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# ASYMPTOTIC BEHAVIOR FOR A VISCOELASTIC WAVE EQUATION WITH PAST HISTORY, DISTRIBUTED DELAY AND BALAKRISHNAN-TAYLOR DAMPING TERMS

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Abstract. A nonlinear viscoelastic wave equation with Balakrishnan-Taylor damping, infinite memory and distributed delay terms is studed. By considered the kernel $h:\mathbb{R}_+\to\mathbb{R}_+$ satisfying

$$h(t)\le \xi(t)H(h(t)),\quad \forall t\in\mathbb{R}_+,$$

where $\xi$ and $H$ are functions satisfying some specific properties, and under this very general hypothesis on the behavior of $h$ at infinity and by drop the boundedness hypothesis in the history data, we show the stability of the system.

## 1. Introduction

Let $\mathcal{H}=\Omega\times(\tau_1,\tau_2)\times(0,\infty)$, in the present work, we consider the following wave equation

$$\begin{cases} u_{tt}-\left(\zeta_0+\zeta_1\|\nabla u\|_2^2+\sigma(\nabla u,\nabla u_t)_{L^2(\Omega)}\right)\Delta u(t)+\displaystyle\int_0^\infty h(\varrho)\Delta u(t-\varrho)d\varrho \\ \quad +\beta_1|u_t(t)|^{m-2}u_t(t)+\displaystyle\int_{\tau_1}^{\tau_2}|\beta_2(s)||u_t(t-s)|^{m-2}u_t(t-s)ds=0. \\ u(x,-t)=u_0(x),\ \ u_t(x,0)=u_1(x),\ \ in\ \ \Omega \\ u_t(x,-t)=f_0(x,t),\ \ in\ \ \Omega\times(0,\tau_2) \\ u(x,t)=0,\ \ in\ \ \partial\Omega\times(0,\infty) \end{cases}\tag{1.1}$$

where $\Omega\in\mathbb{R}^N$ is a bounded domain with sufficiently smooth boundary $\partial\Omega$. $\zeta_0,\zeta_1,\sigma,\beta_1$ are positive constants, $m\ge 1$ for $N=1,2$, and $1<m\le\frac{N+2}{N-2}$ for $N\ge 3$.
$\tau_1<\tau_2$ are non-negative constants such that $\beta_2:[\tau_1,\tau_2]\to\mathbb{R}$ represents distributive time delay, $h$ is positive functions.

Physically, the relationship between the stress and strain history in the beam inspired by Boltzmann theory called viscoelastic damping term, where the kernel of the term of memory (finite or infinite) is the function $h$. There are many works that talk about this topic with a lot of new and innovative results, especially the hypotheses on the kernel and the initial conditions. See ([2],[6],[9],[12],[13],[14],[16],[17],[18],[19],[21],[23],[25],[26]).

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In [4], Balakrishnan and Taylor they proposed a new model of damping called it the Balakrishnan-Taylor damping , as it relates to the span problem and the plate equation. For more depth, here are some papers that focused on the study of this damping ([4],[5] [8],[12],[16],[17],[20],[22],[26] ).
The effect of the delay often appear in many applications and pratical problems and turns a lot of systems into different problems worth studying. Recently, the stability and the asymptotic behavior of evolution systems with time delay especially the distributed delay effect has been studed by many authors. See([7],[9],[10],[12],[13],[24]).
Based on all of the above, the combination of these terms of damping (Infinte memory, Balakrishnan-Taylor damping and the distributed delay terms ) in one particular problem, especially with the addition of the past history and the distributed delay ($\int_{\tau_1}^{\tau_2} |\beta_2(s)||u_t(t-s)|^{m-2}u_t(t-s)ds$) we believe that it constitutes a new problem worthy of study and research different from the above that we will try to shed light on it.
The rest of work is organized as follows: In section 2, we recall some preliminaries and assumptions. In section 3, we prove the main stability result in both cases where $H$ is linear and nonlinear. Finally, we give a conclusion in Section 4.

## 2. Preliminaries

For studying our problem, in this section we will need some materials.
Firstly, to achieve our goal, we suppose the following assumptions:
(**H1**) $h : \mathbb{R}_+ \to \mathbb{R}_+$ is a nonincreasing $C^1$ function satisfing

$$h(0) > 0, \quad , h_0 = \int_0^\infty h(\varrho)d\varrho, \quad \zeta_0 - h_0 = l > 0. \tag{2.1}$$

(**H2**)There exists a $C^1$ function $H : \mathbb{R}_+ \to \mathbb{R}_+$ satisfing $H(0) = H'(0) = 0$.
The function $H(t)$ is linear or it is an increasing strictly convex function of class $C^2(\mathbb{R}_+)$ on $(0, r]$, $r \le h(0)$, such that

$$h'(t) \le -\xi(t)H(h(t)), \ \forall t \ge 0. \tag{2.2}$$

where $\xi(t)$ is a $C^1$ function satisfying

$$\xi(t) > 0, \ \xi'(t) \le 0, \ \ \forall t \ge 0. \tag{2.3}$$

(**H3**) $\beta_2 : [\tau_1, \tau_2] \to \mathbb{R}$ is a bounded function satisfing

$$\int_{\tau_1}^{\tau_2} |\beta_2(s)|ds < \beta_1. \tag{2.4}$$

Let us indroduce

$$(h \circ \psi)(t) := \int_\Omega \int_0^\infty h(\varrho)|\psi(t) - \psi(t-\varrho)|^2 d\varrho dx.$$

and

$$M(t) := \left(\zeta_0 + \zeta_1\|\nabla u\|_2^2 + \sigma(\nabla u(t), \nabla u_t(t))_{L^2(\Omega)}\right).$$

**Lemma 2.1.** *(Sobolev-Poincare inequality* [1]*). Let* $2 \leq q < \infty (n = 1, 2)$ *or* $2 \leq q < \frac{2n}{n-2} (n \geq 3)$*. Then,* $\exists c_* = c(\Omega, q) > 0$ *such that*

$$\|u\|_q \leq c_* \|\nabla u\|_2, \quad \forall u \in H_0^1(\Omega).$$

Secondly, as in [24], taking the following new variables

$$y(x, \rho, s, t) = u_t(x, t - s\rho)$$

which satisfy

$$\begin{cases} s y_t(x, \rho, s, t) + y_\rho(x, \rho, s, t) = 0, \\ y(x, 0, s, t) = u_t(x, t). \end{cases} \tag{2.5}$$

Set an auxiliary variable as in [15]

$$\eta^t(x, \varrho) = u(x, t) - u(x, t - \varrho), \varrho \geq 0.$$

Then,

$$\eta_t^t(x, \varrho) + \eta_\varrho^t(x, \varrho) = u_t(x, t). \tag{2.6}$$

So, problem (1.1) can be written as

$$\begin{cases} u_{tt} - \left(l + \zeta_1 \|\nabla u\|_2^2 + \sigma(\nabla u, \nabla u_t)_{L^2(\Omega)}\right) \Delta u(t) + \int_0^\infty h(\varrho) \Delta \eta^t(\varrho) d\varrho \\ \quad + \beta_1 |u_t(t)|^{m-2} u_t(t) + \int_{\tau_1}^{\tau_2} |\beta_2(s)| |y(x, 1, s, t)|^{m-2} y(x, 1, s, t) ds = 0. \\ s y_t(x, \rho, s, t) + y_\rho(x, \rho, s, t) = 0 \\ \eta_t^t(x, \varrho) + \eta_\varrho^t(x, \varrho) = u_t(x, t), \end{cases} \tag{2.7}$$

where

$$(x, \rho, s, t) \in \Omega \times (0, 1) \times (\tau_1, \tau_2) \times (0, \infty).$$

with the intial data and boundary conditions

$$\begin{cases} u(x, -t) = u_0(x), \;\; u_t(x, 0) = u_1(x), \;\; in \;\; \Omega \\ y(x, \rho, s, 0) = f_0(x, \rho s), \;\; in \;\; \Omega \times (0, 1) \times (0, \tau_2) \\ u(x, t) = \eta^t(x, \varrho) = 0, \;\; x \in \;\; \partial\Omega, \;\; t, \varrho \in (0, \infty), \\ \eta^t(x, 0) = 0, \; \forall t \geq 0, \;\; \eta^0(x, \varrho) = \eta_0(\varrho) = 0, \; \forall \varrho \geq 0, \end{cases} \tag{2.8}$$

Now, we give the energy functional.

**Lemma 2.2.** *The energy functional $E$, defined by*

$$\begin{aligned} E(t) \;\; = \;\; & \frac{1}{2}\|u_t\|_2^2 + \frac{1}{2}\left(\zeta_0 - \int_0^\infty h(\varrho) d\varrho\right) \|\nabla u(t)\|_2^2 + \frac{\zeta_1}{4} \|\nabla u(t)\|_2^4 \\ & + \frac{1}{2}(h \circ \nabla u)(t) + \frac{m-1}{m} \int_0^1 \int_{\tau_1}^{\tau_2} s |\beta_2(s)| \|y(x, \rho, s, t)\|_m^m ds d\rho. \end{aligned} \tag{2.9}$$

*satisfies*

$$E'(t) \leq -\gamma_0 \|u_t(t)\|_m^m + \frac{1}{2}(h' \circ \nabla u)(t) - \frac{\sigma}{4}\left(\frac{d}{dt}\left\{\|\nabla u(t)\|_2^2\right\}\right)^2 \leq 0, \tag{2.10}$$

*where* $\gamma_0 = \beta_1 - \int_{\tau_1}^{\tau_2} |\beta_2(s)| ds > 0$.

*Proof.* Taking the inner product of $(2.7)_1$ with $u_t$, then integrating over $\Omega$, we find

$$\begin{aligned}
&(u_{tt}(t), u_t(t))_{L^2(\Omega)} - (M_1(t)\Delta u(t), u_t(t))_{L^2(\Omega)} \\
&+(\int_0^\infty h(\varrho)\Delta\eta^t(\varrho)d\varrho, u_t(t))_{L^2(\Omega)} + \beta_1(|u_t|^{m-2}u_t, u_t)_{L^2(\Omega)} \\
&+ \int_{\tau_1}^{\tau_2} |\beta_2(s)|(|y(x,1,s,t)|^{m-2}y(x,1,s,t), u_t(t))_{L^2(\Omega)}ds = 0.
\end{aligned} \tag{2.11}$$

where

$$M_1(t) := \bigg(l + \zeta_1\|\nabla u\|_2^2 + \sigma(\nabla u(t), \nabla u_t(t))_{L^2(\Omega)}\bigg).$$

A calculation direct, gives

$$(u_{tt}(t), u_t(t))_{L^2(\Omega)} = \frac{1}{2}\frac{d}{dt}\bigg(\|u_t(t)\|_2^2\bigg), \tag{2.12}$$

using integration by parts, we find

$$\begin{aligned}
&-(M_1(t)\Delta u(t), u_t(t))_{L^2(\Omega)} \\
=\ & -(\bigg(l + \zeta_1\|\nabla u\|_2^2 + \sigma(\nabla u(t), \nabla u_t(t))_{L^2(\Omega)}\bigg)\Delta u(t), u_t(t))_{L^2(\Omega)} \\
=\ & \bigg(l + \zeta_1\|\nabla u\|_2^2 + \sigma(\nabla u(t), \nabla u_t(t))_{L^2(\Omega)}\bigg)\int_\Omega \nabla u(t).\nabla u_t(t)dx \\
=\ & \bigg(l + \zeta_1\|\nabla u\|_2^2 + \sigma(\nabla u(t), \nabla u_t(t))_{L^2(\Omega)}\bigg)\frac{d}{dt}\bigg\{\int_\Omega |\nabla u(t)|^2dx\bigg\} \\
=\ & \frac{d}{dt}\bigg\{\frac{1}{2}\bigg(l + \frac{\zeta_1}{2}\|\nabla u\|_2^2\bigg)\|\nabla u(t)\|_2^2\bigg\} + \frac{\sigma}{4}\frac{d}{dt}\bigg\{\|\nabla u(t)\|_2^2\bigg\}^2.
\end{aligned} \tag{2.13}$$

By (2.6) and integration by parts, we have

$$\begin{aligned}
(\int_0^\infty h(\varrho)\Delta\eta^t(\varrho)d\varrho, u_t(t))_{L^2(\Omega)} &= \int_\Omega \nabla u_t \int_0^\infty h(\varrho)\nabla\eta^t(\varrho)d\varrho dx \\
&= \int_0^\infty h(\varrho)\int_\Omega \nabla u_t\nabla\eta^t(\varrho)dxd\varrho \\
&= \int_0^\infty h(\varrho)\int_\Omega (\nabla\eta_t^t + \nabla\eta_\varrho^t)\nabla\eta^t(\varrho)dxd\varrho \\
&= \int_0^\infty h(\varrho)\int_\Omega \nabla\eta_t^t\nabla\eta^t(\varrho)dxd\varrho \\
&\quad + \int_\Omega\int_0^\infty h(\varrho)\nabla\eta_\varrho^t\nabla\eta^t(\nabla)d\nabla dx \\
&= \frac{1}{2}\frac{d}{dt}(h\circ\nabla u)(t) - \frac{1}{2}(h'\circ\nabla u)(t).
\end{aligned} \tag{2.14}$$

Now, multiplying the equation $(2.7)_2$ by $-y|\beta_2(s)|$, and integrating over $\Omega\times(0,1)\times(\tau_1,\tau_2)$, and using $(2.5)_2$, we get

$$\begin{aligned}
\frac{d}{dt} \quad & \frac{m-1}{m}\int_\Omega\int_0^1\int_{\tau_1}^{\tau_2} s|\beta_2(s)|.|y(x,\rho,s,t)|^m ds d\rho dx \\
= \quad & -(m-1)\int_\Omega\int_0^1\int_{\tau_1}^{\tau_2} |\beta_2(s)|.|y|^{m-1}y_\rho ds d\rho dx \\
= \quad & -\frac{m-1}{m}\int_\Omega\int_0^1\int_{\tau_1}^{\tau_2} |\beta_2(s)|\frac{d}{d\rho}|y(x,\rho,s,t)|^m ds d\rho dx \\
= \quad & \frac{m-1}{m}\int_\Omega\int_{\tau_1}^{\tau_2} |\beta_2(s)|\Big(|y(x,0,s,t)|^m - |y(x,1,s,t)|^m\Big) ds dx \\
= \quad & \frac{m-1}{m}\left(\int_{\tau_1}^{\tau_2}|\beta_2(s)|ds\right)\int_\Omega |u_t(t)|^m dx \\
& -\frac{m-1}{m}\int_\Omega\int_{\tau_1}^{\tau_2} |\beta_2(s)|.|y\,(x,1,s,t)\,|^m ds dx \\
= \quad & \frac{m-1}{m}\left(\int_{\tau_1}^{\tau_2}|\beta_2(s)|ds\right)\|u_t(t)\|_m^m \\
& -\frac{m-1}{m}\int_{\tau_1}^{\tau_2}|\beta_2(s)|\|y\,(x,1,s,t)\,\|_m^m ds. \qquad (2.15)
\end{aligned}$$

and by Young's inequalitie, we have

$$\begin{aligned}
& \int_{\tau_1}^{\tau_2}|\beta_2(s)|\Big(|y(x,1,s,t)|^{m-2}y(x,1,s,t),u_t(t)\Big)_{L^2(\Omega)} ds \qquad (2.16)\\
\leq \quad & \frac{1}{m}\left(\int_{\tau_1}^{\tau_2}|\beta_2(s)|ds\right)\|u_t(t)\|_m^m + \frac{m-1}{m}\int_{\tau_1}^{\tau_2}|\beta_2(s)|\|y\,(x,1,s,t)\,\|_m^m ds.
\end{aligned}$$

By replacement (2.12)-(2.16) into (2.11), we find (2.9) and (2.10). Hence, by $(2.2)-(2.4)$, we get $E$ is a non-increasing function. This completes of the proof. □

Let the vector function $U=(u,u_t,y,\eta^t)^T$.

**Theorem 2.3.** *Suppose that (2.1)-(2.4) are satisfied. Then, for any $U_0\in\mathcal{H}$, there exists a unique solution $U$ of problem (2.7)-(2.8) such that*

$$U\in C(\mathbb{R}_+,\mathcal{H}).$$

*If $U_0\in\mathcal{H}_1$, then $U$ satisfies*

$$U\in C^1(\mathbb{R}_+,\mathcal{H})\cap C(\mathbb{R}_+,\mathcal{H}_1).$$

*where*

$$\begin{aligned}
\mathcal{H} \quad &= \quad H_0^1(\Omega)\times L^2(\Omega)\times L^2(\Omega,(0,1),(\tau_1,\tau_2))\times L_h. \\
\mathcal{H}_1 \quad &= \quad \Big\{U\in\mathcal{H}/u\in u\in H^2\cap H_0^1, u_t\in H_0^1(\Omega), y, y_\rho\in L^2(\Omega,(0,1),(\tau_1,\tau_2)), \\
& \qquad \eta^t\in L_h, \eta^t(x,0)=0, y(x,0,s,t)=u_t\Big\}.
\end{aligned}$$

## 3. Stability result

In this section, we state and prove the asymptotic behavior of the system (2.7)-(2.8). For this goal, we need the following lemmas.
As in [23], we set for any $0 < \kappa < 1$,

**Lemma 3.1.** *Assume that condition (2.1)-(2.2) holds.*

$$\int_\Omega \bigg( \int_0^\infty h(\varrho)(u(t) - u(t-\varrho))d\varrho \bigg)^2 dx \leq C_\kappa (g \circ u)(t). \tag{3.1}$$

*where*

$$\begin{aligned} C_\kappa &:= \int_0^\infty \frac{h^2(\varrho)}{\kappa h(\varrho) - h'(\varrho)} d\varrho \\ g(t) &:= \kappa h(t) - h'(t). \end{aligned}$$

*Proof.*

$$\begin{aligned} & \int_\Omega \bigg( \int_0^\infty h(\varrho)(u(t) - u(t-\varrho))d\varrho \bigg)^2 dx \\ = & \int_\Omega \bigg( \int_{-\infty}^t h(t-\varrho)(u(t) - u(t-\varrho))d\varrho \bigg)^2 dx \\ = & \int_\Omega \bigg( \int_{-\infty}^t \frac{h(t-\varrho)}{\sqrt{\kappa h(t-\varrho) - h'(t-\varrho)}} \sqrt{\kappa h(t-\varrho) - h'(t-\varrho)} \\ & \qquad (u(t) - u(\varrho))d\varrho \bigg)^2 dx \end{aligned} \tag{3.2}$$

by Young's inequality, we obtain (3.1). □

**Lemma 3.2.** *(Jensens inequality). Let $H : [a,b] \to \mathbb{R}$ be a convex function. Assume that the functions $f : \Omega \to [a,b]$ and $g : \Omega \to \mathbb{R}$ are integrable such that $g(x) > 0$, for any $x \in \Omega$ and $\int_\Omega g(x)dx = k > 0$. Then*

$$H\bigg(\frac{1}{k}\int_\Omega f(x)g(x)dx\bigg) < \frac{1}{k}\int_\Omega H(f(x))g(x)dx. \tag{3.3}$$

**Lemma 3.3.** *As in* [18]*, there exist a positive constant $\beta$ such that*

$$I(t) = \int_\Omega \int_t^\infty h(\varrho)|\nabla \eta^t(\sigma)|^2 d\varrho dx \leq \beta \mu(t), \tag{3.4}$$

*where*

$$\mu(t) = \int_0^\infty h(t+\varrho)\bigg(1 + \int_\Omega \nabla u_0^2(x,\varrho)dx\bigg)d\varrho.$$

*Proof.* Since $E(t)$ is decreasing function and using (2.9), then for $t, \varrho \geq 0$,

$$\begin{aligned}
\int_\Omega |\nabla\eta^t(\varrho)|^2 dx &= \int_\Omega (\nabla u(x,t) - u(x,t-\varrho)^2 dx \\
&\leq 2\int_\Omega \nabla u^2(x,t)dx + 2\int_\Omega \nabla u^2(x,t-\varrho)dx \\
&\leq 2\sup_{\varrho>0}\int_\Omega \nabla u^2(x,\varrho)dx + 2\int_\Omega \nabla u^2(x,t-\varrho)dx \\
&\leq \frac{4E(0)}{l} + 2\int_\Omega \nabla u^2(x,t-\varrho)dx, \qquad (3.5)
\end{aligned}$$

then

$$\begin{aligned}
I(t) &\leq \frac{4E(0)}{l}\int_t^\infty h(\varrho)d\varrho + 2\int_t^\infty h(\varrho)\int_\Omega \nabla u^2(x,t-\varrho)dxd\varrho \\
&\leq \frac{4E(0)}{l}\int_0^\infty h(t+\varrho)d\varrho + 2\int_0^\infty h(t+\varrho)\int_\Omega \nabla u_0^2(x,\varrho)dxd\varrho \\
&\leq \beta\mu(t). \qquad (3.6)
\end{aligned}$$

where $\beta = \max\{\frac{4E(0)}{l}, 2\}$ and $\mu(t) = \int_0^\infty h(t+\varrho)(1 + \int_\Omega \nabla u_0^2(x,\varrho)dx)d\varrho$. □

Now, we set

$$\Psi(t) := \int_\Omega u(t)u_t(t)dx + \frac{\sigma}{4}\|\nabla u(t)\|_2^4 \qquad (3.7)$$

and

$$\Phi(t) := -\int_\Omega u_t \int_0^\infty h(\varrho)(u(t) - u(t-\varrho))d\varrho dx \qquad (3.8)$$

and

$$\Theta(t) := \int_0^1\int_{\tau_1}^{\tau_2} se^{-\rho s}|\beta_2(s)|.\|y(x,\rho,s,t)\|_m^m ds d\rho. \qquad (3.9)$$

**Lemma 3.4.** *The functional $\Psi(t)$ defined in (3.7) satisfies, for any $\varepsilon, \delta_1 > 0$*

$$\begin{aligned}
\Psi'(t) \leq\ & \|u_t\|_2^2 - (l - \varepsilon(c_1+c_2) - \delta_1)\|\nabla u\|_2^2 - \zeta_1\|\nabla u\|_2^4 + \frac{c}{\delta_1}C_\kappa(g\circ\nabla u)(t) \\
&+ c(\varepsilon)\bigg(\|u_t\|_m^m + \int_{\tau_1}^{\tau_2}|\beta_2(s)|\|y(x,1,s,t)\|_m^m ds\bigg). \qquad (3.10)
\end{aligned}$$

*Proof.* A differentiation of (3.7) and using $(2.7)_1$, gives

$$\begin{aligned}\Psi'(t) &= \|u_t\|_2^2 + \int_\Omega u_{tt} u dx + \sigma\|\nabla u\|_2^2 \int_\Omega \nabla u_t \nabla u dx \\ &= \|u_t\|_2^2 - \zeta_0\|\nabla u\|_2^2 - \zeta_1\|\nabla u\|_2^4 \underbrace{-\beta_1 \int_\Omega |u_t|^{m-2} u_t u dx}_{J_1} \\ &\quad \underbrace{+ \int_\Omega \nabla u(t) \int_0^\infty h(\varrho)\nabla u(t-\varrho) d\varrho dx}_{J_2} \\ &\quad \underbrace{- \int_\Omega \int_{\tau_1}^{\tau_2} |\beta_2(s)||y(x,1,s,t)|^{m-2} y(x,1,s,t) u ds dx}_{J_3}. \end{aligned} \tag{3.11}$$

We estimate the last 3 terms of the RHS of (3.11). Applying Hölder's, Sobolev-Poincare and Young's inequalities, (2.1) and (2.9), we find

$$\begin{aligned} J_1 &\leq \varepsilon\beta_1^m \|u\|_m^m + c(\varepsilon)\|u_t\|_m^m \\ &\leq \varepsilon\beta_1^m c_p^m \|\nabla u\|_2^m + c(\varepsilon)\|u_t\|_m^m \\ &\leq \varepsilon\beta_1^m c_p^m \left(\frac{E(0)}{l}\right)^{(m-2)/2} \|\nabla u\|_2^2 + c(\varepsilon)\|u_t\|_m^m \\ &\leq \varepsilon c_1 \|\nabla u\|_2^2 + c(\varepsilon)\|u_t\|_m^m. \end{aligned} \tag{3.12}$$

and, By Lemma 3.1, we get for any $\delta_1 > 0$

$$\begin{aligned} J_2 &\leq (\int_0^\infty h(\varrho)d\varrho)\|\nabla u\|_2^2 - \int_\Omega \nabla u(t) \int_0^\infty h(\varrho)(\nabla u(t) - \nabla u(t-\varrho)) d\varrho dx \\ &\leq (\zeta_0 - l + \delta_1)\|\nabla u\|_2^2 + \frac{c}{\delta_1} C_\kappa (g \circ \nabla u)(t). \end{aligned} \tag{3.13}$$

Similarly to $J_1$, we have

$$J_3 \leq \varepsilon c_2 \|\nabla u\|_2^2 + c(\varepsilon) \int_{\tau_1}^{\tau_2} |\beta_2(s)|.\|y(x,1,s,t)\|_m^m ds. \tag{3.14}$$

Combining (3.12)-(3.14) and (3.11), we obtain (3.10). □

**Lemma 3.5.** *The functional $\Phi(t)$ defined in (3.8) satisfies, for any $\delta, \delta_2, \delta_3 > 0$*

$$\begin{aligned} \Phi'(t) &\leq -\Big(h_0 - \delta_3\Big)\|u_t\|_2^2 + \delta\Big(\zeta_0 + h_0^2\Big)\|\nabla u\|_2^2 + \zeta_1\delta\|\nabla u\|_2^4 \\ &\quad + \delta_2 \frac{2\sigma E(0)}{l}\left(\frac{1}{2}\frac{d}{dt}\|\nabla u\|_2^2\right)^2 + c(\delta,\delta_2,\delta_3) C_\kappa (g\circ\nabla u)(t) \\ &\quad + c(\delta)\Big(\|u_t\|_m^m + \int_{\tau_1}^{\tau_2} |\beta_2(s)\|y(x,1,s,t)\|_m^m ds\Big). \end{aligned} \tag{3.15}$$

*Proof.* A differentiation of (3.8) and using $(2.7)_1$, gives

$$
\begin{aligned}
\Phi'(t) &= -\int_\Omega u_{tt} \int_0^\infty h(\varrho)(u(t)-u(t-\varrho))d\varrho dx \\
&\quad -\int_\Omega u_t \frac{\partial}{\partial t}\Big(\int_0^\infty h(\varrho)(u(t)-u(t-\varrho))d\varrho\Big)dx \\
&= \underbrace{(\zeta_0+\zeta_1\|\nabla u\|_2^2)\int_\Omega \nabla u \int_0^\infty h(\varrho)(\nabla u(t)-\nabla u(t-\varrho))d\varrho dx}_{J_1} \\
&\quad \underbrace{+\sigma \int_\Omega \nabla u \nabla u_t dx . \int_\Omega \nabla u \int_0^\infty h(\varrho)(\nabla u(t)-\nabla u(t-\varrho))d\varrho dx}_{J_2} \\
&\quad \underbrace{-\int_\Omega \Big(\int_0^\infty h(\varrho)\nabla u(t-\varrho)d\varrho\Big).\Big(\int_0^\infty h(\varrho)(\nabla u(t)-\nabla u(t-\varrho))d\varrho\Big)dx}_{J_3} \\
&\quad \underbrace{-\beta_1\int_\Omega |u_t|^{m-2}u_t\Big(\int_0^\infty h(\varrho)(\nabla u(t)-\nabla u(t-\varrho))d\varrho\Big)dx}_{J_4} \\
&\quad \underbrace{\begin{aligned}&-\int_\Omega\int_{\tau_1}^{\tau_2}|\beta_2(s)||y(x,1,s,t)|^{m-2}y(x,1,s,t)\\ &\times\int_0^\infty h(\varrho)(\nabla u(t)-\nabla u(t-\varrho))d\varrho\Big)d\varrho dx\end{aligned}}_{J_5} \\
&\quad \underbrace{-\int_\Omega u_t\frac{\partial}{\partial t}\Big(\int_0^\infty h(\varrho)(u(t)-u(t-\varrho))d\varrho\Big)dx}_{J_6} . \qquad (3.16)
\end{aligned}
$$

We estimate the terms of the RHS of (3.16). Applying Hölder's, Sobolev-Poincare and Young's inequalities, (2.1), (2.9) and Lemma 3.1, we find

$$
\begin{aligned}
|J_1| &\le (\zeta_0+\zeta_1\|\nabla u\|_2^2)\Big(\delta\|\nabla u\|_2^2+\frac{1}{4\delta}C_\kappa(g\circ\nabla u)(t)\Big) \\
&\le \delta\zeta_0\|\nabla u\|_2^2+\delta\zeta_1\|\nabla u\|_2^4+\Big(\frac{\zeta_0}{4\delta}+\frac{\zeta_1E(0)}{4l\delta}\Big)C_\kappa(g\circ\nabla u)(t),
\end{aligned}
\qquad (3.17)
$$

and

$$
\begin{aligned}
J_2 &\le \delta_2\sigma\Big(\int_\Omega \nabla u\nabla u_t dx\Big)^2\|\nabla u\|_2^2+\frac{\sigma}{4\delta_2}C_\kappa(g\circ\nabla u)(t) \\
&\le \delta_2\frac{2\sigma E(0)}{l}\Big(\frac{1}{2}\frac{d}{dt}\|\nabla u\|_2^2\Big)^2+\frac{\sigma}{4\delta_2}C_\kappa(g\circ\nabla u)(t), \qquad (3.18)
\end{aligned}
$$

$$
\begin{aligned}
|J_3| &\leq \int_\Omega \bigg(\int_0^\infty h(\varrho)\nabla u(t)d\varrho\bigg)\bigg(\int_0^\infty h(\varrho)(\nabla u(t-\varrho)-\nabla u(t))d\varrho\bigg)dx \\
&\quad -\int_\Omega \bigg(\int_0^\infty h(\varrho)(\nabla u(t)-\nabla u(t-\varrho))d\varrho\bigg)^2 dx \\
&\leq \delta h_0^2\|\nabla u\|_2^2 + \bigg(1+\frac{1}{4\delta}\bigg)C_\kappa(g\circ\nabla u)(t), \qquad (3.19)
\end{aligned}
$$

$$
\begin{aligned}
|J_4| &\leq c(\delta)\|\nabla u_t\|_m^m + \delta\beta_1^m \int_\Omega \bigg(\int_0^\infty h(\varrho)(u(t)-u(t-\varrho))d\varrho\bigg)^m dx \\
&\leq c(\delta)\|\nabla u_t\|_m^m + \delta\bigg(\beta_1^m c_p^m [\frac{4h_0E(0)}{l}]^{(m-2)}\bigg)C_\kappa(g\circ\nabla u)(t) \\
&\leq c(\delta)\|\nabla u_t\|_m^m + \delta c_3 C_\kappa(g\circ\nabla u)(t), \qquad (3.20)
\end{aligned}
$$

Similarly, we have

$$
J_5 \leq c(\delta)\|y(x,1,s,t)\|_m^m + \delta c_4 C_\kappa(g\circ\nabla u)(t), \qquad (3.21)
$$

and, to estimate $J_6$, we have

$$
\begin{aligned}
\frac{\partial}{\partial t}\bigg(\int_0^\infty h(\varrho)(u(t)-u(t-\varrho))d\varrho\bigg) &= \frac{\partial}{\partial t}\bigg(\int_{-\infty}^t h(t-\varrho)(u(t)-u(\varrho))d\varrho\bigg) \\
&= \int_{-\infty}^t h'(t-\varrho)(u(t)-u(\varrho))d\varrho \\
&\quad +\bigg(\int_{-\infty}^t h(t-\varrho)d\varrho\bigg)u_t(t) \\
&= \int_0^\infty h'(\varrho)(u(t)-u(t-\varrho))d\varrho + h_0u_t(t),
\end{aligned}
$$

by (2.2), gives

$$
J_6 \leq -(h_0-\delta_3)\|u_t\|_2^2 + \frac{c}{\delta_3}C_\kappa(g\circ\nabla u)(t). \qquad (3.22)
$$

A substitution of (3.17)-(3.22) into (3.16), we get (3.15). □

**Lemma 3.6.** *The functional $\Theta(t)$ defined in (3.9) satisfies*

$$
\begin{aligned}
\Theta'(t) &\leq -\gamma_1\int_0^1\int_{\tau_1}^{\tau_2} s|\beta_2(s)|.\|y(x,\rho,s,t)\|_m^m ds d\rho \\
&\quad -\gamma_1\int_{\tau_1}^{\tau_2}|\beta_2(s)|.\|y\,(x,1,s,t)\,\|_m^m ds + \beta_1\|u_t(t)\|_m^m. \qquad (3.23)
\end{aligned}
$$

*Proof.* By differentiating of $\Theta(t)$, and using $(2.7)_2$, we have

$$\begin{aligned}\Theta'(t) &= -m\int_\Omega\int_0^1\int_{\tau_1}^{\tau_2} e^{-s\rho}|\beta_2(s)|.|y|^{m-1}y_\rho\left(x,\rho,s,t\right)dsd\rho dx\\ &= -\int_\Omega\int_0^1\int_{\tau_1}^{\tau_2} se^{-s\rho}|\beta_2(s)|.|y(x,\rho,s,t)|^m dsd\rho dx\\ &\quad -\int_\Omega\int_{\tau_1}^{\tau_2}|\beta_2(s)|\left[e^{-s}|y\left(x,1,s,t\right)|^m-|y\left(x,0,s,t\right)|^m\right]dsdx.\end{aligned}$$

Applying $y(x,0,s,t)=u_t(x,t)$, and $e^{-s}\leq e^{-s\rho}\leq 1$, for any $0<\rho<1$, and we set $\gamma_1=e^{-\tau_2}$, we obtain

$$\begin{aligned}\Theta'(t) &\leq -\gamma_1\int_\Omega\int_0^1\int_{\tau_1}^{\tau_2} s|\beta_2(s)|.|y(x,\rho,s,t)|^m dsd\rho dx\\ &\quad -\gamma_1\int_\Omega\int_{\tau_1}^{\tau_2}|\beta_2(s)||y(x,1,s,t)|^m dsdx+\int_{\tau_1}^{\tau_2}|\beta_2(s)|ds\int_\Omega|u_t|^m(t)dx,\end{aligned}$$

using (2.4), we find (3.23).

□

Now, we define the functional

$$F(t):=\int_\Omega\int_0^t f(t-\varrho)\nabla u(\varrho)^2 d\varrho dx \tag{3.24}$$

where $f(t)=\int_t^\infty h(\varrho)d\varrho$.

**Lemma 3.7.** *Assume that (2.1)-(2.2) hold. Then, the functional $F_3$ satisfies,*

$$\begin{aligned}F'(t) &\leq -\frac{1}{2}(h\circ\nabla u)(t)+3(\zeta_0-l)\int_\Omega\nabla u^2 dx\\ &\quad+\frac{1}{2}\int_\Omega\int_t^\infty h(\varrho)(\nabla u(t)-\nabla u(t-\varrho))^2 d\varrho dx.\end{aligned} \tag{3.25}$$

*Proof.* It follows from the proof of Lemma 3.4 in [23] and Lemma 3.7 in [19]. □

We are now ready to prove the following result.

**Theorem 3.8.** *Assume (2.1)-(2.4), there exist positive constants $\tau_i, i=1,2,3$ and $\tau_4(t)$ be a positive function, such that the energy functional given by (2.9) satisfies*

$$E\left(t\right)\leq\tau_1 H_2^{-1}\left(\frac{\tau_2+\tau_3\int_0^t\xi(\varsigma)H_4(\tau_4(\varsigma)\mu(\varsigma))d\varsigma}{\int_0^t\xi(\varsigma)d\varsigma}\right), \tag{3.26}$$

*where*

$$H_2(t)=tH'(\varepsilon_0 t),\quad H_3(t)=tH'^{-1}(t),\quad H_4(t)=\overline{H}_3^*(t), \tag{3.27}$$

*which are convex and increasing functions on $(0,r]$*

*Proof.* Firstly, we introduce the functional

$$\mathcal{G}(t) \;:=\; NE(t)+N_1\Psi(t)+N_2\Phi(t)+N_3\Theta(t), \tag{3.28}$$

for some positive constants $N, N_i, i = 1, 2, 3$ to be determined.
A differentiation of (3.33), using 2.10, the Lemmas 3.4, 3.5 and 3.6

$$\begin{aligned}
\mathcal{G}'(t) &:= NE'(t) + N_1\Psi'(t) + N_2\Phi'(t) + N_3\Theta'(t) \\
&\leq -\left\{N_2(h_0 - \delta_3) - N_1\right\}\|u_t\|_2^2 - \left\{N_3\zeta_1 - N_2\zeta_1\delta\right\}\|\nabla u\|_2^4 \\
&\quad -\left\{N_1(l - \varepsilon(c_1 + c_2) - \delta_1) - N_2\delta(\zeta_0 + h_0^2)\right\}\|\nabla u\|_2^2 \\
&\quad -\left\{\frac{N\sigma}{4} - N_2\delta_2\frac{2\sigma E(0)}{l}\right\}\left(\frac{1}{2}\frac{d}{dt}\|\nabla u\|_2^2\right)^2 \\
&\quad +\left\{N_1c(\delta_1) + N_2c(\delta, \delta_2, \delta_3)\right\}C_\kappa(g \circ \nabla u)(t) + \frac{N}{2}(h' \circ \nabla u)(t) \\
&\quad -\left\{\gamma_0 N - N_1c(\varepsilon) - N_2c(\delta) - N_3\beta_1\right\}\|u_t\|_m^m \\
&\quad -\left(\gamma_1 N_3 - N_1c(\varepsilon) - N_2c(\delta)\right)\int_{\tau_1}^{\tau_2}|\beta_2(s)|\|y(x, 1, s, t)\|_m^m ds\Big) \\
&\quad -N_3\gamma_1\int_0^1\int_{\tau_1}^{\tau_2}s|\beta_2(s)|.\|y(x, \rho, s, t)\|_m^m ds d\rho. \qquad (3.29)
\end{aligned}$$

At this stage, we choose the various constants so that the values inside the parentheses are positive.
First, setting

$$\delta_3 = \frac{h_0}{2},\quad \varepsilon = \frac{l}{4(c_1 + c_2)},\quad \delta_1 = \frac{l}{4},\quad \delta_2 = \frac{lN}{16E(0)N_2},\quad N_1 = \frac{h_0}{4}N_2.$$

Thus, we arrive at

$$\begin{aligned}
\mathcal{G}'(t) &\leq -\frac{h_0}{4}N_2\|u_t\|_2^2 - \zeta_1N_2\left(\frac{h_0}{4} - \delta\right)\|\nabla u\|_2^4 \\
&\quad -N_2\left(\frac{lh_0}{8} - \delta(\zeta_0 + h_0^2)\right)\|\nabla u\|_2^2 - \frac{N\sigma}{8}\left(\frac{1}{2}\frac{d}{dt}\|\nabla u\|_2^2\right)^2 \\
&\quad +N_2c(\delta, \delta_1, \delta_2, \delta_3)C_\kappa(g \circ \nabla u)(t) + \frac{N}{2}(h' \circ \nabla u)(t) \\
&\quad -\left(\gamma_0 N - N_2c(\delta, \varepsilon) - N_3\beta_1\right)\|u_t\|_m^m \\
&\quad -\left(\gamma_1 N_3 - N_2c(\delta, \varepsilon)\right)\int_{\tau_1}^{\tau_2}|\beta_2(s)|\|y(x, 1, s, t)\|_m^m ds\Big) \\
&\quad -N_3\gamma_1\int_0^1\int_{\tau_1}^{\tau_2}s|\beta_2(s)|.\|y(x, \rho, s, t)\|_m^m ds d\rho. \qquad (3.30)
\end{aligned}$$

Next, we choose $\delta$ so small that

$$\delta < \min\left\{\frac{h_0}{4}, \frac{lh_0}{8(\zeta_0 + h_0^2)}\right\}.$$

Then, we pick $N_2$ large enough such that

$$N_2\left(\frac{lh_0}{8}-\delta(\zeta_0+h_0^2)\right)>4h_0=4(\zeta_0-l),$$

then we choose $N_3$ large enough such that

$$\gamma_1 N_3-N_2c(\delta,\varepsilon)>0.$$

Therefore, (3.30) becomes, for positive constants $d_i, i=1,2,3,4$

$$\begin{aligned}\mathcal{G}'(t) &\le -d_1\|u_t\|_2^2-d_2\|\nabla u\|_2^4-4h_0\|\nabla u\|_2^2-\frac{N\sigma}{8}\left(\frac{1}{2}\frac{d}{dt}\|\nabla u\|_2^2\right)^2\\ &\quad -\left(\frac{N}{2}-d_3C_\kappa\right)(g\circ\nabla u)(t)+\frac{N\kappa}{2}(h\circ\nabla u)(t) \qquad (3.31)\\ &\quad -(\gamma_0 N-c)\|u_t\|_m^m-d_4\int_0^1\int_{\tau_1}^{\tau_2}\varrho|\beta_2(s)|.\|y(x,\rho,s,t)\|_m^m dsd\rho.\end{aligned}$$

As $\frac{\kappa h^2(\varrho)}{\kappa h(\varrho)-h(\varrho)}\le h(\varrho)$, it follows from the Lebesgue Dominated Convergence

$$\lim_{\kappa\to 0^+}\kappa C_\kappa=\lim_{\kappa\to 0^+}\int_0^\infty\frac{\kappa h^2(\varrho)}{\kappa h(\varrho)-h(\varrho)}d\varrho=0 \qquad (3.32)$$

Consequently, there exists $0<\kappa_0<1$ such that if $\kappa<\kappa_0$, then

$$\kappa C_\kappa\le\frac{1}{d_3} \qquad (3.33)$$

On the other hand, from (3.7)-(3.9), by using Hölder, Young's and poincare inequalities, we get

$$\begin{aligned}|\mathcal{G}(t)-NE(t)| &\le \frac{N_1}{2}\left(\|u_t(t)\|_2^2+c_p\|\nabla u(t)\|_2^2\right)+\sigma\frac{N_1}{4}\|\nabla u(t)\|_2^4\\ &\quad +\frac{N_2}{2}\|u_t(t)\|_2^2+\frac{N_2}{2}c_pC_\kappa(g\circ\nabla u)(t)\\ &\quad +N_3\int_0^1\int_{\tau_1}^{\tau_2}se^{-\rho s}|\beta_2(s)|.\|y(x,\rho,s,t)\|_m^m dsd\rho. \qquad (3.34)\end{aligned}$$

Using the fact that $e^{-\rho s}<1$ and (2.2), we find

$$\begin{aligned}|\mathcal{G}(t)-NE(t)| &\le \frac{N_1}{2}\left(\|u_t(t)\|_2^2+c_p\|\nabla u(t)\|_2^2\right)+\sigma\frac{N_1}{4}\|\nabla u(t)\|_2^4\\ &\quad +\frac{N_2}{2}\|u_t(t)\|_2^2+\frac{N_2c}{2}C_\kappa(h\circ\nabla u)(t)\\ &\quad +N_3\int_0^1\int_{\tau_1}^{\tau_2}se^{-\rho s}|\beta_2(s)|.\|y(x,\rho,s,t)\|_m^m dsd\rho\\ &\le C(N_1,N_2,N_3)E(t)=C_1E(t). \qquad (3.35)\end{aligned}$$

that is

$$\left(N-C_1\right)E\left(t\right)\le\mathcal{G}\left(t\right)\le\left(N+C_1\right)E\left(t\right) \qquad (3.36)$$

Now, by choosing $N$ large enough, and set $\kappa = \frac{1}{2N}$, such that

$$N - C_1 > 0, \ , \ \ \gamma_0 N - c > 0, \ \ \frac{1}{2}N - \frac{1}{2\kappa_0} > 0, \ \ \kappa = \frac{1}{2N} < \kappa_0,$$

we find

$$\mathcal{G}'(t) \leq -k_2 E(t) + \frac{1}{4}(h \circ \nabla u)(t) \tag{3.37}$$

for some $k_2 > 0$, and

$$c_5 E(t) \leq \mathcal{G}(t) \leq c_6 E(t), \forall t \geq 0 \tag{3.38}$$

for some $c_5, c_6 > 0$, we have

$$\mathcal{G}(t) \sim E(t).$$

Secondly, we consider the following two cases.

**Case 3.9.** $H$ **is linear**. Multiplying (3.37) by $\xi(t)$, we find

$$\xi(t)\mathcal{G}'(t) \leq -k_2 \xi(t) E(t) + \frac{1}{4}\xi(t)(h \circ \nabla u)(t). \tag{3.39}$$

The last term in (3.39), we have

$$\begin{aligned} \frac{\xi(t)}{4}(h \circ \nabla u)(t) &= \frac{\xi(t)}{4}\int_\Omega \int_0^\infty h(\sigma)|\nabla \eta^t(\varrho)|^2 d\varrho dx \\ &= \underbrace{\frac{\xi(t)}{4}\int_\Omega \int_0^t h(\varrho)|\nabla \eta^t(\varrho)|^2 d\varrho dx}_{I_1} \\ &\quad + \underbrace{\frac{\xi(t)}{4}\int_\Omega \int_t^\infty h(\varrho)|\nabla \eta^t(\varrho)|^2 d\varrho dx}_{I_2} \end{aligned} \tag{3.40}$$

To estimate $I_1$, using (2.9),

$$\begin{aligned} I_1 &\leq \frac{1}{4}\int_\Omega \int_0^t \xi(\varrho) h(\varrho)|\nabla \eta^t(\varrho)|^2 d\varrho dx \\ &= -\frac{1}{4}\int_\Omega \int_0^t h'(\varrho)|\nabla \eta^t(\varrho)|^2 d\varrho dx \\ &\leq -\frac{1}{2l}E'(t), \end{aligned} \tag{3.41}$$

and by (3.4), we get

$$I_2 \leq \frac{\beta}{4}\xi(t)\mu(t). \tag{3.42}$$

Hence,

$$\xi(t)\mathcal{G}'(t) \leq -k_2 \xi(t) E(t) - \frac{1}{2l}E'(t) + \widehat{\beta} v(t), \tag{3.43}$$

where $\widehat{\beta} = \frac{\beta}{4}$ and $v(t) = \xi(t)\mu(t)$.
Using $\xi'(t) \leq 0$, we have

$$\mathcal{G}_1'(t) \leq -k_2 \xi(t) E(t) + \widehat{\beta} v(t), \tag{3.44}$$

with

$$\mathcal{G}_1(t) = \xi(t)\mathcal{G}(t) + \frac{1}{2l}E(t) \sim E(t),$$

we have

$$k_4 E(t) \leq \mathcal{G}_1(t) \leq k_5 E(t), \tag{3.45}$$

then, from (3.44) for all $T \geq 0$, we have

$$\begin{aligned} k_2 E(T)\int_0^T \xi(t)dt &\leq k_2\int_0^T \xi(t)E(t)dt \\ &\leq \mathcal{G}_1(0) - \mathcal{G}_1(T) + \widehat{\beta}\int_0^T v(t)dt \\ &\leq \mathcal{G}_1(0) + \widehat{\beta}\int_0^T \xi(t)\mu(t)dt. \end{aligned}$$

Hence

$$E(T) \leq \frac{1}{k_2}\Big(\frac{\mathcal{G}_1(0) + \widehat{\beta}\int_0^T \xi(t)\mu(t)dt}{\int_0^T \xi(t)dt}\Big),$$

Since $H$ is linear, we deduce that $H_2, H_2$ and $H_4$ are linear functions. Then, we can write

$$E(T) \leq \lambda_1 H_2^{-1}\Big(\frac{\frac{\mathcal{G}_1(0)}{k_2} + \frac{\widehat{\beta}}{k_2}\int_0^T \xi(t)\mu(t)dt}{\int_0^T \xi(t)dt}\Big), \tag{3.46}$$

which gives (3.26) with $\tau_1 = \lambda_1$, $\tau_2 = \frac{\mathcal{G}_1(0)}{k_2}$, $\tau_3 = \frac{\widehat{\beta}}{\lambda_2 k_2}$ , and $\tau_4(t) = 1$. This completes the proof.

**Case 3.10.** ***H* is nonlinear**. First, According (3.25) and (3.37). Let the functional

$$\mathcal{G}_2(t) = \mathcal{G}(t) + F(t)$$

is positive and satisfies, for some $k_3 > 0$ and $\forall t \geq 0$,

$$\mathcal{G}_2'(t) \leq -k_3 E(t) + \frac{1}{2}\int_\Omega\int_t^\infty h(\varrho)(\nabla u(t) - \nabla u(t-\varrho))^2 d\varrho dx, \tag{3.47}$$

by using (3.4), this gives us

$$\begin{aligned} k_3\int_0^t E(y)dy &\leq \mathcal{G}_2(0) - \mathcal{G}_2(t) + \frac{\beta}{2}\int_0^t \mu(\varsigma)d\varsigma \\ &\leq \mathcal{G}_2(0) + \frac{\beta}{2}\int_0^t \mu(\varsigma)d\varsigma. \end{aligned} \tag{3.48}$$

Therefore

$$\int_0^t E(y)dy \leq k_6\mu_0(t), \tag{3.49}$$

where $k_6 = \max\{\frac{\mathcal{G}_2(0)}{k_3}, \frac{\beta}{2k_3}\}$ and $\mu_0(t) = 1 + \int_0^t \mu(\varsigma)d\varsigma$.

**Corollary 3.11.** *From (2.9) and (3.49), we have*

$$\begin{aligned}
&\int_0^t \int_\Omega |\nabla u(t) - \nabla u(t-\varrho)|^2 dx d\varrho \\
\le\ & 2\int_0^t \int_\Omega \nabla u^2(t) - \nabla u^2(t-\varrho) dx d\varrho \\
\le\ & \frac{4}{l}\int_0^t E(t) - E(t-\varrho) d\varrho \\
\le\ & \frac{8}{l}\int_0^t E(y) dy \le \frac{8k_6}{l}\mu_0(t). \qquad (3.50)
\end{aligned}$$

Now, we define $\zeta(t)$ by

$$\zeta(t) := \mathcal{B}(t)\int_0^t \int_\Omega |\nabla u(t) - \nabla u(t-\varrho)|^2 dx d\varrho \qquad (3.51)$$

where $\mathcal{B}(t) = \frac{\mathcal{B}_0}{\mu_0(t)}$ and $0 < \mathcal{B}_0 < \min\{1, \frac{l}{8k_6}\}$.
Then, by (3.49), we have

$$\zeta(t) < 1, \quad \forall t > 0, \qquad (3.52)$$

we further assume that $\zeta(t) > 0, \quad \forall t > 0$.
Also, we define another functional $\Gamma$ by

$$\Gamma(t) := -\int_0^t h'(\varrho)\int_\Omega |\nabla u(t) - \nabla u(t-\varrho)|^2 dx d\varrho \qquad (3.53)$$

Clearly, $\Gamma(t) \le -cE'(t)$. Since $H(t)$ is strictly convex on $(0, r]$ and $H(0) = 0$, then

$$H(\lambda x) \le \lambda H(x), \quad 0 < \lambda < 1, \ x \in (0, r]. \qquad (3.54)$$

By using (2.3) and (3.52), we get

$$\begin{aligned}
\Gamma(t) &= \frac{-1}{\mathcal{B}(t)\zeta(t)}\int_0^t \zeta(t)(h'(\varrho))\int_\Omega \mathcal{B}(t)|\nabla u(t) - \nabla u(t-\varrho)|^2 dx d\varrho \\
&\ge \frac{1}{\mathcal{B}(t)\zeta(t)}\int_0^t \zeta(t)\xi(\varrho)H(h(\varrho))\int_\Omega \mathcal{B}(t)|\nabla u(t) - \nabla u(t-\varrho)|^2 dx d\varrho \\
&\ge \frac{\xi(t)}{\mathcal{B}(t)\zeta(t)}\int_0^t H(\zeta(t)h(\varrho))\int_\Omega \mathcal{B}(t)|\nabla u(t) - \nabla u(t-\varrho)|^2 dx d\varrho \\
&\ge \frac{\xi(t)}{\mathcal{B}(t)} H\bigg(\frac{1}{\zeta(t)}\int_0^t \zeta(t)h(\varrho)\int_\Omega \mathcal{B}(t)|\nabla u(t) - \nabla u(t-\varrho)|^2 dx d\varrho\bigg) \\
&= \frac{\xi(t)}{\mathcal{B}(t)} H\bigg(\mathcal{B}(t)\int_0^t h(\varrho)\int_\Omega |\nabla u(t) - \nabla u(t-\varrho)|^2 dx d\varrho\bigg) \\
&= \frac{\xi(t)}{\mathcal{B}(t)} \overline{H}\bigg(\mathcal{B}(t)\int_0^t h(\varrho)\int_\Omega |\nabla u(t) - \nabla u(t-\varrho)|^2 dx d\varrho\bigg) \qquad (3.55)
\end{aligned}$$

where $\overline{H}$ is a $C^2$-extension of $H$, that is strictly increasing and strictly convex on $\mathbb{R}_+$. From (3.55), we get

$$\int_0^t h(\varrho)\int_\Omega |\nabla u(t) - \nabla u(t-\varrho)|^2 dx d\varrho \le \frac{1}{\mathcal{B}(t)}\overline{H}^{-1}\bigg(\frac{\mathcal{B}(t)\Gamma(t)}{\xi(t)}\bigg) \qquad (3.56)$$

Substituting (3.56)and (3.4) into (3.37), we get for some $k_6 > 0$

$$\mathcal{G}'(t) \leq -k_2 E(t) + \frac{k_6}{\mathcal{B}(t)} \overline{H}^{-1}\left(\frac{\mathcal{B}(t)\Gamma(t)}{\xi(t)}\right) + k_6 \mu(t) \tag{3.57}$$

Now, for $\varepsilon_0 < r$, we define the function $\mathcal{K}_1(t)$ by

$$\mathcal{K}_1(t) = \overline{H}'\left(\varepsilon_0 \frac{\mathcal{B}(t)E(t)}{E(0)}\right)\mathcal{G}(t) + E(t) \tag{3.58}$$

which is equivalent to $E(t)$. In view of $E'(t) \leq 0, \overline{H}' > 0$, and $\overline{H}'' > 0$, and using (3.57), we conclude that

$$\begin{aligned} \mathcal{K}_1'(t) &= \varepsilon_0\left(\frac{\mathcal{B}(t)E'(t)}{E(0)} + \frac{\mathcal{B}'(t)E(t)}{E(0)}\right)\overline{H}''\left(\varepsilon_0 \frac{\mathcal{B}(t)E(t)}{E(0)}\right)\mathcal{G}(t) \\ &\quad + \overline{H}'\left(\varepsilon_0 \frac{\mathcal{B}(t)E(t)}{E(0)}\right)\mathcal{G}'(t) + E'(t) \\ &\leq -k_2 E(t)\overline{H}'\left(\varepsilon_0 \frac{\mathcal{B}(t)E(t)}{E(0)}\right) + k_6\mu(t)\overline{H}'\left(\varepsilon_0 \frac{\mathcal{B}(t)E(t)}{E(0)}\right) \\ &\quad + \frac{k_6}{\mathcal{B}(t)}\overline{H}^{-1}\left(\frac{\mathcal{B}(t)\Gamma(t)}{\xi(t)}\right)\Bigg)\overline{H}'\left(\varepsilon_0 \frac{\mathcal{B}(t)E(t)}{E(0)}\right) + E'(t) \end{aligned} \tag{3.59}$$

As in [3] , we define the conjugate function of $\overline{H}$ by $\overline{H}^*$, which satisfies

$$AB \leq \overline{H}^*(A) + \overline{H}(B) \tag{3.60}$$

For $A = \overline{H}'(\varepsilon_0(\mathcal{B}(t)E(t))/(E(0))))$ and $B = \overline{H}^{-1}((\mathcal{B}(t)\Gamma(t))/(\xi(t)))$ , and using (3.59), we get

$$\begin{aligned} \mathcal{K}_1'(t) &\leq -k_2 E(t)\overline{H}'\left(\varepsilon_0 \frac{\mathcal{B}(t)E(t)}{E(0)}\right) + k_6\mu(t)\overline{H}'\left(\varepsilon_0 \frac{\mathcal{B}(t)E(t)}{E(0)}\right) \\ &\quad + \frac{k_6}{\mathcal{B}(t)}\overline{H}^*\left(\overline{H}'\left(\varepsilon_0 \frac{\mathcal{B}(t)E(t)}{E(0)}\right)\right) + \frac{k_6}{\mathcal{B}(t)}\frac{\mathcal{B}(t)\Gamma(t)}{\xi(t)} + E'(t) \\ &\leq -k_2 E(t)\overline{H}'\left(\varepsilon_0 \frac{\mathcal{B}(t)E(t)}{E(0)}\right) + k_6\mu(t)\overline{H}'\left(\varepsilon_0 \frac{\mathcal{B}(t)E(t)}{E(0)}\right) \\ &\quad + \frac{k_6}{\mathcal{B}(t)}\varepsilon_0 \frac{\mathcal{B}(t)E(t)}{E(0)}\overline{H}'\left(\varepsilon_0 \frac{\mathcal{B}(t)E(t)}{E(0)}\right) + \frac{k_6\Gamma(t)}{\xi(t)} + E'(t) \end{aligned} \tag{3.61}$$

Multiplying (3.61) by $\xi(t)$, we see that

$$\begin{aligned} \xi(t)\mathcal{K}_1'(t) &\leq -k_2\xi(t)E(t)\overline{H}'\left(\varepsilon_0 \frac{\mathcal{B}(t)E(t)}{E(0)}\right) + k_6\xi(t)\mu(t)\overline{H}'\left(\varepsilon_0 \frac{\mathcal{B}(t)E(t)}{E(0)}\right) \\ &\quad + \frac{k_6\xi(t)}{\mathcal{B}(t)}\varepsilon_0 \frac{\mathcal{B}(t)E(t)}{E(0)}\overline{H}'\left(\varepsilon_0 \frac{\mathcal{B}(t)E(t)}{E(0)}\right) + k_6\Gamma(t) + \xi(t)E'(t) \\ &\leq -k_2\xi(t)E(t)\overline{H}'\left(\varepsilon_0 \frac{\mathcal{B}(t)E(t)}{E(0)}\right) + k_6\xi(t)\mu(t)\overline{H}'\left(\varepsilon_0 \frac{\mathcal{B}(t)E(t)}{E(0)}\right) \\ &\quad + \frac{k_6\xi(t)}{\mathcal{B}(t)}\varepsilon_0 \frac{\mathcal{B}(t)E(t)}{E(0)}\overline{H}'\left(\varepsilon_0 \frac{\mathcal{B}(t)E(t)}{E(0)}\right) - cE'(t) \end{aligned} \tag{3.62}$$

where we used the fact that as
$\varepsilon_0(\mathcal{B}(t)E(t)/E(0)) < r$, $\overline{H}'\left(\varepsilon_0(\mathcal{B}(t)E(t)/E(0))\right) = H'\left(\varepsilon_0(\mathcal{B}(t)E(t)/E(0))\right)$,
and we define the functional $\mathcal{K}_2(t)$ by

$$\mathcal{K}_2(t) = \xi(t)\mathcal{K}_1(t) + cE(t) \tag{3.63}$$

It is easy to obtain that $\mathcal{K}_2(t) \sim E(t)$, i.e., there exist two positive constants $m_1$ and $m_2$ such that

$$m_1\mathcal{K}_2(t) \leq E(t) \leq m_2\mathcal{K}_2(t), \tag{3.64}$$

we obtain

$$\begin{aligned} \mathcal{K}_2'(t) &\leq -\beta_3\xi(t)\frac{E(t)}{E(0)}H'\left(\varepsilon_0\frac{\mathcal{B}(t)E(t)}{E(0)}\right) + k_6\xi(t)\mu(t)H'\left(\varepsilon_0\frac{\mathcal{B}(t)E(t)}{E(0)}\right) \\ &= -\beta_3\frac{\xi(t)}{\mathcal{B}(t)}H_2\left(\frac{\mathcal{B}(t)E(t)}{E(0)}\right) + k_6\xi(t)\mu(t)H'\left(\varepsilon_0\frac{\mathcal{B}(t)E(t)}{E(0)}\right) \end{aligned} \tag{3.65}$$

where $\beta_3 = (k_2E(0) - \varepsilon_0k_6)$ and $H_2(t) = tH'(\varepsilon_0t)$.
Choosing $\varepsilon_0$ so small such that $\beta_3 > 0$, since $H_2'(t) = H'(\varepsilon_0t) + \varepsilon_0tH''(\varepsilon_0t)$, then, using the strict convexity of $H$ on $(0, r]$, we know that $H_2'(t), H_2(t) > 0$ on $(0, 1]$.
Using the generalized Young inequality (3.60) on the last term in (3.65)
with $A = H'\left(\varepsilon_0\frac{\mathcal{B}(t)E(t)}{E(0)}\right)$ and $B = \frac{k_6}{\delta}\mu(t)$, we find

$$\begin{aligned} k_6\mu(t)H'\left(\varepsilon_0\frac{\mathcal{B}(t)E(t)}{E(0)}\right) &= \frac{\delta}{\mathcal{B}(t)}\left(\frac{k_6}{\delta}\mathcal{B}(t)\mu(t)\right)\left(H'\left(\varepsilon_0\frac{\mathcal{B}(t)E(t)}{E(0)}\right)\right) \\ &< \frac{\delta}{\mathcal{B}(t)}H_3^*\left(\frac{k_6}{\delta}\mathcal{B}(t)\mu(t)\right) + \frac{\delta}{\mathcal{B}(t)}H_3\left(H'\left(\varepsilon_0\frac{\mathcal{B}(t)E(t)}{E(0)}\right)\right) \\ &< \frac{\delta}{\mathcal{B}(t)}H_4\left(\frac{k_6}{\delta}\mathcal{B}(t)\mu(t)\right) \\ &\quad + \frac{\delta}{\mathcal{B}(t)}\left(\varepsilon_0\frac{\mathcal{B}(t)E(t)}{E(0)}\right)H'\left(\varepsilon_0\frac{\mathcal{B}(t)E(t)}{E(0)}\right) \\ &< \frac{\delta}{\mathcal{B}(t)}H_4\left(\frac{k_6}{\delta}\mathcal{B}(t)\mu(t)\right) + \frac{\delta\varepsilon_0}{\mathcal{B}(t)}H_2\left(\varepsilon_0\frac{\mathcal{B}(t)E(t)}{E(0)}\right). \end{aligned} \tag{3.66}$$

Combining (3.65) and (3.66) and we select $\delta$ small enough such that $\beta_3 - \delta\varepsilon_0 > 0$, we obtain

$$\mathcal{K}_2'(t) \leq -\beta_4\frac{\xi(t)}{\mathcal{B}(t)}H_2\left(\frac{\mathcal{B}(t)E(t)}{E(0)}\right) + \frac{\delta\xi(t)}{\mathcal{B}(t)}H_4\left(\frac{k_6}{\delta}\mathcal{B}(t)\mu(t)\right). \tag{3.67}$$

where $\beta_4 = \beta_3 - \delta\varepsilon_0 > 0$, $H_3(t) = tH'^{-1}(t)$ and $H_4(t) = \overline{H}_3^*(t)$.
Since $E' < 0$ and $\mathcal{B}' < 0$, then $H_2(\frac{\mathcal{B}(t)E(t)}{E(0)})$ is decreasing. Hence, for $0 \leq t \leq T$, we have

$$H_2\left(\frac{\mathcal{B}(T)E(T)}{E(0)}\right) < H_2\left(\frac{\mathcal{B}(t)E(t)}{E(0)}\right). \tag{3.68}$$

Combining (3.67) with (3.68) and multiplying by $\mathcal{B}(t)$, we get

$$\mathcal{B}(t)\mathcal{K}_2'(t) + \beta_4\xi(t)H_2\left(\frac{\mathcal{B}(T)E(T)}{E(0)}\right) < \delta\xi(t)H_4\left(\frac{k_6}{\delta}\mathcal{B}(t)\mu(t)\right). \tag{3.69}$$

Since $\mathcal{B}' < 0$, then for any $0 < t < T$

$$(\mathcal{B}\mathcal{K}_2)'(t) + \beta_4\xi(t)H_2\left(\frac{\mathcal{B}(T)E(T)}{E(0)}\right) < \delta\xi(t)H_4\left(\frac{k_6}{\delta}\mathcal{B}(t)\mu(t)\right). \tag{3.70}$$

By integration of (3.70) over $[0, T]$ and we use $\mathcal{B}(0) = 1$, we have

$$H_2\left(\frac{\mathcal{B}(T)E(T)}{E(0)}\right)\int_0^T \xi(t)dt < \frac{\mathcal{K}_2(0)}{\beta_4} + \frac{\delta}{\beta_4}\int_0^T \xi(t)H_4\left(\frac{k_6}{\delta}\mathcal{B}(t)\mu(t)\right)dt. \tag{3.71}$$

Therefor,

$$H_2\left(\frac{\mathcal{B}(T)E(T)}{E(0)}\right) < \frac{\frac{\mathcal{K}_2(0)}{\beta_4} + \frac{\delta}{\beta_4}\int_0^T \xi(t)H_4(\frac{k_6}{\delta}\mathcal{B}(t)\mu(t))dt}{\int_0^T \xi(t)dt}. \tag{3.72}$$

Then,

$$\left(\frac{\mathcal{B}(T)E(T)}{E(0)}\right) < H_2^{-1}\left(\frac{\frac{\mathcal{K}_2(0)}{\beta_4} + \frac{\delta}{\beta_4}\int_0^T \xi(t)H_4(\frac{k_6}{\delta}\mathcal{B}(t)\mu(t))dt}{\int_0^T \xi(t)dt}\right). \tag{3.73}$$

Hence,

$$E(T) < \frac{E(0)}{\mathcal{B}(T)}H_2^{-1}\left(\frac{\frac{\mathcal{K}_2(0)}{\beta_4} + \frac{\delta}{\beta_4}\int_0^T \xi(t)H_4(\frac{k_6}{\delta}\mathcal{B}(t)\mu(t))dt}{\int_0^T \xi(t)dt}\right). \tag{3.74}$$

which gives (3.26) with $\tau_1 = \frac{E(0)}{\mathcal{B}(T)}$, $\tau_2 = \frac{\mathcal{K}_2(0)}{\beta_4}$, $\tau_3 = \frac{\delta}{\beta_4}$ , and $\tau_4(t) = \frac{k_6}{\delta}\mathcal{B}(t)$. This ends the proof of Theorem 3.8.

□

## 4. conclusion

In this work, we impose a several dissipations (Infinite memory, distributed delay and Balakrishnan-Taylor damping terms) on the viscoelastic wave equation. This type of damping mechanisms is found to be effective in various other systems and problems especially (Infinite memory or distributed delay terms) like Timoshenko (see [10],[19]), porous system (see [13]), Bress (see[9]), Kirchhoff equation ([8],[12],[20]) and others. Under this very general hypothesis on the behavior of $h$ at infinity and by drop the boundedness hypothesis in the history data, we obtain a general decay result. We strongly believe that the same result holds if the damping terms is moved to the Kirchhoff equation or coupled system of nonlinear viscoelastic wave equation.

## References

[1] R. Adams, J. Fourier.: *Sobolev Space.* Academic Press. New York (2003).

[2] A.M. Al-Mehdi, M. M. Al-Gharabli, S. A. Messaoudi.: New general decay result for a system of viscoelastic wave equation with past hystory.*Communications On Pure and Applied Analysis.* V 20,(1)(2021). Doi:10.3934/cpaa.2020273.

[3] V. I. Arnold.:*Mathematical Methods of Classical Mechanics.* Springer-Verlag, New York, NY, USA, 1989.

[4] AV. Balakrishnan, LW. Taylor.: Distributed parameter nonlinear damping models for flight structures. In: Proceedings (Damping 89), Flight Dynamics Lab and Air Force Wright Aeronautical Labs, Washington: WPAFB; 1989.

[5] RW. Bass, D. Zes.: Spillover nonlinearity, and flexible structures. The Fourth NASA Workshop on Computational Control of Flexible Aerospace Systems.Washington:*NASA Conference Publication* 10065 ( 1991)1–14.

[6] D.R. Bland.: The theory of linear viscoelasticity. Mineola: Courier Dover Publications(2016).

[7] S. Boulaaras, A. Choucha, D. Ouchenane, B. Cherif.: Blow up of solutions of two singular nonlinear viscoelastic equations with general source and localized frictional damping terms, *Adv. Differ. Equ.* (2020) 310.

[8] S. Boulaaras, A. Draifia, Kh. Zennir.: General decay of nonlinear viscoelastic Kirchhoff equation with Balakrishnan-Taylor damping and logarithmic nonlinearity,*Math Meth Appl Sci.* 42(2019)4795–4814.

[9] A. Choucha, D. Ouchenane, Kh. Zennir and B. Feng.: Global well-posedness and exponential stability results of a class of Bresse-Timoshenko-type systems with distributed delay term. *Math Meth Appl Sci.* (2020) 1–26. DOI: 10.1002/mma.6437.

[10] A. Choucha, D. Ouchenane and S. Boulaaras.: Well Posedness and Stability result for a Thermoelastic Laminated Timoshenko Beam with distributed delay term.*Math Meth Appl Sci.* (2020) 1–22. DOI: 10.1002/mma.6673.

[11] A. Choucha, D. Ouchenane and S. Boulaaras.: Blow-up of a nonlinear viscoelastic wave equation with distributed delay combined with strong damping and source terms.*J. Nonlinear Funct. Anal.* (2020), Article ID 31 https://doi.org/10.23952/jnfa.2020.31.

[12] A. Choucha, S. Boulaaras, D. Ouchenane and S. Beloul.: General decay of nonlinear viscoelastic Kirchhoff equation with Balakrishnan-Taylor damping, logarithmic nonlinearity and distributed delay terms.*Math Meth Appl Sci.* (2020) 1–22. DOI:10.1002/mma.7121.

[13] A. Choucha, S. M. Boulaaras, D. Ouchenane, B.B. Cherif and M. Abdalla.: Exponential Stability of Swelling Porous Elastic with a Viscoelastic Damping and Distributed Delay Term.*JFS.* V( 2021), Article ID 5581634, https://doi.org/10.1155/2021/5581634.

[14] B.D. Coleman and W. Noll.: Foundations of linear viscoelasticity.*Reviews of Modern Physics*, 33(2)(1961)239.

[15] C. M. Dafermos:. Asymptotic stability in viscoelasticity. Arch.*Rational Mech. Anal.* 37(9-10)(1970)297–308.

[16] B. Feng, A. Soufyane.: Existence and decay rates for a coupled Balakrishnan-Taylor viscoelastic system with dynamic boundary conditions.*Math. Methods. Sci.* 43(2020) 3375–3391.

[17] B. Gheraibia, N. Boumaza.: General decay result of solution for viscoelastic wave equation with Balakrishnan-Taylor damping and a delay term.*ZAMP.*71(2020)198. Doi.org/10.1007/s00033-020-01426-1.

[18] A. Guesmia.: New general decay rates of solutions for two viscoelastic wave equations with infinite memory.*Math. Model. Anal.*, 25 (2020) 351–373.

[19] A. Guesmia and N. Tatar. Some well-posedness and stability results for abstract hyperbolic equations with infinite memory and distributed time delay, Hal-Inria , (2015).

[20] W. Liu, B. Zhu, G. Li, D. Wang.: General decay for a viscoelastic Kirchhoof equation with Balakrishnan-Taylor damping, dynamic boundary conditions and a time-varying delay term. *Evol. Equ. Control Theory.* 6(2017) 239–260.

[21] F. Mesloub, S. Boulaaras.: General decay for a viscoelastic problem with not necessarily decreasing kernel.*J Appl Math Comput.* 58(2018) 647–665. doi:10.1007/S12190-017-1161-9.

[22] C. Mu, J. Ma.: On a system of nonlinear wave equations with Balakrishnan-Taylor damping. *Z Angew Math Phys.* 65(2014) 91–113.

[23] M. I. Mustafa.: General decay result for nonlinear viscoelastic equations. *J. Math. Anal. Appl.* 457 (1)(2018) 134– 152.

[24] A. S. Nicaise and C. Pignotti.: Stabilization of the wave equation with boundary or internal distributed delay. *Diff. Int. Equs.*, 21(9-10)(2008) 935–958.

[25] D. Ouchenane, S. Boulaaras, F. Mesloub.: General decay for a viscoelastic problem with not necessarily decreasing kernel. *Applicable Analysis.* 98(44)(2018) 1–17. DOI: 10.1080/00036811.2018.1437421.

[26] A. Zarai, N. Tatar.: Global existence and polynomial decay for a problem with Balakrishnan-Taylor damping. *Arch Math(BRNO)*. (2010), 46: 157–176.

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