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STABILITY ANALYSIS FROM FOURTH ORDER NONLINEAR EVOLUTION EQUATION FOR GRAVITY WAVES INCLUDING THE EFFECT OF THIN THERMOCLINE AND UNIFORM WIND FLOW

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ABSTRACT. An evolution equation that is appropriate up to fourth order in wave steepness has been ascertained for the deep water gravity waves in a situation of thin thermocline when the effect of wind blowing over water has been taken into account. Later it has been employed to discuss the stability of the uniform gravity wavetrain. Along with the wave number at marginal stability, the expression for maximum instability growth rate has been acquired immediately. From the graphs, it has been investigated that for fixed values of wind velocity, the effect of a thin thermocline appearing at depth h produces a stabilizing effect. Here the maximum instability growth rate diminishes with the increment of thermocline depth h. Moreover, it has been additionally inferred that the maximum instability growth rate enhances with the increment of perturbation angle θ and wind velocity u for fixed value of thermocline depth h.

1. Introduction

The bulk of fascinating papers have been emerged on nonlinear interaction between internal waves and surface gravity waves in recent times, among which many papers are dealing with the procedure of formation of internal waves by coupling with surface gravity waves. Phillips [1], Gargett and Hughes [2] have exhibited that the appearance of an internal wave would, in general, tend to modulate a surface wave spectrum, which is by virtue of non-uniform surface current induced by the internal wave that is refracting the surface waves. Obviously, the interaction is mutual since a non-uniform surface wave spectrum originates the internal wave. Longuet-Higgins and Stewart [3] have demonstrated that the nonuniformities in the surface wave spectrum produce corresponding nonuniformities in the radiation pressure and, therefore the surface forces are acting on the internal wave. Ball [4], Thorpe [5], Watson et al. [6] have cultivated interactions of two surface waves including one internal wave. Dysthe and Das [7] have explored a modulational instability mechanism for the creation of the lowest internal wave mode employing a pattern of the three-layer ocean. Recently, Manna and Dhar [8] have studied the effect of vorticity on Peregrine breather for finite amplitude interfacial gravity waves. In another paper [9], they have also obtained the analytical solutions from coupled nonlinear evolution equations for obliquely interacting Stokes waves in deep water.

Now it is of remarkable significance to investigate the reverse problem. In this premise, we want to investigate how the surface gravity waves' amplitude is modulated by the interaction with internal waves. Das [10] has derived the fourth-order nonlinear evolution equation of a three-dimensional surface gravity wave packet for a two-layer fluid considering the impact of its interaction with internal waves. Rizk and Ko [11] have considered similar sort of interactions between small scale surface waves and extensive scale internal waves in a two-layer liquid in which their analysis is confined to one-dimensional wave packets along with the variance of small density ratio.

The nonlinear evolution equation of fourth-order, which was first derived by Dysthe [12], is a good beginning point to investigate the stability analysis of nonlinear water waves in deep water. Brinch-Nielsen and Jonsson [13] have likewise determined the evolution equation of the fourth-order for Stokes wave on unrestricted water depth. Further, Janssen [14] has manifested on the Dysthe's [12] approach by exploring the impact of wave-induced flow on the extended characteristic of Benjamin-Feir instability. Additionally, he has utilized this equation to the homogeneous random field of gravity waves and has found the nonlinear energy transfer function that has been invented by Dungey and Hui [15]. Since then a number of authors Dhar and

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Das [16, 17], Dhar and Mondal [18], Dhar and Manna [19], Debsarma and Das [20, 21], Gramstad and Trulsen [22], Hogan [23] and Stiassnie [24] have established the nonlinear evolution of fourth-order in different contexts. Purkait and Debsarma [25] have recently established a fourth order nonlinear evolution equations for two obliquely interacting wave trains including the effect of thin pycnocline.

Considering the significance of the nonlinear evolution equation of fourth-order derived by Dysthe [12], in this paper, we have extended the investigation of Dysthe [12] in a situation of thin thermocline and taking into account the effect of uniform wind flow. Accordingly, the present paper deals with the wind effect on Benjamin-Feir instability. In this paper, we have derived a nonlinear evolution equation of fourth-order which is appropriate in case of the velocity of wind lesser than a critical velocity, which is characterized by the fact that the wave turns out to be linearly unstable as the velocity of wind exceeds this critical velocity.

2. Basic assumptions and equations

When we deal with the mutual interaction between surface waves and internal waves in the ocean, we consider the following assumptions. It appears from some articles (e.g. Watson et.al. [6], Olbers and Herterich [26]) that the strongest interaction appears for the lowest internal mode. Thus being interested mainly in the lowest internal mode (Phillips [27], P-211), we have considered the following simple model of the ocean for convenience as displayed in figure-1,

(i) a homogeneous layer of thickness h,

(ii) a layer of thickness 2ε below this layer, with a shallow thermocline region of changing density. Thus our model agrees with the seasonal thermocline,

(iii) the ocean below the thermocline region is further taken to be homogeneous.

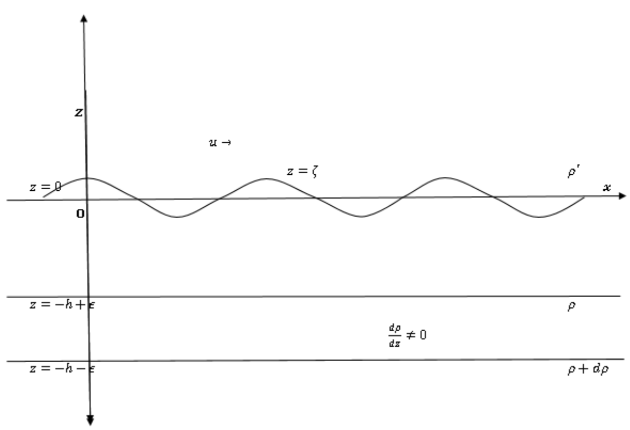


FIGURE 1. **Ocean model in a situation of thin thermocline at a depth** h

We have chosen the interface of air and water in the undisturbed state as the $z = 0$ plane and wind blows over water with velocity u along the x axis. The equation $z = \zeta(x, y, t)$ is considered as undulating common interface at any time t in the perturbed state. Taking ρ and ρ' as the densities of water and air respectively, we now consider non-dimensional quantities ϕ^* as perturbed velocity potential in water, ${\phi_a}^*$ as the perturbed velocity potential in air, ζ^* the wavy surface elevation of the air-water interface, (x^*,y^*,z^*) for the space coordinates, t^*, u^*, w^*, γ^*, as the time, air flow velocity, vertical component of velocity inside the thermocline and air to water density ratio respectively, which are connected to the corresponding dimensionless quantities as follows:

$$\phi^* = \sqrt{\frac{k_0^3}{g}}\phi,\ \phi_a{}^* = \sqrt{\frac{k_0^3}{g}}\phi_a,\ \zeta^* = k_0\zeta,\ (x^*,y^*,z^*) = (k_0x,k_0y,k_0z),$$
$$h^* = kh,\ t^* = \omega t,\ u^* = \sqrt{\frac{k_0}{g}}u,\ \gamma^* = \frac{\rho'}{\rho}\ ,\ w^* = \sqrt{\frac{k_0}{g}}w \tag{2.1}$$

Here, k_0 is the characteristic wave number. Later we have used all those non-dimensional quantities by removing their asterisks. The velocity potential of water ϕ and the elevation of the free surface ζ can be written into two parts given by,

$$\phi = \phi_0 + \phi_l,\ \zeta = \zeta_0 + \zeta_l\ , \tag{2.2}$$

where ϕ_0, ζ_0 are real functions gradually varying with space and time, expressing the perturbations formed by the internal wave due to the presence of a thin thermocline and by spatial non-uniformity of the surface wave radiation stress and ϕ_l, ζ_l correspond to the surface gravity waves [7]. As surface gravity waves and internal waves interact non-linearly, we write the full expressions for ϕ and ζ through nonlinear terms. The governing equations for ϕ_l, ϕ_a and ζ_l in which we have not considered the existence of the thermocline are the following equations (2.3) to (2.8):

$$\nabla^2\phi_l = 0,\ -\infty < z < \zeta \tag{2.3}$$

$$\nabla^2\phi_a = 0,\ \zeta < z < \infty \tag{2.4}$$

$$\frac{\partial\phi_l}{\partial z} - \frac{\partial\zeta_l}{\partial t} = (\nabla_{xy}\ \phi).(\nabla_{xy}\ \zeta),\ \text{on}\ z=\zeta \tag{2.5}$$

$$\frac{\partial\phi_a}{\partial z} - \frac{\partial\zeta_l}{\partial t} - u\frac{\partial\zeta_l}{\partial x} = (\nabla_{xy}\ \phi_a).(\nabla_{xy}\ \zeta),\ \text{on}\ z=\zeta \tag{2.6}$$

$$\frac{\partial\phi_l}{\partial t} - \gamma\frac{\partial\phi_a}{\partial t} + (1-\gamma)\zeta - \gamma u\frac{\partial\phi_a}{\partial x} = -\frac{1}{2}(\nabla\phi)^2 + \frac{\gamma}{2}(\nabla\phi_a)^2,\ \text{on}\ z=\zeta \tag{2.7}$$

$$\frac{\partial\phi_l}{\partial z} \to 0\ \text{as}\ z\to-\infty,\frac{\partial\phi_a}{\partial z} \to 0\ \text{as}\ z\to\infty\ , \tag{2.8}$$

where $\nabla_{xy} \equiv (\frac{\partial}{\partial x},\frac{\partial}{\partial y},0)$ is the horizontal gradient operator. We consider that the thermocline thickness 2ε is small so that it is confined between the planes $z=-h+\varepsilon$ and $z=-h-\varepsilon$. The terms responsible for internal waves of long wavelength are taken to be sufficiently small and hence their product and higher degree terms can be neglected [7]. The equations for ϕ_0,ϕ_0',ζ_0, where ϕ_0 and ϕ_0' are the velocity potentials above and below the thermocline respectively, can be expressed as follows

$$\nabla^2\phi_0 = 0,\ -h+\varepsilon < z < \zeta \tag{2.9}$$

$$\nabla^2\phi_0' = 0,\ -\infty < z < -h-\varepsilon \tag{2.10}$$

$$\left(\frac{\partial\phi_0}{\partial z}\right)_{z=-h+\varepsilon} = \left(\frac{\partial\phi_0'}{\partial z}\right)_{z=-h-\varepsilon} \tag{2.11}$$

$$\frac{\partial^2(\nabla^2 w)}{\partial t^2} + N^2(z)(\nabla_{xy}^2\ w) = 0,\ -h-\varepsilon < z < -h+\varepsilon \tag{2.12}$$

$$\frac{\partial\phi_0'}{\partial z} \to 0\ \text{as}\ z\to-\infty \tag{2.13}$$

$$\left(\frac{\partial\phi_0}{\partial z}\right)_{z=0} - \frac{\partial\zeta_0}{\partial t} = (\nabla_{xy}\ \phi).(\nabla_{xy}\ \zeta) \tag{2.14}$$

(2.15) $$\frac{\partial \phi_0}{\partial t} + \zeta_0 = -\frac{1}{2}(\nabla\phi)^2, \text{ on } z = \zeta$$

where w is the vertical velocity component inside thermocline [1, 7] and $N^2(z) = -\frac{1}{\rho}\frac{d\rho}{dz}$ is the square of the Brunt-Vaisala frequency. Equation (2.12) is found if the Boussinesq approximation is used (Phillips [27], P-211) and the internal wave is assumed to have a sufficiently small amplitude so that the nonlinear terms can be neglected. As the thickness of the thermocline is considered as very small compared to the depth of the ocean we have the limit $\varepsilon \to 0$, so that the equation (2.11) reduces to,

(2.16) $$\left(\frac{\partial \phi_0}{\partial z}\right)_{z=-h} = \left(\frac{\partial \phi_0'}{\partial z}\right)_{z=-h}$$

Next integrating equation (2.12) with respect to z between the limits $-h-\varepsilon$ to $-h+\varepsilon$ and then proceeding the limit $\varepsilon \to 0$, we have

(2.17) $$\left(\frac{\partial^4 \phi_0}{\partial z^2 \partial t^2}\right)_{z=-h} - \left(\frac{\partial^4 \phi_0'}{\partial z^2 \partial t^2}\right)_{z=-h} = -\Gamma\nabla_{xy}^2\left(\frac{\partial \phi_0}{\partial z}\right)_{z=-h},$$

where $\Gamma = \frac{\delta\rho}{\rho}$ and $\delta\rho = \rho(-h-0) - \rho(-h+0)$ is the increment of density across the thermocline. As the disturbance is considered to be a progressive wave, we take the solutions of the governing equations in the following form,

(2.18)
$$\phi = \phi_0 + \sum_{i=1}^{\infty}(\phi_n e^{in\psi} + \phi_n^* e^{-in\psi}),\ \zeta = \zeta_0 + \sum_{i=1}^{\infty}(\zeta_n e^{in\psi} + \zeta_n^* e^{-in\psi}),\ \phi_a = \phi_{a0} + \sum_{i=1}^{\infty}(\phi_{an} e^{in\psi} + \phi_{an}^* e^{-in\psi}),$$

where $\psi = kx - \omega t$ and ϕ_0, ϕ_{a0}, ϕ_n, ϕ_{an}, ϕ_n^*, ϕ_{an}^* are functions of z, $\xi = \delta(x - c_g t)$, $\eta = \delta y$, $\tau = \delta^2 t$, furthermore ζ_0, ζ_n, ζ_n^* are functions of ξ, η, τ. Here, asterisk denotes the complex conjugate, δ is a small ordering parameter indicating the slowness of the space-time variation of the amplitude of different harmonics of the wave and $c_g = \frac{d\omega}{dk}$ is the group velocity. The linear dispersion relation is given by,

(2.19) $$(1+\gamma)\omega^2 - 2\gamma\omega u + \gamma u^2 - (1-\gamma) = 0$$

From this equation we have two values of ω given by

(2.20) $$\omega = \frac{\gamma u \pm \sqrt{(1-\gamma^2-\gamma u^2)}}{1+\gamma}.$$

For linear stability the wind velocity u must satisfy the condition $|u| < \sqrt{\frac{1-\gamma^2}{\gamma}}$.

3. Derivation of evolution equation

Now, if we substitute the expansions (2.18) in equations (2.3) and (2.4) and equate the coefficients of $e^{in\psi}$, n=1,2 on both sides, we arrive at the following equations:

(3.1) $$\frac{d^2\phi_n}{dz^2} - \Delta_n^2\phi_n = 0\ ,\ \frac{d^2\phi_{a_n}}{dz^2} - \Delta_n^2\phi_{a_n} = 0\ ,$$

where Δ_n is the operator given by,

(3.2) $$\Delta_n^2 = \left(nk - i\delta\frac{\partial}{\partial\xi}\right)^2 - \delta^2\frac{\partial^2}{\partial\eta^2}$$

The solutions of (3.1) satisfying (2.8) can be written in the form

(3.3) $$\phi_n = e^{z\Delta_n}A_n,\ \phi_{an} = e^{-z\Delta_n}A_n',$$

where A_n, A_n' are functions of ξ, η, τ. We now employ the Fourier transforms of equations (2.9), (2.10), (2.13) and (2.4), (2.8) for n=0 with respect to ξ, η and obtain the following equations

(3.4) $$\frac{d^2\overline{\phi}_0}{dz^2} - \delta^2\overline{k}^2\overline{\phi}_0 = 0$$

$$\frac{d^2\overline{\phi'}_0}{dz^2}-\delta^2\overline{k}^2\overline{\phi'}_0=0,\ \frac{d\overline{\phi'}_0}{dz}\to 0 \text{ as } z\to-\infty \tag{3.5}$$

$$\frac{d^2\overline{\phi}_{a_0}}{dz^2}-\delta^2\overline{k}^2\overline{\phi}_{a_0}=0,\ \frac{d\overline{\phi}_{a_0}}{dz}\to 0 \text{ as } z\to\infty \tag{3.6}$$

In the above equations, $\overline{\phi}_0$, $\overline{\phi'}_0$, $\overline{\phi}_{a_0}$ are Fourier transforms of ϕ_0, ϕ_0', ϕ_{a_0} respectively, defined by

$$(\overline{\phi}_0,\overline{\phi'}_0,\overline{\phi}_{a_0})=\frac{1}{2\pi}\iint_{-\infty}^{\infty}(\phi_0,\phi'_0,\phi_{a_0})\mathrm{e}^{-\mathrm{i}(\mathrm{k}_\xi\xi+\mathrm{k}_\eta\eta)}\,\mathrm{d}\xi\mathrm{d}\eta \tag{3.7}$$

The solution of equations (3.4), (3.5), (3.6) are given by,

$$\overline{\phi}_0=\overline{A}_0e^{\delta\bar{k}z}+\overline{B}_0e^{-\delta\bar{k}z},\ \overline{\phi'}_0=\overline{C}_0e^{\delta\bar{k}z},\ \overline{\phi}_{a_0}=\overline{D}_0e^{-\delta\bar{k}z}, \tag{3.8}$$

where $\overline{k}^2=k_\xi^2+k_\eta^2$ and $\overline{A}_0$, $\overline{B}_0$, $\overline{C}_0$, $\overline{D}_0$ are functions of k_ξ, k_η and τ.

If we now substitute the expansions (2.18) in the Taylor expanded forms of the equations given by (2.5), (2.6) and (2.7) about $z=0$ and then equate the coefficients of $e^{in\psi}$, for $n=1,2$ on both sides, we get equations for A_1, A_1', ζ_1, A_2, A_2', ζ_2 available in (A1) to (A6) in which we substitute the solutions given by (3.3) for ϕ_1, ϕ_2, ϕ_{a_1}, ϕ_{a_2}. Again substituting expansions (2.18) on right hand side of the equations (2.14), (2.15) and considering only ψ free terms we obtain equations (A7), (A8). Now taking the Fourier transform of equations (2.16), (2.17) with respect to ξ, η and then inserting the solutions for $\overline{\phi}_0$, $\overline{\phi'}_0$ given by (3.8) we obtain another two equations (A9), (A10). Eliminating $\overline{C}_0$ from (A9), (A10) we obtain the new equation (A11). All these equations (A1)-(A11) has been provided in Appendix A. Thus we obtain three sets of equations: (A1), (A2), (A3) comprising the first set, the equations (A4), (A5), (A6) constituting the second set and finally the equations (A7), (A8), (A11) to constitute the third set. To solve the aforesaid three sets of equations we employ the following perturbation expansions for the quantities A_1, A_1', A_2, A_2', ζ_1, ζ_2, $\overline{M}$, $\overline{N}$, ζ_0 as follows:

$$(A_n,A_n')=\sum_{j=n}^{\infty}\varepsilon^j(A_{nj},A'_{nj}),\ \zeta_n=\sum_{j=n}^{\infty}\varepsilon^j\zeta_{nj},\ (n\geqslant 1);\ (\overline{M},\overline{N})=\sum_{j=n}^{\infty}\varepsilon_j(\overline{M}_j,\overline{N}_j),\ \zeta_0=\sum_{j=n}^{\infty}\varepsilon^j\zeta_{0j},\ (n\geqslant 1) \tag{3.9}$$

We now substitute the expansions given by (3.9) in those aforesaid three sets of equations and equating the coefficients of several powers of δ on both sides, we obtain a sequence of equations. Now from the first(i.e. lowest order) and second order equations of (A1) and (A2) we find the solutions for A_{11}, A'_{11} and A_{12}, A'_{12} respectively. After that, from the first order and second order equations of (A4), (A5), and (A6) we get the solutions for A_{22}, A'_{22}, ζ_{22} and A_{23}, A'_{23}, ζ_{23} respectively. Again from first, second and third order equations of (A8), we find solutions for ζ_{01}, ζ_{02} and ζ_{03}. Finally from the first order and second order equations of (A7) we find solutions for N_1 and N_2. Equations (A3) and (A11), which have not been employed in getting the above perturbation solutions can be expressed in the following two coupled fourth order nonlinear evolution equations after eliminating A_1, A_1' by the use of the equations (A1) and (A2).

$$\begin{aligned}&i\frac{\partial\zeta}{\partial\tau}+\frac{1}{2}\frac{dc_g}{dk}\frac{\partial^2\zeta}{\partial\xi^2}+\frac{(1-\gamma)}{4[(1+\gamma)\omega-\gamma u]}\frac{\partial^2\zeta}{\partial\eta^2}+\frac{i}{2}\left[\frac{(1+\gamma)c_g-\gamma u}{(1+\gamma)\omega-\gamma u}\right]\frac{dc_g}{dk}\frac{\partial^3\zeta}{\partial\xi^3}\\&+\frac{i(1-\gamma)}{4[(1+\gamma)\omega-\gamma u]}\left[1+\frac{(1+\gamma)c_g-\gamma u}{(1+\gamma)\omega-\gamma u}\right]\frac{\partial^3\zeta}{\partial\xi\partial\eta^2}=\left[\frac{\omega^2+\gamma(\omega-u)^2+\frac{\{\omega^2-\gamma(\omega-u)^2\}^2}{1-\gamma}}{(1+\gamma)\omega-\gamma u}\right]\zeta^2\zeta^*\\&\qquad+\mu_1\zeta\zeta^*\frac{\partial\zeta}{\partial\xi}+\mu_2\zeta^2\frac{\partial\zeta^*}{\partial\xi}+\frac{c_g\{\omega^2-\gamma(\omega-u)^2\}}{\{(1+\gamma)\omega-\gamma u\}}\zeta\frac{\partial M}{\partial\xi}\end{aligned} \tag{3.10}$$

$$\frac{\partial^2M}{\partial\xi^2}-\frac{\Gamma h}{c_g^2}\left(\frac{\partial^2}{\partial\xi^2}+\frac{\partial^2}{\partial\eta^2}\right)M=\frac{4\Gamma}{c_g^2}\frac{\partial}{\partial\xi}(\zeta\zeta^*)+4\frac{\partial}{\partial\xi}\left[H\left(\frac{\partial}{\partial\xi}(\zeta\zeta^*)\right)\right] \tag{3.11}$$

Here $\zeta=\zeta_{11}+\delta\zeta_{12}$, $M=M_1+\delta M_2$ and the operator H is the two dimensional version of the Hilbert transform defined as

$$H\psi=\frac{1}{2\pi}\iint_{-\infty}^{\infty}\frac{(\xi'-\xi)}{[(\xi'-\xi)^2+(\eta'-\eta)]^{\frac{3}{2}}}\psi(\xi',\eta')\mathrm{d}\xi'\mathrm{d}\eta' \tag{3.12}$$

In the absence of the seasonal thermocline we have $\Gamma = 0$ and hence the equation (3.11) reduces to

$$\frac{\partial M}{\partial \xi} = 4H\left[\frac{\partial}{\partial \xi}(\zeta\zeta^*)\right] \tag{3.13}$$

Inserting the value of $\frac{\partial M}{\partial \xi}$ in (3.10) we obtain the single non-linear evolution equation as follows:

$$\begin{aligned} &i\frac{\partial \zeta}{\partial \tau} + \frac{1}{2}\frac{dc_g}{dk}\frac{\partial^2\zeta}{\partial\xi^2} + \frac{(1-\gamma)}{4[(1+\gamma)\omega-\gamma u]}\frac{\partial^2\zeta}{\partial\eta^2} + \frac{i}{2}\left[\frac{(1+\gamma)c_g-\gamma u}{(1+\gamma)\omega-\gamma u}\right]\frac{dc_g}{dk}\frac{\partial^3\zeta}{\partial\xi^3} \\ &+ \frac{i(1-\gamma)}{4[(1+\gamma)\omega-\gamma u]}\left[1+\frac{(1+\gamma)c_g-\gamma u}{(1+\gamma)\omega-\gamma u}\right]\frac{\partial^3\zeta}{\partial\xi\partial\eta^2} = \left[\frac{\omega^2+\gamma(\omega-u)^2+\frac{\{\omega^2-\gamma(\omega-u)^2\}^2}{1-\gamma}}{(1+\gamma)\omega-\gamma u}\right]\zeta^2\zeta^* \\ &+ \mu_1\zeta\zeta^*\frac{\partial\zeta}{\partial\xi} + \mu_2\zeta^2\frac{\partial\zeta^*}{\partial\xi} + \frac{4c_g\{\omega^2-\gamma(\omega-u)^2\}}{\{(1+\gamma)\omega-\gamma u\}}\zeta H\left[\frac{\partial}{\partial\xi}(\zeta\zeta^*)\right] \end{aligned} \tag{3.14}$$

If we consider $\gamma = 0$, $u = 0$, then equation (3.14) reduces to the equation equivalent to equation (2) of Janssen [14] and equation (2.20) of Hogan [23] for $\kappa = 0$. Now, we want to establish a nonlinear evolution equation of fourth order from the aforesaid two coupled equations (3.10) and (3.11) in which we consider that the space variation of amplitudes taking place along a direction, forming an angle θ with the the direction of propagation of the wave. By a linear orthogonal transformation of the horizontal coordinates ξ, η employing the relations (3.15) the equations (3.10) and (3.11) are then reduce to the following two coupled equations on the assumption that ζ, M depend only on ξ' but not on η'.

$$\xi' = \xi\cos\theta + \eta\sin\theta,\ \eta' = -\xi\sin\theta + \eta\cos\theta \tag{3.15}$$

(3.16)

$$\begin{aligned} &i\frac{\partial\zeta}{\partial\tau} + \left[\frac{\cos^2\theta}{2}\frac{dc_g}{dk} + \frac{(1-\gamma)\sin^2\theta}{4[(1+\gamma)\omega-\gamma u]}\right]\frac{\partial^2\zeta}{\partial\xi'^2} + \left[\frac{i(1-\gamma)\sin^2\theta\cos\theta}{4[(1+\gamma)\omega-\gamma u]}\left\{1+\frac{(1+\gamma)c_g-\gamma u}{(1+\gamma)\omega-\gamma u}\right\}\right]\frac{\partial^3\zeta}{\partial\xi'^3} \\ &+ \left[\frac{i\cos^3\theta}{2}\left\{\frac{(1+\gamma)c_g-\gamma u}{(1+\gamma)\omega-\gamma u}\right\}\frac{dc_g}{dk}\right]\frac{\partial^3\zeta}{\partial\xi'\partial\eta'^2} = \left[\frac{\omega^2+\gamma(\omega-u)^2+\frac{\{\omega^2-\gamma(\omega-u)^2\}^2}{1-\gamma}}{(1+\gamma)\omega-\gamma u}\right]\zeta^2\zeta^* + \mu_1\cos\theta\ \zeta\zeta^*\frac{\partial\zeta}{\partial\xi'} \\ &+ \mu_2\cos\theta\ \zeta^2\frac{\partial\zeta^*}{\partial\xi'} + \frac{c_g\cos\theta\ \{\omega^2-\gamma(\omega-u)^2\}}{\{(1+\gamma)\omega-\gamma u\}}\zeta\frac{\partial M}{\partial\xi'} \end{aligned}$$

$$\frac{\partial M}{\partial\xi'} = \frac{4\Gamma\cos\theta}{c_g^2\cos^2\theta\ -\Gamma h}\zeta\zeta^* + \frac{4c_g^2\cos^3\theta}{c_g^2\cos^2\theta\ -\Gamma h}\frac{1}{\pi}P\int_{-\infty}^{\infty}\frac{d\tilde{\xi}}{(\tilde{\xi}-\xi')}\frac{\partial}{\partial\tilde{\xi}}(\zeta\zeta^*) \tag{3.17}$$

Eliminating $\frac{\partial M}{\partial\xi'}$ between the equations (3.16) and (3.17) and dropping the primes on ξ, we obtain a fourth order nonlinear evolution equation for ζ in the presence of a thin thermocline and in the case of air water interface given by:

$$i\frac{\partial\zeta}{\partial\tau} - \beta_1\frac{\partial^2\zeta}{\partial\xi^2} + i\beta_2\frac{\partial^3\zeta}{\partial\xi^3} = \Lambda_1\zeta^2\zeta^* + i\Lambda_2\zeta\zeta^*\frac{\partial\zeta}{\partial\xi} + i\Lambda_3\zeta^2\frac{\partial\zeta^*}{\partial\xi} + \Lambda_4\zeta H\left\{\frac{\partial}{\partial\xi}(\zeta\zeta^*)\right\}, \tag{3.18}$$

where the Hilbert transform operator H is given by $H\psi = \frac{1}{\pi}P\int_{-\infty}^{\infty}\frac{\psi(\xi')}{(\xi'-\xi)}d\xi'$ and the coefficients β_1, β_2, Λ_1, Λ_2, Λ_3, Λ_4 have been provided in Appendix B.

Since in the expressions for Λ_1 and Λ_4 containing the factor $c_g^2\cos^2\theta\ -\Gamma h$ in their denominators, the nonlinear evolution equation of fourth order given by (3.18) remains invalid for $c_g^2\cos^2\theta\ = \Gamma h$. This is the second harmonic condition of resonance which is satisfied when the group velocity component of the surface gravity wave along a line creating an angle θ with the propagation direction of wave is equal to the phase velocity of the internal wave of long wavelength.

4. Stability of a finite amplitude wave train

The solution of the uniform wave train of equation given by (3.18) can be written as

$$\zeta = \alpha_0 e^{(-i\Lambda_1 \alpha_0{}^2 \tau)} \equiv \zeta^{(0)} , \tag{4.1}$$

where α_0 is a real constant that indicates the wave steepness. We now consider the following perturbation,

$$\zeta = \zeta^{(0)}[1+\tilde{\zeta}(\xi,\tau)] , \tag{4.2}$$

where $\tilde{\zeta}$ is a complex quantity given by $\tilde{\zeta} = \tilde{\zeta}_r + i\tilde{\zeta}_i$, $\tilde{\zeta}_r$, $\tilde{\zeta}_i$ being real. Inserting (4.2) in the nonlinear evolution equation (3.18), linearising with respect to $\tilde{\zeta}$ and $\tilde{\zeta}^*$ and then separating into real and imaginary parts we obtain two equations for $\tilde{\zeta}_r$ and $\tilde{\zeta}_i$. Next taking Fourier transforms with respect to ξ we obtain the following equations,

$$\left[-\frac{\partial(\bar{\tilde{\zeta}}_i)}{\partial \tau} + i(\beta_2\lambda^3 + \alpha_0{}^2\Lambda_2\lambda - \alpha_0{}^2\Lambda_3\lambda)(\bar{\tilde{\zeta}}_i)\right] + \left[\lambda^2\beta_1 - 2\alpha_0{}^2\Lambda_1 + 2\alpha_0{}^2\Lambda_4|\lambda|\right](\bar{\tilde{\zeta}}_r) = 0 \tag{4.3}$$

$$\beta_1\lambda^2(\bar{\tilde{\zeta}}_i) + \left[\frac{\partial(\bar{\tilde{\zeta}}_r)}{\partial \tau} - i(\beta_2\lambda^3 + \alpha_0{}^2\Lambda_2\lambda + \alpha_0{}^2\Lambda_3\lambda)(\bar{\tilde{\zeta}}_r)\right] = 0 , \tag{4.4}$$

where $(\bar{\tilde{\zeta}}_r)$ and $(\bar{\tilde{\zeta}}_i)$ indicate Fourier transforms of $\tilde{\zeta}_r$ and $\tilde{\zeta}_i$ respectively with respect to ξ and defined them as

$$[(\bar{\tilde{\zeta}}_r),(\bar{\tilde{\zeta}}_i)] = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}[\tilde{\zeta}_r,\tilde{\zeta}_i]\mathrm{e}^{-\mathrm{i}\lambda\xi}\mathrm{d}\xi \tag{4.5}$$

If we now suppose that the τ dependence of $(\bar{\tilde{\zeta}}_r)$ and $(\bar{\tilde{\zeta}}_i)$ be of the form $e^{-i\tilde{\Omega}\tau}$, then the nontrivial solution of (4.3) and (4.4) gives the following nonlinear dispersion relation

$$\tilde{\Omega} = -\beta_2\lambda^3 - \alpha_0{}^2\Lambda_2\lambda \pm \sqrt{\beta_1\lambda^2(\beta_1\lambda^2 - 2\alpha_0{}^2\Lambda_1 + 2\alpha_0{}^2\Lambda_4|\lambda|)} \tag{4.6}$$

Instability occurs for

$$\beta_1\lambda^2(\beta_1\lambda^2 - 2\alpha_0{}^2\Lambda_1 + 2\alpha_0{}^2\Lambda_4|\lambda|) < 0 \tag{4.7}$$

When condition (4.7) is satisfied the maximum instability growth rate takes the form

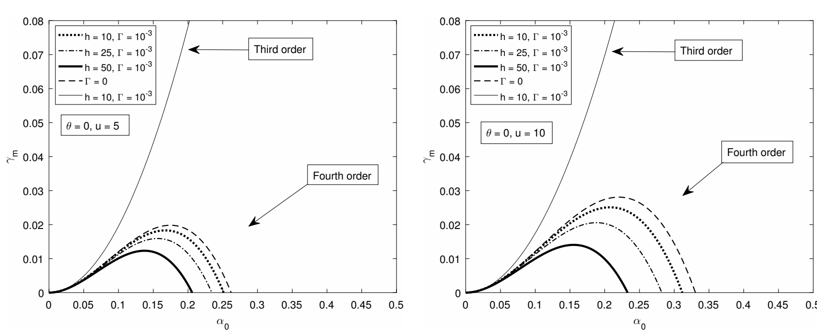
$$\gamma_m = \Lambda_1\alpha_0{}^2\left(1 - \frac{\Lambda_4\alpha_0}{\sqrt{\beta_1\Lambda_1}}\right), \tag{4.8}$$

which occurs at the wave-number of perturbation λ_m given by

$$\lambda_m = \sqrt{\frac{\Lambda_1}{\beta_1}}\alpha_0 - \frac{3\Lambda_4}{4\beta_1}\alpha_0{}^2 \tag{4.9}$$

At marginal stability $\tilde{\Omega}$ is real and from (4.6) the wave number λ is given by

$$\lambda = \sqrt{\frac{\Lambda_1}{\beta_1}}\alpha_0 - \frac{\Lambda_4}{\beta_1}\alpha_0{}^2 \tag{4.10}$$



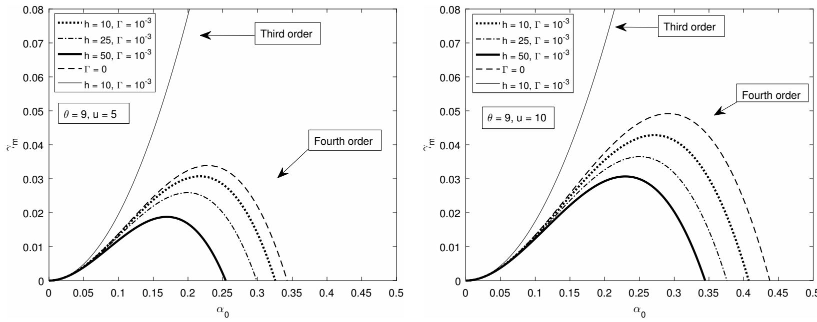


FIGURE 2. **Maximum instability growth rate γ_m as a function of dimensionless wave steepness α_0. Here γ= 0.00129 for all the graphs.**

Graphs have been plotted indicating the variation of maximum instability growth rate γ_m from equation (4.8) against non-dimensional wave steepness α_0 for different values of non-dimensional thermocline depth h and for two fixed values of perturbation angle θ and uniform wind velocity u. In figures 2 and 4 we have considered $\Gamma = 10^{-3}$ for seasonal thermocline. The maximum instability growth rates γ_m, which have been observed from third order, are also displayed in each of the figures. Further in the absence of thermocline, that is, for $\Gamma = 0$, γ_m have also been displayed. From figure 2 we have observed that a thin thermocline produces a stabilizing effect on the instability of surface gravity wave trains and γ_m decreases with the increase of thermocline depth h for fixed values of perturbation angle θ and wind velocity u. But γ_m increases with the increase of perturbation angle θ and wind velocity u for fixed value of thermocline depth h.

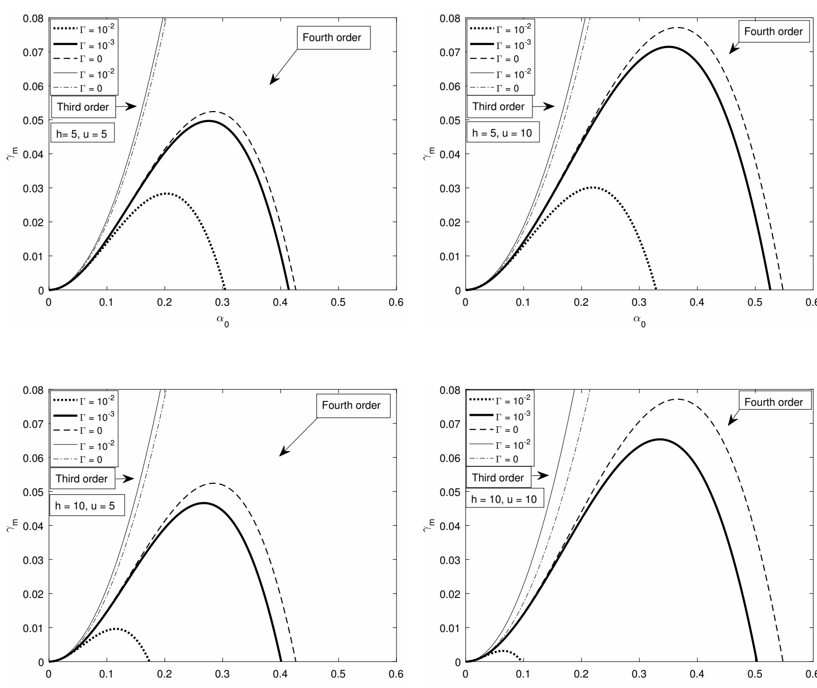


FIGURE 3. **Maximum instability growth rate γ_m as a function of dimensionless wave steepness α_0. Here γ= 0.00129 and $\theta = 10^\circ$ for all the graphs.**

Graphs have also been portrayed in figure 3 pointing out the variations of maximum instability growth rate γ_m as a function of wave steepness α_0 for various thermocline values Γ and for some fixed values of h and u. From the figures we have observed that γ_m decreases with the increase of Γ.

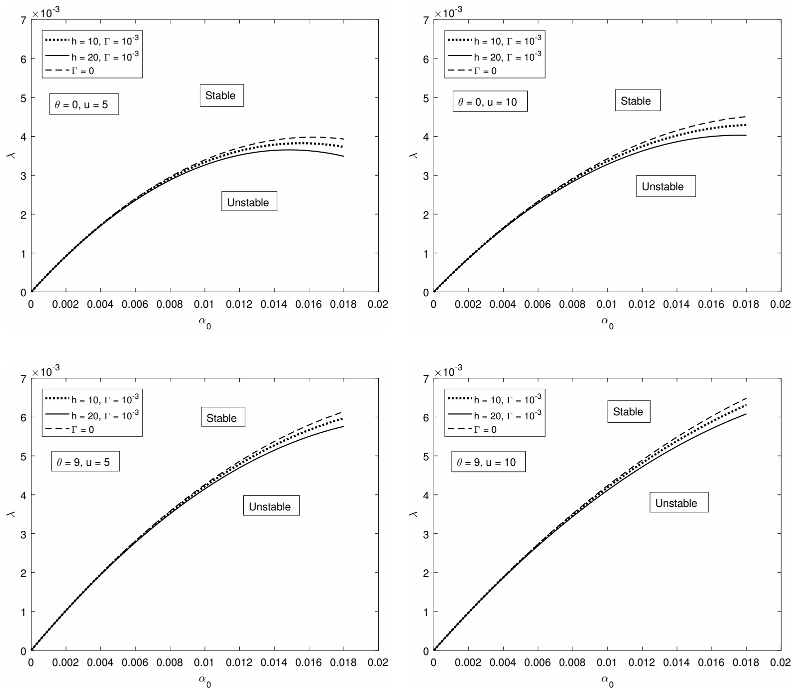


FIGURE 4. **Perturbed wave number λ at marginal stability as a function of dimensionless wave steepness α_0. Here γ= 0.00129 for all the graphs.**

Finally at marginal instability the wave number of perturbation λ from equation (4.10) have been displayed in figure 4 as a function of α_0 indicating stable and unstable regions for several values of non-dimensional thermocline depth h and for two fixed values of θ and u.

5. Discussion and conclusion

To explore the effect caused by a thin thermocline on the Benjamin-Feir instability for a surface gravity wave incorporating the case of air-water interface, we have set up two coupled nonlinear evolution equations which are appropriate up to fourth-order. Here we have pursued the simple three-layer model of the ocean, including a shallow thermocline region of non-vanishing density gradient discriminating homogeneous regions above and below. Thus the present model fulfills the criteria for seasonal thermocline. The significance of beginning from the nonlinear evolution equation of the fourth order was first exhibited by Dysthe [12]. Two coupled evolution equations are then concise to a single evolution equation by assuming the space variation of the amplitude ensuing towards an arbitrary fixed direction. It is to be noted that the single evolution equation ceases to be valid due to the occurrence of second harmonic resonance. The aspect of the maximum instability growth rate together with the instability condition is then acquired from the evolution equation, as mentioned earlier. It has been investigated that for fixed values of the wind velocity, the thin thermocline has a stabilizing influence. Furthermore, for fixed values of perturbation angle θ and uniform wind velocity u, the maximum instability growth rate diminishes as the thermocline depth increases. Further, the maximum instability growth rate enhances with the increment of perturbation angle θ and wind velocity u for a fixed value of thermocline depth h. Moreover, it has been additionally inferred that the maximum instability growth rate diminishes with the enhancement of thermocline value Γ for fixed values of h and u. Graphs have also been portrayed, demonstrating the variation of the perturbed wave number λ at marginal stability versus wave steepness α_0 for different values of non dimensional thermocline depth h and for two fixed values of wind velocity u along with perturbation angle θ.

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Appendix A

$$\Delta_1 A_1 + i\omega_1\zeta_1 = a_1 \tag{A1}$$

$$-\Delta_1 A_1' + i\omega_1\zeta_1 - iuk_1\zeta_1 = b_1 \tag{A2}$$

$$-i\omega_1 A_1 + (1-\gamma)\zeta_1 + i\omega_1\gamma A_1' - i\gamma u k_1 A_1' = c_1 \tag{A3}$$

$$\Delta_2 A_2 + i\omega_2\zeta_2 = a_2 \tag{A4}$$

$$-\Delta_2 A_2' + i\omega_2\zeta_2 - iuk_2\zeta_2 = b_2 \tag{A5}$$

$$-i\omega_2 A_2 + (1-\gamma)\zeta_2 + i\omega_2\gamma A_2' - i\gamma u k_2 A_2' = c_2 \tag{A6}$$

$$\delta\bar{k}\bar{N} - i\delta c_g k_\xi \bar{\zeta}_0 - \delta^2\frac{\partial\bar{\zeta}_0}{\partial\tau} = \bar{a}_0 \tag{A7}$$

$$i\delta c_g k_\xi \bar{M} + \delta^2\frac{\partial\bar{M}}{\partial\tau} + \bar{\zeta}_0 = \bar{b}_0\ , \tag{A8}$$

where $\bar{M} = \bar{A_0} + \bar{B_0}$, $\bar{N} = \bar{A_0} - \bar{B_0}$ and $\bar{a}_0$ and $\bar{b}_0$ denotes the Fourier transforms of a_0 and b_0.

$$\bar{A}_0 e^{-\delta\bar{k}h} - \bar{B}_0 e^{\delta\bar{k}h} = \bar{C}_0 e^{-\delta\bar{k}h} \tag{A9}$$

$$\begin{aligned}\frac{\delta\rho}{\rho}\bar{k}^3\delta(\bar{A}_0 e^{-\delta\bar{k}h} - \bar{B}_0 e^{\delta\bar{k}h}) + \delta^2 k_\xi^2 c_g^2 \bar{k}^2(\bar{A}_0 e^{-\delta\bar{k}h} + \bar{B}_0 e^{\delta\bar{k}h} - \bar{C}_0 e^{-\delta\bar{k}h})\\ = 2ic_g k_\xi \bar{k}^2\delta^3\left(\frac{\partial\bar{A}_0}{\partial\tau}e^{-\delta\bar{k}h} + \frac{\partial\bar{B}_0}{\partial\tau}e^{\delta\bar{k}h} - \frac{\partial\bar{C}_0}{\partial\tau}e^{-\delta\bar{k}h}\right)\end{aligned} \tag{A10}$$

$$2ic_g k_\xi \bar{k}^2\delta^3\left(\frac{\partial\bar{M}}{\partial\tau} + \frac{\partial\bar{N}}{\partial\tau}\right)e^{\delta\bar{k}h} + \delta^2 k_\xi^2 c_g^2\bar{k}^2(\bar{M}-\bar{N})e^{\delta\bar{k}h} = \frac{\delta\rho}{\rho}\bar{k}^3\delta(\bar{N}\cosh\delta\bar{k}h - \bar{M}\sinh\delta\bar{k}h) \tag{A11}$$

Appendix B

$\beta_1 = -\left[\frac{\cos^2\theta}{2}\frac{dc_g}{dk} + \frac{(1-\gamma)\sin^2\theta}{4[(1+\gamma)\omega-\gamma u]}\right]$

$\beta_2 = \left[\frac{\cos^3\theta}{2}\left\{\frac{(1+\gamma)c_g-\gamma u}{(1+\gamma)\omega-\gamma u}\right\}\frac{dc_g}{dk} + \frac{(1-\gamma)\sin^2\theta\cos\theta}{4[(1+\gamma)\omega-\gamma u]}\left\{1+\frac{(1+\gamma)c_g-\gamma u}{(1+\gamma)\omega-\gamma u}\right\}\right]$

$\Lambda_1 = \frac{1}{(1+\gamma)\omega-\gamma u}\left[\omega^2+\gamma(\omega-u)^2+\frac{\{\omega^2-\gamma(\omega-u)^2\}^2}{1-\gamma}+\frac{4c_g\Gamma\cos^2\theta\{\omega^2-\gamma(\omega-u)^2\}}{c_g^2\cos^2\theta-\Gamma h}\right]$

$\Lambda_2 = -\mu_1\cos\theta = \frac{\cos\theta}{f\omega}\left[\delta_3-\delta_2 c_g+\frac{4(1+\gamma)c_g\delta_1}{f\omega}\right]$

$\Lambda_3 = -\mu_2\cos\theta = \frac{\cos\theta}{f\omega}\left[\delta_4+\frac{2(1+\gamma)c_g\delta_1}{f\omega}\right]$

$\Lambda_4 = \frac{4c_g^3\cos^4\theta\{\omega^2-\gamma(\omega-u)^2\}}{(c_g^2\cos^2\theta-\Gamma h)\{(1+\gamma)\omega-\gamma u\}}$

$\delta_1 = 2\left[\omega^2+\gamma(\omega-u)^2+\frac{\{\omega^2+\gamma(\omega-v)^2\}^2}{1-\gamma}\right]$

$\delta_2 = 4\left[\omega-\gamma(\omega-u)+\frac{2\{\omega-\gamma(\omega-u)\}\{\omega^2-\gamma(\omega-u)^2\}}{1-\gamma}-\frac{2\{\omega-\gamma(\omega-u)\}\{\omega^2-\gamma(\omega-u)^2\}^2}{(1-\gamma)^2}\right]$

$$\begin{aligned}\delta_3 = -2\Bigg[&\omega^2+\gamma(\omega-u)^2+2\{\omega^2+\gamma(\omega-u)(\omega-2u)\}+\frac{\{(1-\gamma)+4\gamma u(\omega-u)\}\{\omega^2-\gamma(\omega-u)^2\}^2}{(1-\gamma)^2}\\ &+\frac{\{\omega^2-\gamma(\omega-u)^2\}\{\omega^2-\gamma(\omega-u)(\omega-3u)\}}{1-\gamma}-\frac{\{\omega-\gamma(\omega-u)\}\{\omega^2-\gamma(\omega-u)^2\}^2}{(1-\gamma)^2}\\ &+\frac{2\{\omega^2-\gamma(\omega-u)(\omega-2u)\}\{\omega^2-\gamma(\omega-u)^2\}}{1-\gamma}\Bigg]\end{aligned}$$

$\delta_4 = -2\left[\omega^2+\gamma(\omega-u)^2+\frac{\{\omega^2+\gamma(\omega-u)^2\}^2}{1-\gamma}\right]$

$c_g = \frac{d\omega}{dk} = -\frac{f_k}{f_\omega}.$

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