$.
There exist $M \geq 1$, $M_\eta \geq 0$ and $\alpha > 0$ such that

$$\begin{aligned} \|S(t)\| &\leq M, && \text{for} \quad t \geq 0, \\ \|A^\eta S(t)\| &\leq \frac{M_\eta}{t^\eta} e^{-\alpha t}, && \text{for} \quad t, > 0, \\ A^\eta S(t) z &= S(t) A^\eta z, && \text{for} \quad z \in \mathcal{X}^\eta, \\ S(t) z &\in \mathcal{D}(A), && z \in \mathcal{X}, t > o. \end{aligned}$$

for more details see [8, 9, 23].

A normal space to work with impulsive differential systems is given by

$$\begin{aligned} \mathcal{PW}^\eta_{[-r,T]} \quad = \quad & \{z : [-r, T] \to \mathcal{X}^\eta : z\Big|_{[-r,0]} \in \mathcal{PW}^\eta_{[-r,0]}, \quad z\Big|_{[0,T]} \in C([0, T] \setminus \{t_i\}_{i=1}^N; \mathcal{X}^\eta), \exists\, z(t_k^+), \\ & z(t_k^-) \quad \text{s.t.} \quad z(t_k) = z(t_k^-)\}, \end{aligned}$$

endowed with the norm

$$\|z\| = \sup_{t\in[-r,T]} \|z(t)\|_\eta,$$

with $\mathcal{PW}^\eta_{[-r,0]}$ given as follows:

$$\mathcal{PW}^\eta_{[-r,0]} \quad = \quad \{\Phi : [-r, 0] \to \mathcal{X}^\eta : \Phi \quad \text{is piece-wise continuous}\},$$

endowed with the norm

$$\|\Phi\| = \sup_{s\in[-r,0]} \|\Phi(s)\|_\eta.$$

Also, we shall denote the Banach space $\mathcal{X}^\eta_q$ as follows

$$\mathcal{X}^\eta_q = \underbrace{\mathcal{X}^\eta \times \mathcal{X}^\eta \times \cdots \mathcal{X}^\eta}_{q \quad times} = \prod_{i=1}^q \mathcal{X}^\eta,$$

endowed with the norm

$$\|y\|^\eta_q = \sum_{i=1}^q \|y_i\|_\eta, \quad y = (y_1, y_2, \ldots, y_q)^T \in \mathcal{X}^\eta,$$

and the space $\mathcal{PW}^\eta_{rq}$ given as follows:

$$\mathcal{PW}^\eta_{rq} \quad = \quad \{g : [-r, 0] \to \mathcal{X}^\eta_q : g \quad \text{is piece-wise continuous}\},$$

endowed with the norm

$$\|y\|_q = \sup_{t\in[-r,0]} \|y(t)\|^\eta_q = \sup_{t\in[-r,0]} \left(\sum_{i=1}^q \|y_i(t)\|_\eta\right), \quad \forall y \in \mathcal{PW}^\eta_{rq}.$$

To prove the existence of solutions we need the following results.

Definition 2.1. Let y be a function belongs to $\mathcal{PW}^\eta_{[0,T]}$, and $i = 0, 1, 2, \ldots, N$ we define the function $\tilde{y}_i \in C([t_i, t_{i+1}]; \mathcal{X}^\eta)$ such that

$$\tilde{y}_i(t) = \begin{cases} y(t), & \text{for} \quad t \in [t_i, t_{i+1}), \\ y(t^-_{i+1}), & \text{for} \quad t = t_{i+1}. \end{cases} \tag{2.1}$$

For $W \subset \mathcal{PW}^\eta_{[0,T]}$ and $i = 0, 1, 2, \ldots, N$, we define $\tilde{W}_i = \{\tilde{y}_i : y \in W\}$. Using the following Ascoli-Arzela Theorem we can get a characterization of compactness in $\mathcal{PW}^\eta_{[0,T]}$ (see Theorem 1.1.1 from [13]).

THEOREM 2.1. A set $W \subset \mathcal{PW}^{\eta}_{[0,T]}$ is relatively compact in $\mathcal{PW}^{\eta}_{[0,T]}$ if, and only if, each set $\tilde{W}_i$ with $i = 0, 1, 2, \dots, N$ is relatively compact in $C([t_i, t_{i+1}]; \mathcal{X}^{\eta})$ where $t_0 = 0$ and $t_{N+1} = T$. Note that the space $\mathcal{PW}^{\eta}_{[0,T]}$ is correspond to the following space $\mathcal{PW}^{\eta}_{[-r,T]}\Big|_{[0,T]}$.

THEOREM 2.2. (**G. L. Karakostas [10]**) Let Z and Y be Banach spaces and D be a closed convex subset of Z, let $\mathcal{C} : D \to Y$ be a continuous operator such that $\mathcal{C}(D)$ is a relatively compact subset of Y, and

$$\Psi : D \times \overline{\mathcal{C}(D)} \to D \tag{2.2}$$

a continuous operator such that the family $\{\Psi(\cdot, y) : y \in \overline{\mathcal{C}(D)}\}$ is equicontractive. Then the operator equation

$$\Psi(z, \mathcal{C}(z)) = z, \tag{2.3}$$

admits a solution on D.

3. Existence and uniqueness result

In this section, the proof of the existence and uniqueness of solutions for system (1.1) is presented. A characterization for its solution is given through the following definition,

Definition 3.1. A function $z(\cdot) \in \mathcal{PW}^{\eta}_{[-r,T]}$ is called a mild solution for the system (1.1) if it satisfies the following integral-algebraic equation

$$z(t) = \begin{cases} \phi(t) - \mathcal{H}(z_{\tau_1}, \dots, z_{\tau_q})(t), & t \in [-r, 0], \\ S(t)\{\phi(0) - \mathcal{H}(z_{\tau_1}, \dots, z_{\tau_q})(0)\} & t \in [0, t_1], \\ + \displaystyle\int_0^t S(t-s)F(s, z_s)ds, & \\ \mathcal{G}_i(t, z(t_i^-)), & t \in (t_i, s_i], i = 1, 2, \dots, N, \\ S(t - s_i)\mathcal{G}_i(s_i, z(t_i^-)) & t \in (s_i, t_{i+1}], \ i = 1, 2, \dots, N, \\ + \displaystyle\int_{s_i}^t S(t-s)F(s, z_s)ds. & \end{cases} \tag{3.1}$$

Now, we suppose the following hypotheses on F, $\mathcal{H}$, and $\mathcal{G}_i$ in the aim of demonstrating the existence of solutions

I) The function $F : [0, T] \times \mathcal{PW}^{\eta}_{[-r,0]} \to \mathcal{X}$ satisfies the following conditions

   a) $\|F(t, \nu_1) - F(t, \nu_2)\| \leq \mathcal{N}(\|\nu_1\|, \|\nu_2\|)\|\nu_1 - \nu_2\|, \quad \nu_1, \nu_2 \in \mathcal{PW}^{\eta}_{[-r,0]}$.

   b) $\|F(t, \nu)\| \leq \xi(\|\nu\|), \quad \nu \in \mathcal{PW}^{\eta}_{[-r,0]}$,

   where $\mathcal{N} : I\!R_+ \times I\!R_+ \to I\!R_+$ and $\xi : I\!R_+ \to I\!R_+$ are continuous and non-decreasing functions of their arguments.

II) There exist constants $l_q, h_i > 0, \quad i = 1, 2, \dots, N$ such that
   a) For $t \in [-r, 0]$, $y \in \mathcal{PW}^{\eta}_{[-r,0]}$ the mapping $\mathcal{H}(y)(t)$ is completely continuous.

$$\|\mathcal{H}(y)(t) - \mathcal{H}(v)(t)\|_{\eta} \leq q l_q \|y - v\|_q, \quad \forall y, v \in \mathcal{PW}^{\eta}_{rq}$$

$\|\mathcal{H}(0)\| \leq \gamma_1$, such that $\gamma_1 \geq 0$, and

$$Mql_q \leq Mh_i < \frac{1}{2}$$

b) for all $i = 1, 2, 3, \cdots, N$ and $y, z \in \mathcal{X}^\eta$, $l, t \in (t_i, s_i]$ we have that

$$\|\mathcal{G}_i(l, y) - \mathcal{G}_i(t, z)\|_\eta \leq h_i\{|l - t| + \|y - z\|_\eta\},$$

and $\|\mathcal{G}_i(l, 0)\| \leq \gamma_2$ with $\gamma_2 \geq 0$.

III) For $\mu > 0$, $t_{i+1} \in (0, T]$ and $i = 1, 2, 3, \cdots, N$ the following inequality is satisfied:

$$Mh_i[\|\tilde{\phi}\| + \mu] + M\gamma + \frac{M_\eta (t_{i+1})^{1-\eta}}{1-\eta}\xi(\|\tilde{\phi}\| + \mu) \leq \mu,$$

such that $\gamma = \max(\gamma_1, \gamma_2)$, and the function $\tilde{\phi}$ is given as follows:

$$\tilde{\phi}(t) = \begin{cases} \phi(t), & t \in [-r, 0], \\ S(t)\phi(0), & t \in [0, t_1], \\ 0, & t \in (t_i, s_i], \\ 0, & t \in (s_i, t_{i+1}]. \end{cases} \tag{3.2}$$

IV) The next inequality is satisfied for μ and for $t_{i+1} \in (0, T]$

$$Mh_i + M_\eta \frac{(t_{i+1})^{1-\eta}}{1-\eta}\mathcal{N}(\|\tilde{\phi}\| + \mu, \|\tilde{\phi}\| + \mu) < 1.$$

Note that the spaces where our problem is set, the value of η is $\eta = 1/2$. However, in order to make our result more general, we use throughout this paper the η-notation spaces.

THEOREM 3.1. If the assumptions $I) - III)$ hold, the system (1.1) has at least one mild solution on $[-r, T]$.

Proof Let us assume that $I) - III)$ hold and define the following two operators:

$$\Psi : \mathcal{PW}^\eta_{[-r,T]} \times \mathcal{PW}^\eta_{[-r,T]} \to \mathcal{PW}^\eta_{[-r,T]},$$

and

$$\mathcal{C} : \mathcal{PW}^\eta_{[-r,T]} \to \mathcal{PW}^\eta_{[-r,T]},$$

given as follows

$$\Psi(z, y)(t) = \begin{cases} \phi(t) - \mathcal{H}(z_{\tau_1}, z_{\tau_2}, \ldots, z_{\tau_q})(t), & t \in [-r, 0], \\ y(t) + S(t)\{\phi(0) - \mathcal{H}(z_{\tau_1}, \ldots, z_{\tau_q})(0), & t \in [0, t_1], \\ \mathcal{G}_i(t, z(t_i^-)), & t \in (t_i, s_i], \\ y(t) + S(t - s_i)\mathcal{G}_i(s_i, z(t_i^-)), & t \in (s_i, t_{i+1}], \end{cases}$$

and

$$\mathcal{C}(y)(t) = \begin{cases} \phi(t), & t \in [-r, 0], \\ \displaystyle\int_0^t S(t - s)F(s, y_s)ds, & t \in [0, t_1], \\ 0, & t \in (t_i, s_i], \\ \displaystyle\int_{s_i}^t S(t - s)F(s, y_s)ds, & t \in (s_i, t_{i+1}], \end{cases}$$

with $i = 1, 2, \cdots N$. Take into account that the composition of the previous operators results in the solution of the system (1.1). Hence, our problem for finding at least a mild solution of the system (1.1) is equivalent to solve the fixed-point problem given by the following equation:

$$\Psi(z, \mathcal{C}(z)) = z. \tag{3.3}$$

Next, we make use of Karakostas fixed point theorem (Theorem 2.2) to solve this problem. In this regard, let us verify that Ψ and $\mathcal{C}$ satisfy the conditions required by Theorem 2.2 which is done by the following steps.

Step 1: $\mathcal{C}$ is a continuous operator.
Let $z, y \in \mathcal{PW}^{\eta}_{[-r,T]}$ and $t \in (s_i, t_{i+1}]$ with $i = 0, 1, 2, \cdots N$, by assumption I), we obtain

$$\begin{aligned}
\|\mathcal{C}(z)(t) - \mathcal{C}(y)(t)\|_\eta &\leq \int_{s_i}^{t} \|A^\eta S(t-s)[F(s, z_s) - F(s, y_s)]\| ds \\
&\leq \int_{s_i}^{t} \frac{M_\eta}{(t-s)^\eta} \mathcal{N}(\|z_s\|, \|y_s\|)\|z_s - y_s\| ds \\
&\leq M_\eta \frac{((t_{i+1}) - s_i)^{1-\eta}}{1-\eta} \mathcal{N}(\|z\|, \|y\|)\|z - y\| \\
&\leq M_\eta \frac{(t_{i+1})^{1-\eta}}{1-\eta} \mathcal{N}(\|z\|, \|y\|)\|z - y\| \\
&\leq M_\eta \frac{T^{1-\eta}}{1-\eta} \mathcal{N}(\|z\|, \|y\|)\|z - y\|.
\end{aligned}$$

On the other side, when $t \in (t_i, s_i]$ with $i = 1, 2, 3, \cdots N$ or $t \in [-r, 0]$, we have the following estimate

$$\|\mathcal{C}(z)(t) - \mathcal{C}(y)(t)\|_\eta = 0.$$

Thus, $\mathcal{C}$ is continuous. Moreover, $\mathcal{C}$ is locally Lipschitz.

Step 2: $\mathcal{C}$ maps bounded sets of $\mathcal{PW}^{\eta}_{[-r,T]}$ into bounded set of $\mathcal{PW}^{\eta}_{[-r,T]}$.
Let $B_R = \{z \in \mathcal{PW}^{\eta}_{[-r,T]} : \|z\| \leq R\}$. Thus, to show that $\mathcal{C}$ maps bounded sets of $\mathcal{PW}^{\eta}_{[-r,T]}$ into bounded set of $\mathcal{PW}^{\eta}_{[-r,T]}$ it is enough to show that for any $R > 0$ there exists $\delta > 0$ such that $\|\mathcal{C}(y)\| \leq \delta$ for each $y \in B_R$.
Let $R > 0$ and take $y \in B_R$, then we have the following estimates holds:
If $t \in [-r, 0]$

$$\|\mathcal{C}(y)(t)\|_\eta = \|\phi(t)\|_\eta \leq \|\phi\| = \delta_1,$$

If $t \in (s_i, t_{i+1}]$ with $i = 0, 1, 2, \cdots N$, by assumption $I)$, we obtain

$$\begin{aligned}
\|\mathcal{C}(y)(t)\|_\eta &\leq \int_{s_i}^{t} \|A^\eta S(t-s)F(s, y_s))\| ds \\
&\leq \int_{s_i}^{t} \frac{M_\eta}{(t-s)^\eta} \xi(\|y_s\|) ds \\
&\leq M_\eta \frac{(t_{i+1}-s_i)^{1-\eta}}{1-\eta} \xi(\|y\|) \\
&\leq M_\eta \frac{(t_{i+1})^{1-\eta}}{1-\eta} \xi(R) \\
&\leq M_\eta \frac{T^{1-\eta}}{1-\eta} \xi(R) \\
&= \delta_2.
\end{aligned}$$

If $t \in (t_i, s_i]$ with $i = 1, 2, \cdots N$, $\|\mathcal{C}(y)(t)\|_\eta = 0$. Taking $\delta = \max\{\delta_1, \delta_2\}$, we obtain that $\|\mathcal{C}(y)\| \leq \delta$ for all $y \in B_R$.

Step 3: $\mathcal{C}$ maps bounded sets into equicontinuous sets of $\mathcal{PW}^\eta_{[-r,T]}$.
Let $R > 0$ and take $y \in B_R$ where B_R is given as in the previous step. Then, we have to prove that the family of functions $\mathcal{C}(B_R)$ is equicontinuous on the interval $[-r, T]$.
Let $i = 0, \ldots, N$ and $0 \leq \pi_1 < \pi_2$, then we have the following estimates:
In $[-r, 0]$, $\mathcal{C}(y(t)) = \phi(t)$, which is trivially equicontinuous. On the other side, when $\pi_1, \pi_2 \in (s_i, t_{i+1}]$ with $0 < \pi_1 < \pi_2$, we have

$$\begin{aligned}
\|\mathcal{C}(y)(\pi_2) - \mathcal{C}(y)(\pi_1)\|_\eta &\leq \int_{s_i}^{\pi_1-\varepsilon} \|A^\eta [S(\pi_2 - s) - S(\pi_1 - s)]F(s, y_s)\| ds \\
&\quad + \int_{\pi_1-\varepsilon}^{\pi_1} \|A^\eta [S(\pi_2 - s) - S(\pi_1 - s)]F(s, y_s)\| ds \\
&\quad + \int_{\pi_1}^{\pi_2} \|A^\eta S(\pi_2 - s)F(s, y_s)\| ds \\
&\leq \|S(\pi_2 - \pi_1 + \varepsilon) - S(\varepsilon)\| \int_{s_i}^{\pi_1-\varepsilon} \|A^\eta S(\pi_1 - s - \varepsilon)F(s, y_s)\| ds \\
&\quad + M_\eta \frac{\xi(\|y\|)}{1-\eta} [(\pi_2 - \pi_1 + \varepsilon)^{1-\eta} - (\pi_2 - \pi_1)^{1-\eta} + (\varepsilon)^{1-\eta}] \\
&\quad + M_\eta \frac{\xi(\|y\|)}{1-\eta} (\pi_2 - \pi_1)^{1-\eta}
\end{aligned}$$

$$\begin{aligned}
&\leq \|S(\pi_2 - \pi_1 + \varepsilon) - S(\varepsilon)\| M_\eta \frac{\xi(R)}{1-\eta} (\pi_1 - \varepsilon)^{1-\eta} \\
&\quad + M_\eta \frac{\xi(R)}{1-\eta} [(\pi_2 - \pi_1 + \varepsilon)^{1-\eta} - (\pi_2 - \pi_1)^{1-\eta} + (\varepsilon)^{1-\eta}] \\
&\quad + M_\eta \frac{\xi(R)}{1-\eta} (\pi_2 - \pi_1)^{1-\eta}.
\end{aligned}$$

By the definition of $\mathcal{C}(\cdot)$, for all $\pi_1, \pi_2 \in (t_i, s_i]$, we have

$$\|\mathcal{C}(y)(\pi_2) - \mathcal{C}(y)(\pi_1)\|_\eta = 0.$$

Since the semigroup $(S(t))_{t\geq 0}$ is uniformly continuous away from zero and by the previous estimates, we obtain the equicontinuity of $\mathcal{C}(B_R)$ on $[0,T]$. Indeed, it is equicontinuous on $[-r,T]$.

Step 4: $\mathcal{C}$ maps bounded sets into relatively compact sets in $\mathcal{PW}^{\eta}_{[0,T]}$.
Let $W=\{\mathcal{C}(y): y\in B_R\}$, since $\mathcal{C}(y)(t)=\phi(t)$ for $t\in[-r,0]$, we only consider $W\subset\mathcal{PW}^{\eta}_{[0,T]}$. Hence, by Theorem 2.1 is enough to prove that $\tilde{W}_i=\{\mathcal{C}(\tilde{y}_i): y\in B_R\}$ is relatively compact in $C([t_i,t_{i+1}];\mathcal{X}^{\eta})$ for all $i=0,1,2,\ldots,N$, where $t_0=0$ and $t_{N+1}=T$.

If $t\in(t_i,s_i]$ with $i=1,2,3,\cdots N$, we have

$$W(t)=\Big\{0\Big\}. \tag{3.4}$$

Clearly, the set $\tilde{W}_i$ given in (3.4) is relatively compact in $\mathcal{X}^{\eta}$. Now, if $t\in[s_i,t_{i+1}]$, with $i=0,1,2,\cdots,N$, we consider the set

$$\tilde{U}_i(t)=\tilde{W}_i(t),$$

$$\tilde{U}_i=\Big\{\tilde{u}_i(t)=\int_{s_i}^{t}S(t-s)F(s,\tilde{y}_s)\,ds : y\in B_R\Big\}.$$

Consider $\varepsilon\in(0,t)$, and the set

$$\begin{aligned}\tilde{U}_{i,\varepsilon} &= \Big\{\tilde{u}_{i,\varepsilon}(t)=\int_{s_i}^{t-\varepsilon}S(t-s)F(s,\tilde{y}_s)\,ds : y\in B_R\Big\}\\ &= \Big\{u_{i,\varepsilon}(t)=S(\varepsilon)\int_{s_i}^{t-\varepsilon}S(t-\varepsilon-s)F(s,\tilde{y}_s)ds : y\in B_R\Big\},\end{aligned}$$

By the compactness of $S(\varepsilon)$, we get that $\tilde{U}_{i,\varepsilon}$ is relatively compact in $\mathcal{X}^{n}$. Thus, $\tilde{U}_i(t)$ is relatively compact in $\mathcal{X}^{\eta}$ for all $\epsilon>0$. On the other hand, we have

$$\begin{aligned}\|\tilde{u}_i(t)-\tilde{u}_{i,\varepsilon}(t)\|_{\eta} &\leq \int_{t-\varepsilon}^{t}\|A^{\eta}S(t-s)F(s,\tilde{y}_s)\|ds\\ &\leq \int_{t-\varepsilon}^{t}\frac{M_{\eta}\xi(\|y\|)}{(t-s)^{\eta}}ds\\ &\leq M_{\eta}\frac{\xi(\|y\|)}{1-\eta}(\varepsilon)^{1-\eta}\\ &\leq M_{\eta}\frac{\xi(R)}{1-\eta}(\varepsilon)^{1-\eta}.\end{aligned}$$

Hence, we have a sequence of relatively compact sets arbitrarily close to $\tilde{U}_i$, therefore $\tilde{U}_i$ is relatively compact in $\mathcal{X}^{\eta}$.
Step 5:The family $\{\Psi(\cdot,y): y\in\overline{\mathcal{C}(D)}\}$ is equicontractive
D is a closed and convex set defined in (3.5). Let $v,w\in\mathcal{PW}^{\eta}_{[-r,T]}$. For $t\in[-r,0]$ we have

$$\begin{aligned}\|\Psi(v,\mathcal{C}(y))(t)-\Psi(w,\mathcal{C}(y))(t)\|_{\eta} &\leq \|\mathcal{H}(v_{\tau_1},v_{\tau_2},\ldots,v_{\tau_q})(t)-\mathcal{H}(w_{\tau_1},w_{\tau_2},\ldots,w_{\tau_q})(t)\|\\ &\leq ql_q\|v-w\|.\end{aligned}$$

For $t\in[0,t_1]$, we have

$$\begin{aligned}\|\Psi(v,\mathcal{C}(y))(t)-\Psi(w,\mathcal{C}(y))(t)\|_{\eta} &\leq \|A^{\eta}S(t)[\mathcal{H}(v_{\tau_1},v_{\tau_2},\ldots,v_{\tau_q})(t)-\mathcal{H}(w_{\tau_1},w_{\tau_2},\ldots,w_{\tau_q})(t)]\|\\ &\leq Mql_q\|v-w\|.\end{aligned}$$

For $t \in (t_i, s_i]$ with $i = 1, 2, 3, \cdots N$, we get

$$\begin{aligned}\|\Psi(v, \mathcal{C}(y))(t) - \Psi(w, \mathcal{C}(y))(t)\|_\eta &\leq \|\mathcal{G}_i(t, v(t_i^-)) - \mathcal{G}_i(t, w(t_i^-))\|_\eta \\ &\leq h_i\|v - w\|.\end{aligned}$$

For $t \in (s_i, t_{i+1}]$ with $i = 1, 2, 3, \cdots N$, we have the following estimate

$$\begin{aligned}\|\Psi(v, \mathcal{C}(y))(t) - \Psi(w, \mathcal{C}(y))(t)\|_\eta &\leq \|A^\eta S(t - s_i)[\mathcal{G}_i(s_i, v(t_i^-)) - \mathcal{G}_i(s_i, w(t_i^-))]\| \\ &\leq Mh_i\|v - w\|.\end{aligned}$$

Since $ql_q \leq Mql_q \leq h_i \leq Mh_i$, by assumption $II)$, we obtain

$$\|\Psi(v, \mathcal{C}(y))(t) - \Psi(w, \mathcal{C}(y))(t)\| \leq Mh_i\|v - w\| \leq \frac{1}{2}\|v - w\| \quad \forall t \in [-r, T].$$

Thus, it is a contraction independently of $y \in \overline{\mathcal{C}(D)}$.
Step 6: $\Psi(\cdot, \mathcal{C})D(\mu, T, \phi) \subset D(\mu, T, \phi)$.
Consider the following closed and convex set

$$D = D(\mu, T, \phi) = \{y \in \mathcal{PW}^\eta_{[-r,T]} : \|y - \tilde{\phi}\| \leq \mu\}, \tag{3.5}$$

Let us prove that,

$$\Psi(\cdot, \mathcal{C})D(\mu, T, \phi) \subset D(\mu, T, \phi).$$

Let $z \in D(\mu, T, \phi)$ and $t \in [-r, 0]$. Then,

$$\Psi(z, \mathcal{C}(z))(t) = \phi(t) - \mathcal{H}(z_{\tau_1}, z_{\tau_2}, \ldots, z_{\tau_q})(t).$$

For $t \in [0, t_1]$, we have

$$\Psi(z, \mathcal{C}(z))(t) = S(t)\{\phi(0) - \mathcal{H}(z_{\tau_1}, \ldots, z_{\tau_q})(0)\} + \int_0^t S(t - s)F(s, z_s)ds.$$

For $t \in (t_i, s_i]$, $i = 1, 2, \cdots, N$, we have

$$\Psi(z, \mathcal{C}(z))(t) = \mathcal{G}_i(t, z(t_i^-)).$$

For $t \in (s_i, t_{i+1}]$, $i = 1, 2, \cdots, N$, we have

$$\Psi(z, \mathcal{C}(z))(t) = S(t - s_i)\mathcal{G}_i(s_i, z(t_i^-)) + \int_{s_i}^t S(t - s)F(s, z_s)ds.$$

Hence, using the assumption $III)$ in each sub-interval on $[-r, T]$, we obtain the following estimates

If $t \in [-r, 0]$, we get

$$\begin{aligned}\|\Psi(z, \mathcal{C}(z))(t) - \tilde{\phi}(t)\|_\eta &= \|\mathcal{H}(z_{\tau_1}, z_{\tau_2}, \ldots, z_{\tau_q})(t)\|_\eta \\ &\leq ql_q\|z\| + \gamma_1 \leq ql_q(\|\tilde{\phi}\| + \mu) + \gamma_1 < \mu.\end{aligned}$$

If $t \in [0, t_1]$, we get

$$\begin{aligned}
\|\Psi(z,\mathcal{C}(z))(t) - \tilde{\phi}(t)\|_\eta &\leq M\|\mathcal{H}(z_{\tau_1}, z_{\tau_2}, \ldots, z_{\tau_q})(0)\|_\eta + \int_0^t \|A^\eta S(t-s)F(s,z_s)\|ds \\
&\leq Mql_q\|z\| + M\gamma_1 + \int_0^t \frac{M_\eta}{(t-s)^\eta}\|F(s,z_s)\|ds \\
&\leq Mql_q(\|\tilde{\phi}\| + \mu) + M\gamma_1 + \frac{M_\eta (t_1)^{1-\eta}}{1-\eta}\xi(\|z\|) \\
&\leq Mql_q(\|\tilde{\phi}\| + \mu) + M\gamma_1 + \frac{M_\eta (t_1)^{1-\eta}}{1-\eta}\xi(\|\tilde{\phi}\| + \mu) \\
&< \mu.
\end{aligned}$$

If $t \in (t_i, s_i]$,$i = 1, 2, \cdots, N$, we get

$$\begin{aligned}
\|\Psi(z,\mathcal{C}(z))(t) - \tilde{\phi}(t)\|_\eta &\leq \|\mathcal{G}_i(t, z(t_i^-))\|_\eta \\
&\leq h_i\|z\| + \gamma_2 \\
&\leq h_i(\|\tilde{\phi}\| + \mu) + \gamma_2 \\
&< \mu.
\end{aligned}$$

If $t \in (s_i, t_{i+1}]$, $i = 1, 2, \cdots, N$, we get

$$\begin{aligned}
\|\Psi(z,\mathcal{C}(z))(t) - \tilde{\phi}(t)\|_\eta &\leq \|A^\eta S(t-s_i)[\mathcal{G}_i(s_i, z(s_i))]\| + \int_{s_i}^t \|A^\eta S(t-s)F(s,z_s)\|ds \\
&\leq M\|A^\eta[\mathcal{G}_i(s_i, z(s_i))]\| + \int_{s_i}^t \frac{M_\eta}{(t-s)^\eta}\|F(s,z_s)\|ds \\
&\leq M(h_i\|z\|) + M\gamma_2 + \frac{M_\eta (t_{i+1} - s_i)^{1-\eta}}{1-\eta}\xi(\|z\|) \\
&\leq Mh_i(\|\tilde{\phi}\| + \mu) + M\gamma_2 + \frac{M_\eta (t_{i+1})^{1-\eta}}{1-\eta}\xi(\|\tilde{\phi}\| + \mu) \\
&\leq \mu.
\end{aligned}$$

Therefore, The conditions of Theorem2.2 are satisfied, it follows that the equation $\Psi(z, \mathcal{C}(z)) = z$ has at least one mild solution, which means that at least there exist a solution for the system (1.1).

Next, we present the following uniqueness result

Theorem 3.2. Additionally to the assumptions of Theorem 3.1, let us assume that $IV)$ holds. Then the system (1.1) has an unique solution on $[-r, T]$.

Proof . Suppose that z_1 and z_2 are two solutions to (1.1).
For $t \in [-r, 0]$, we have

$$\|z_2(t) - z_1(t)\|_\eta \leq \|\mathcal{H}(z_{2,\tau_1}, z_{2,\tau_2}, \ldots, z_{2,\tau_q})(t) - \mathcal{H}(z_{1,\tau_1}, z_{1,\tau_2}, \ldots, z_{1,\tau_q})(t)\|_\eta \leq ql_q\|z_2 - z_1\|_\eta.$$

Additionally, $t \in (t_i, s_i]$ with $i = 1, 2, 3, \cdots N$, we have

$$\|z_2(t) - z_1(t)\|_\eta \leq \|\mathcal{G}_i(t, z_2(t)) - \mathcal{G}_i(t, z_1(t))\|_\eta \leq h_i\|z_2 - z_1\|_\eta.$$

And, $t \in [0, t_1]$, we get

$$\begin{aligned}\|z_2(t) - z_1(t)\|_\eta &\leq \|A^\eta S(t)[\mathcal{H}(z_{2,\tau_1}, z_{2,\tau_2}, \ldots, z_{2,\tau_q})(t) - \mathcal{H}(z_{1,\tau_1}, z_{1,\tau_2}, \ldots, z_{1,\tau_q})(t)]\| \\ &\quad + \int_0^t \|A^\eta S(t-s)[F(s, z_{2,s}) - F(s, z_{1,s})]\| ds \\ &\leq \left(Mql_q + M_\eta \frac{(t_1)^{1-\eta}}{1-\eta} \mathcal{N}(\|\tilde{\phi}\| + \mu, \|\tilde{\phi}\| + \mu)\right) \|z_2 - z_1\|. \end{aligned} \tag{3.6}$$

Lastly, if $t \in (s_i, t_{i+1}]$ with $i = 1, 2, 3, \cdots N$, similarly, it yields

$$\begin{aligned}\|z_2(t) - z_1(t)\|_\eta &\leq \|A^\eta S(t - s_i)[\mathcal{G}_i(s_i, z_2(s_i)) - \mathcal{G}_i(s_i, z_1(s_i))]\| \\ &\quad + \int_{s_i}^t \|A^\eta S(t-s)[F(s, z_{2,s}) - F(s, z_{1,s})]\| ds \\ &\leq \left(Mh_i + M_\eta \frac{(t_{i+1})^{1-\eta}}{1-\eta} \mathcal{N}(\|\tilde{\phi}\| + \mu, \|\tilde{\phi}\| + \mu)\right) \|z_2 - z_1\| \end{aligned} \tag{3.7}$$

Combining (3.6), (3.7) and using assumptions $II)$ and $IV)$ we obtain $z_1 = z_2$.

4. Application To The Benjamin-Bona-Mahony Equation

In this section we shall apply our previous results to the **BBM** equation with non-instantaneous impulses, delay and nonlocal conditions (1.2). In this regard, we first present the abstract formulation of the problem, then we establish the existence and uniqueness of the solution. Throughout this paper we use the following notations: $0 < \lambda_1 < \lambda_2 < ... < \lambda_n \to \infty$ are the eigenvalues of the operator $\tilde{A}\psi = -\psi_{xx}$ where $\tilde{A}$ is a sectorial operator. According to [17] $-\tilde{A} : \mathcal{D}(\tilde{A}) \subset \mathcal{X} \to \mathcal{X}$ is the generator of a Strongly continuous analytic semigroup $(S(t))_{t\geq 0}$ on $\mathcal{X}$. Moreover, the operator $\tilde{A}$ and the semigroup $(S(t))_{t\geq 0}$ can be represented as follows:

$$\tilde{A}z = \sum_{n=1}^{\infty} \lambda_n \langle z, \phi_n \rangle \phi_n, \quad z \in \mathcal{X},$$

where $\{\phi_n\}$ is an orthonormal set of eigenvectors of $\tilde{A}$ and $\langle \cdot, \cdot \rangle$ is the inner product in $\mathcal{X}$. Thus, the strongly continuous semigroup $(S(t))_{t\geq 0}$ generated by $-\tilde{A}$ is compact and given by

$$S(t)z = \sum_{n=1}^{\infty} e^{-n^2 t} \langle z, \phi_n \rangle \phi_n, \quad z \in \mathcal{X}.$$

As a consequence, we have the following estimate:

$$\| S(t) \| \leq e^{-t}, \quad t \geq 0.$$

Consequently, system (1.2) can be written as an abstract functional differential system with non-instantaneous impulses in $\mathcal{X}$:

$$\begin{cases} z' + a\tilde{A}z' + b\tilde{A}z = \mathcal{F}(t, z_t), & t \in \quad \cup_{i=0}^{N} (s_i, t_{i+1}], \\ z(s) + \mathcal{H}(z_{\tau_1}, \ldots, z_{\tau_q})(s) = \phi(s), & t \in \quad [-r, 0], \\ z(t) = \mathcal{G}_i(t, z(t_i^-)), & t \in \quad \cup_{i=1}^{N} (t_i, s_i], \end{cases} \tag{4.1}$$

where the function $z_t \in C([-r, 0]; \mathcal{X}^{1/2})$ is defined by $z_t(s) = z(t+s)$ with $s \in [-r, 0]$ and $\mathcal{G}_i : (t_i, s_i] \times \mathcal{X}^{1/2} \longrightarrow \mathcal{X}^{1/2}$, $\mathcal{H} : \mathcal{PW}_{rq}^{1/2} \longrightarrow \mathcal{PW}_{[-r,0]}^{1/2}$ and $\mathcal{F} : [0, T] \times \mathcal{PW}_{[-r,0]}^{1/2} \longrightarrow \mathcal{X}$, are defined as follows

$$\mathcal{F}(t,\phi)(x) = \phi(-r,x)\phi_x(-r,x) + f(t,\phi(-r,x)) \qquad \text{in } [0,\pi],$$
$$\mathcal{H}(z_{\tau_1},\ldots,z_{\tau_q})(s)(x) = h(z(\tau_1+s,x),\ldots,z(\tau_q+s,x)) \qquad \text{in } [0,\pi],$$
$$\mathcal{G}_i(t,z(t_i^-))(x) = G_i(t,z(t_i^-,x)) \quad \textit{for} \quad i=1,\ldots,N \qquad \text{in } [0,\pi].$$

Note that the spaces here are the same spaces defined in the previous section with $\eta = 1/2$. Now, we have that $(I + a\tilde{A}) = a(\tilde{A} - (-\frac{1}{a})I)$ and $-\frac{1}{a} \in \rho(\tilde{A})$, where $\rho(\tilde{A})$ is the resolvent set of $\tilde{A}$, then the operator:

$$I + a\tilde{A} : \mathcal{D}(\tilde{A}) \to \mathcal{X}$$

is invertible with bounded inverse

$$(I + a\tilde{A})^{-1} : \mathcal{X} \to \mathcal{D}(\tilde{A}).$$

Hence, for $z \in \mathcal{X}$ and $t \in [0,T]$ the system (4.1) can be presented as follows:

$$\begin{cases} z' + b(I+a\tilde{A})^{-1}\tilde{A}z = (I+a\tilde{A})^{-1}\mathcal{F}(t,z_t), & t \in \quad \cup_{i=0}^{N}(s_i,t_{i+1}], \\ z(s) + \mathcal{H}(z_{\tau_1},\ldots,z_{\tau_q})(s) = \phi(s), & t \in \quad [-r,0], \\ z(t) = \mathcal{G}_i(t,z(t_i^-)), & t \in \quad \cup_{i=1}^{N}(t_i,s_i], \end{cases} \tag{4.2}$$

Additionally, $(I + a\tilde{A})$ and $(I + a\tilde{A})^{-1}$ can be written in terms of the eigenvalues of $\tilde{A}$

$$(I + a\tilde{A})z = \sum_{n=1}^{\infty}(1 + a\lambda_n)\, E_n z,$$
$$(I + a\tilde{A})^{-1}z = \sum_{n=1}^{\infty}\frac{1}{1 + a\lambda_n}\, E_n z,$$

where $\{E_n\}$ is a set of orthogonal projections in $\mathcal{X}$ given by $E_n z = \langle z, \phi_n\rangle\phi_n$ such that $z = \sum_{n=1}^{\infty} E_n z$. Hence, if we put $\mathbb{B} = (I + a\tilde{A})^{-1}$ and $F(t,\phi) = (I + a\tilde{A})^{-1}\mathcal{F}(t,\phi)$, the systems (4.2) can be written as follows

$$\begin{cases} z' + b\mathbb{B}\tilde{A}z = F(t,z_t), & t \in \quad \cup_{i=0}^{N}(s_i,t_{i+1}], \\ z(s) + \mathcal{H}(z_{\tau_1},\ldots,z_{\tau_q})(s) = \phi(s), & t \in \quad [-r,0], \\ z(t) = \mathcal{G}_i(t,z(t_i^-)), & t \in \quad \cup_{i=1}^{N}(t_i,s_i], \end{cases} \tag{4.3}$$

In what follows, we formulate a simple proposition.

Proposition 4.1. [19] The operators $b\mathbb{B}\tilde{A}$ is the generator of astrongly continuous analytic semigroup $S(t) = e^{-b\mathbb{B}\tilde{A}t}$, given by the following expressions

$$b\mathbb{B}\tilde{A}z = \sum_{n=1}^{\infty}\frac{b\lambda_n}{1 + a\lambda_n}E_n z,$$
$$S(t)z = e^{-b\mathbb{B}\tilde{A}t}z = \sum_{n=1}^{\infty} e^{\frac{-b\lambda_n}{1+a\lambda_n}t}E_n z.$$

Moreover, the following estimate holds

$$\| S(t) \| \leq e^{-\beta t}, \quad t \geq 0,$$

where

$$\beta = \inf_{n\geq 1}\left\{\frac{b\,\lambda_n}{1 + a\,\lambda_n}\right\} = \frac{b}{1+a}.$$

Then, the system (4.3) can be presented as an abstract Cauchy problem with Non-instantaneous impulses

$$\begin{cases} z' = -\mathbb{A}z + F(t, z_t), & t \in \quad \cup_{i=0}^{N} (s_i, t_{i+1}], \\ z(s) + \mathcal{H}(z_{\tau_1}, \dots, z_{\tau_q})(s) = \phi(s), & t \in \quad [-r, 0], \\ z(t) = \mathcal{G}_i(t, z(t_i^-)), & t \in \quad \cup_{i=1}^{N} (t_i, s_i]. \end{cases} \tag{4.4}$$

with $\mathbb{A} = b\mathbb{B}\tilde{A}$.

Finally, if we assume that the functions $F, \mathcal{H}$ and $\mathcal{G}_i$ satisfy the assumptions $I) - IV)$. Then the system (4.4) has only one mild solution on $[-r, T]$. Moreover, we have can prove the following result:

Proposition 4.2. The functions F defined above satisfy:

$$\|F(t, \phi_1) - F(t, \phi_2)\| \leq K\{\|\phi_1 - \phi_2\|_{\mathcal{C}} + L\}\|\phi_1 - \phi_2\|_{\mathcal{C}},$$
$$\|F(t, \phi)\| \leq K\|\phi\|_{\mathcal{C}}^2 + 4K\|a\|_{L_\infty}\|\phi\|_{\mathcal{C}} + 4K\|b\|_{L_\infty}\sqrt{\mu(\Omega)},$$

where

$$|f(t, z) - f(t, w)| \leq L|z - w|, \quad t \in [0, \tau], z, w \in I\!R,$$

$K = \|(I + a\tilde{A})^{-1}\|$ and $\sqrt{\mu(\Omega)} = 1$.

5. Controllability Result

Analogous to the abstract formulation of the existence problem, the control problem (1.3) can be presented as follows

$$\begin{cases} z' = -\mathbb{A}z + \mathcal{B}u(t) + F(t, z_t, u), & t \in \quad \cup_{i=0}^{N} (s_i, t_{i+1}], \\ z(s) + \mathcal{H}(z_{\tau_1}, \dots, z_{\tau_q})(s) = \phi(s), & t \in \quad [-r, 0], \\ z(t) = \mathcal{G}_i(t, z(t_i^-), u(t)), & t \in \quad \cup_{i=1}^{N} (t_i, s_i]. \end{cases} \tag{5.1}$$

where $\mathbb{A} = b\mathbb{B}A$ and $\mathcal{B} = \mathbb{B}B_\omega$.
Before studying the controllability of the nonlinear system (5.1), we begin by stating the controllability of the unperterbed linear system. We notice that for an arbitrary initial state $w_0 \in \mathcal{X}$ and $u \in L^2(0, T; U)$, $U = L^2(0, \pi)$ the initial value problem

$$\begin{cases} w' = -\mathbb{A}w + \mathcal{B}u(t), & t \in (t_0, T], \\ w(t_0) = w_0, \end{cases} \tag{5.2}$$

admits only one mild solution given by

$$w(t) = w(t, t_0, w_0, u) = S(t - t_0)w_0 + \int_{t_0}^{t} S(t - s)\mathcal{B}u(s)ds, \;\; t \in [t_0, T], \quad 0 \leq t_0 \leq T. \tag{5.3}$$

Definition 5.1. (Approximate Controllability) The system (5.1) is said to be approximately controllable on $[t_0, T]$, if for all $z_0, z^1 \in \mathcal{X}$, an initial state and a final state respectively, and $\varepsilon > 0$ there exists $u \in L^2(0, T; U)$ such that the mild solution $z(t)$ of (5.1) corresponding to u verifies

$$\left\|z(T) - z^1\right\|_{\mathcal{X}} < \varepsilon,$$

where

$$\left\|z(T) - z^1\right\|_{\mathcal{X}} = \left(\int_0^\pi \left|z(T, x) - z^1(x)\right|^2 \mathrm{d}x\right)^{1/2}.$$

Definition 5.2. For $\tau \in [0, T)$ we define the controllability map for the system (5.2) as follows: $G_{T\tau} : L^2(T - \tau, T; U)) \to \mathcal{X}$ defined by

$$G_{T\tau}u = \int_{T-\tau}^{T} S(T - s)\mathcal{B}u(s)ds, \quad u \in L^2(T - \tau, T; U), \tag{5.4}$$

Its adjoint operator $G^*_{T\tau} : \mathcal{X} \to L^2(T-\tau, T; U)$ is given by

$$(G^*_{T\tau} z)(t) = \mathcal{B}^* S^*(T-t)z, \quad t \in [T-\tau, T]. \tag{5.5}$$

The Gramian controllability operator is given by:

$$Q_{T\tau} = G_{T\tau} G^*_{T\tau} = \int_{T-\tau}^{T} S(T-t)\mathcal{B}\mathcal{B}^* S^*(T-t)dt. \tag{5.6}$$

Lemma 5.3. *The following statements are equivalent to the approximate controllability of the linear system* (5.2) *on* $[T-\tau, T]$,

a) $\overline{\text{Rang}(G_{T\tau})} = \mathcal{X}$.
b) $Ker(G^*_{T\tau}) = \{0\}$.
c) $\langle Q_{T\tau} z, z\rangle > 0$, $z \neq 0$ *in* $\mathcal{X}$.
d) $\lim_{\alpha \to 0^+} \alpha(\alpha I + Q_{T\tau})^{-1} z = 0$.
e) For all $z \in \mathcal{X}$ *we have* $G_{T\tau} u_\alpha = z - \alpha(\alpha I + Q_{T\tau})^{-1} z$, *where*

$$u_\alpha = G^*_{T\tau}(\alpha I + Q_{T\tau})^{-1} z, \quad \alpha \in (0, 1].$$

So, $\lim_{\alpha \to 0} G_{T\tau} u_\alpha = z$ *and the error* $E_{T\tau} z$ *of this approximation is given by the formula*

$$E_{T\tau} z = \alpha(\alpha I + Q_{T\tau})^{-1} z, \quad \alpha \in (0, 1].$$

REMARK 5.1. The foregoing Lemma implies that the family of linear operators $\Psi_\alpha : Z \to W$, defined for $0 < \alpha \leq 1$ by

$$\Psi_\alpha z = G^*_{T\tau}(\alpha I + Q_{T\tau})^{-1} z, \tag{5.7}$$

is an approximate inverse for the right of the operator $G_{T\tau}$, in the sense that

$$\lim_{\alpha \to 0} G_{T\tau} \Psi_\alpha = I. \tag{5.8}$$

in the strong topology.

The above characterization holds in general for a linear bounded operator $G : W \to Z$ between Hilbert spaces W and Z (see [5, 6, 20]).

Lemma 5.4. *[16] Given an initial state* w_0 *and a final state* z^* *we can find a sequence of controls* $\{u^\delta_\beta\}_{0<\beta\leq 1} \subset L^2(T-\delta, T; U)$

$$u^\delta_\beta = G^*_{T\delta}(\beta I + G_{T\delta} G^*_{T\delta})^{-1}(z^* - S(T)w_0), \quad \beta \in (0, 1],$$

such that the solutions $\omega(t) = \omega(t, T-\delta, w_0, u^\delta_\beta)$ *of the initial value problem*

$$\begin{cases} \omega' = -\mathbb{A}\omega + \mathcal{B}u^\delta_\beta(t), & W \in Z, \quad t > 0, \\ \omega(T-\tau) = w_0, \end{cases} \tag{5.9}$$

satisfies

$$\lim_{\beta \to 0^+} \omega(T, T-\delta, \omega_0, u^\delta_\beta) = z^*.$$

e.i.,

$$\lim_{\alpha \to 0^+} \omega(T) = \lim_{\alpha \to 0^+} \left\{ S(T)w_0 + \int_{T-\delta}^{T} S(T-s)\mathcal{B}u^\delta_\beta(s)ds \right\} = z^*.$$

Now, we are ready to prove the interior approximate controllability of the **BBM** equation with non instantaneous impulses, delay and non-local conditions (5.1). Our main assumptions will be the following

V) According to the above section, we suppose that F, $\mathcal{H}$, and $\mathcal{G}_i$, $i = 1, \ldots, N$ are smooth enough, such that for all $\phi \in \mathcal{PW}^\eta_{[-r,0]}$ and $u \in L^2(0, T; U)$ the problem (5.1) has only one mild solution on

$[-r,T]$. And there exists $\varphi \in L^2(I\!R^+)$ which for all $(t,\phi,u) \in [0,T] \times \mathcal{PW}^{\eta}_{[-r,0]} \times L^2(0,T;U)$, the following inequality holds

$$\|F(t,\Phi,u)\|_{\mathcal{X}} \le \varphi(\|\Phi(-r)\|_{\mathcal{X}}), \tag{5.10}$$

Definition 5.5. For all $\phi \in \mathcal{PW}^{\eta}_{[-r,0]}$ and $u \in L^2(0,T;U)$ a function $z(\cdot) \in \mathcal{PW}_{[-r,T]}$ is a mild solution for the system (5.1) if it satisfies the following integral-algebraic equation

$$z(t) = \begin{cases} \phi(t) - \mathcal{H}(z_{\tau_1},\ldots,z_{\tau_q})(t), & t \in [-r,0], \\ S(t)\{\phi(0) - \mathcal{H}(z_{\tau_1},\ldots,z_{\tau_q})(0)\} & t \in [0,t_1], \\ + \displaystyle\int_0^t S(t-s)\Big(\mathcal{B}u(s) + F(s,z_s(-r),u(s))\Big)ds, & \\ \mathcal{G}_i(t,z(t_i^-),u(t)), & t \in (t_i,s_i], i=1,2,\ldots,N, \\ S(t-s_i)\mathcal{G}_i(s_i,z(t_i^-),u(s_i)) & t \in (s_i,t_{i+1}], \, i=1,2,\ldots,N, \\ + \displaystyle\int_{s_i}^t S(t-s)\Big(\mathcal{B}u(s) + F(s,z_s(-r),u(s))\Big)ds. & \end{cases} \tag{5.11}$$

By the previous definition the evaluation in T of the mild solution for the system (5.1) leads us to the following expression

$$z(T) = S(T-s_N)\mathcal{G}_N(s_N,z(t_N^-),u(s_N)) + \int_{s_N}^T S(T-s)\Big(\mathcal{B}u(s) + F(s,z_s(-r),u(s))\Big)\mathrm{d}s$$

THEOREM 5.1. If the function F satisfies assumption (5.10), the system (5.1) is approximately controllable on $[0,T]$. Precisely, Given $\phi \in \mathcal{PW}^{\eta}_{[-r,0]}$, a final state z^* and $\epsilon > 0$, there exists $0 < \delta < \min\{T-s_N, T-r, r, \frac{\epsilon}{2Q}\}$ small enough,were $Q = \min_{t\in[0,T]}\{\|S(t)\|\varphi(\|z(t)\|)\}$, such that there exists a control $\tilde{u}^\epsilon \in L^2(0,T;U)$, in such a way that the corresponding solution z^ϵ of (5.1) satisfies

$$\|z^\epsilon(T) - z^*\| \le \epsilon.$$

Proof. Consider any $u \in L^2(0,T;U)$ and the corresponding solution $z(t) = z(t,\phi,u)$ of the initial value problem (5.1). For $\beta \in (0,1]$ we define the control $\tilde{u}^\epsilon \in L^2(0,\tau;U)$ as follows

$$\tilde{u}^\epsilon(t) = \begin{cases} u(t), & \text{if} \quad 0 \le t \le T-\delta, \\ u_\delta(t), & \text{if} \quad T-\delta < t \le T, \end{cases} \tag{5.12}$$

where u_δ is the control giving by **Lemma**5.4 that steers the unpeturbed linear system (5.2) from the initial state $z(T-\delta)$ to the final state z^* on $[T-\delta;T]$. The corresponding solution $z^\epsilon = z(t,s_N,\tilde{u}^\epsilon)$ of the initial value problem (5.1) at time T can be written as follows

$$\begin{aligned} z^\epsilon(T) &= S(T-s_N)\mathcal{G}_N(s_N,z^\epsilon(t_N^-),\tilde{u}^\epsilon(s_N)) \\ &\quad + \int_{s_N}^T S(T-s)\Big(\mathcal{B}\tilde{u}^\epsilon(s) + F(s,z_s^\epsilon(-r),\tilde{u}_\beta^\delta(s))\Big)\mathrm{d}s \\ &= S(\delta)\Big\{S(T-s_N-\delta)\,\mathcal{G}_N(s_N,z^\epsilon(t_N^-),u(s_N)) + \int_{s_N}^{T-\delta} S(T-s-\delta)\Big[\mathcal{B}u(s) \\ &\quad + F(s,z_s^\epsilon(-r),u(s))\Big]\mathrm{d}s\Big\} \\ &\quad + \int_{T-\delta}^T S(T-s)\Big[\mathcal{B}u_\delta(s) + F(s,z_s^\epsilon(-r),u_\delta(s))\Big]\mathrm{d}s, \end{aligned}$$

then

$$z^{\epsilon}(T) \;=\; S(\delta)z(T-\delta)+\int_{T-\delta}^{T} S(T-s)\Big[\mathcal{B}u_{\delta}(s)+F(s,z_s^{\epsilon}(-r),u_{\delta}(s))\Big]\mathrm{d}s.$$

On the other hand, the corresponding solution $w^{\delta}(t)=w\left(t,T-\delta,z(T-\delta),u_{\delta}\right)$ of the unperterbed linear system (5.2) at time T is given by

$$w^{\delta}(T)=S(\delta)z(T-\delta)+\int_{T-\delta}^{T} S(T-s)\mathcal{B}u_{\delta}(s)\mathrm{d}s, \tag{5.13}$$

therefore,

$$z^{\epsilon}(T)-w^{\delta}(T)=\int_{T-\delta}^{T} S(T-s)F(s,z^{\epsilon}(s-r),u_{\delta}(s))\mathrm{d}s,$$

by the assumption (5.10), we obtain

$$\begin{aligned}\left\|z^{\epsilon}(T)-w^{\delta}(T)\right\| &\leq \int_{T-\delta}^{T} \|S(T-s)\|\,\|F(s,z^{\epsilon}(s-r),u_{\delta}(s))\|\mathrm{d}s,\\ &\leq \int_{T-\delta}^{T} \|S(T-s)\|\,\varphi(\|z^{\epsilon}(s-r)\|)\mathrm{d}s.\end{aligned}$$

Now, since $0<\delta<r$ and $T-\delta\leq s\leq T$, then $0\leq s-r\leq T-r<T-\delta$ then

$$z^{\epsilon}(s-r)=z(s-r).$$

Therefore, for such small δ, we obtain

$$\begin{aligned}\left\|z^{\epsilon}(T)-w^{\delta}(T)\right\| &\leq \int_{T-\delta}^{T} \|S(T-s)\|\,\varphi(\|z(s-r)\|)\mathrm{d}s\\ &\leq \delta Q<\frac{\varepsilon}{2}.\end{aligned}$$

Furthermore, from **Lemma** 5.4 there exist a solution of the linear system (5.2) w^{δ} such that

$$\|w^{\delta}(T)-z^{*}\|\leq \frac{\epsilon}{2}.$$

Therefore,

$$\begin{aligned}\|z^{\epsilon}(T)-z^{*}\| &\leq \left\|z^{\epsilon}(T)-w^{\delta}(T)\right\|+\left\|w^{\delta}(T)-z^{*}\right\|\\ &< \frac{\varepsilon}{2}+\frac{\varepsilon}{2}=\varepsilon.\end{aligned}$$

This completes the proof of the Theorem. □

6. Conclusions and Remarks

In this work, the existence and uniqueness results were proved for a semilinear non-instantaneous impulsive system with delay and non-local conditions. As an application we considered the Benjamin-Bona-Mahony equation (1.2). The used technique is based on Karakostas' fixed-point theorem, which merited transforming the problem of existence of solutions into the problem of existence of a fixed point for a certain operator that satisfies a specified conditions. Also, the approximate controllability for the system (1.3) were established using Bashirov technique [2].

In real-life problems, it is not possible to cover each aspect of the dynamical system separately with instantaneous or non-simultaneous impulses. This is the main reason behind dealing with both impulses in one system, see for instance [11]. It would be of much interest to investigate our system in this case.

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