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# SKEW SEMI-INVARIANT SUBMANIFOLDS IN A GOLDEN RIEMANNIAN MANIFOLD

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ABSTRACT. In this paper, we define and study skew semi-invariant submanifolds of a golden Riemannian manifold. We investigate some characterizations for submanifolds to be skew semi-invariant submanifolds. Moreover, we obtain many interesting results on sectional curvature of skew semi-invariant submanifolds. We also construct an example of skew semi-invariant submanifolds.

## 1. Introduction

The notion of semi-invariant submanifolds was first defined by A. Bejancu and N. Papaghuic [7] as an analogous to that of CR-submanifolds in almost complex manifolds. Since then it becomes a popular topic in differential geometry. This notion has been extended to several ambient manifolds. Semi-invariant submanifolds have been defined and studies in different ambient manifolds by several geometers, such as nearly Kenmotsu manifold ([1], [3], [4]), Lorentzian para Sasakian manifold [2], nearly Sasakian manifolds [10], Trans-Sasakian manifolds [18], Quasi-Sasakian manifolds [19], locally product manifold [20].

In recent years, C.E. Hretcanu and M. Crasmreanu [8] introduced and studied golden Riemannian manifolds by using golden ratio. They also studied invariant submanifolds in [9] Riemannian manifold with golden structure. Gezer A. et al [13] discussed the integerability conditions of golden Riemannian manifolds. M. Ahmad and M. A. Qayyoom [5], Hretcanu C. E. [16] studies submanifolds in Riemannian manifolds with golden structure. Erdogan F. E. and Yildrim C. [12] studied semi-invariant submanifolds of a golden Riemannian manifolds. Some properties of golden Riemannian manifolds has been studied in [11], [14], [15], [17].

In this paper, we defined and discussed a new class of submanifolds of golden Riemannian manifolds, i.e, skew semi-invariant submanifolds which contain semi-invariant submanifolds as a special case. The paper is organized as follows: In section 2, we give definition of golden Riemannian manifold and skew semi-invariant submanifolds of golden Riemannian manifold. In section 3, we investigate some interesting properties of skew semi-invariant submanifolds and construct an example of skew semi-invariant submanifolds.

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## 2. Definition and preliminaries

In this section, we give a brief information of golden Riemannian manifolds.

**Definition 2.1.** [8] Let $(\overline{M}, g)$ be a Riemannian manifold. A golden structure on $(\overline{M}, g)$ is a non-null tensor J of type (1,1) which satisfies the equation

$$J^2 = J + I, \tag{2.1}$$

where $I$ is the identity transformation. We say that the metric $g$ is $J$-compatible if

$$g(JX, Y) = g(X, JY) \tag{2.2}$$

for all X,Y vector fields on $\overline{M}$. If we substitute $JX$ into $X$ in (2.2), then we have

$$g(JX, JY) = g(JX, Y) + g(X, Y).$$

The Riemannian metric (2.2) is called $J$-compatible and $(\overline{M}, J, g)$ is called a Golden Riemannian manifold.

**Proposition 2.2.** [8] *A golden structure on the manifold $\overline{M}$ has the power*

$$J^n = F_n J + F_{n-1} I \tag{2.3}$$

*for any integer n, where $(F_n)$ is the Fibonacci sequence.*
*Using an explicit expression for the Fibonacci sequence namely the Binet's formula*

$$F_n = \frac{J^n - (1-J)^n}{\sqrt{5}},$$

*we obtain a new form for the equality* (2.3) *as*

$$J^n = \left(\frac{\phi^n - (1-\phi)^n}{\sqrt{5}}\right)J + \left(\frac{\phi^{n-1} - (1-\phi)^{n-1}}{\sqrt{5}}\right)I.$$

*The straight forward computations yield:*

**Proposition 2.3.** [8] $(i)$ *The eigen values of a golden structure $J$ are the golden ratio $\phi$ and $1-\phi$.*
$(ii)$ *A golden structure $J$ is an isomorphism on the tangent space $T_x\overline{M}$ of the manifold $\overline{M}$ for every $x \in \overline{M}$.*
$(iii)$ *It follows that $J$ is invertible and its inverse $\widehat{J} = J^{-1}$ satisfies*

$$\widehat{\phi}^2 = -\widehat{\phi} + 1.$$

**Definition 2.4.** A submanifold $M$ of a golden Riemannian manifold $\overline{M}$ is callled a skew semi-invariant submanifold if there exist an integer $k$ and constant functions $\alpha_i$, $1 \leq i \leq k$, defined on $M$ with values in $(0, 1)$ such that
(i) Each $\alpha_i$, $1 \leq i \leq k$, is a distinct eigenvalue of $P^2$ with

$$T_pM = D_p^0 \bigoplus D_p^1 \bigoplus D_p^{\alpha_1} \bigoplus \ldots\ldots \bigoplus D_p^{\alpha_k}$$

for $p \in M$.
(ii) The dimension of $D_p^0$, $D_p^1$ and $D_k^1$, $1 \leq i \leq k$, are independent of $p \in M$.

*Remark* 2.5. Above definition implies $D_p^0$, $D_p^1$, and $D_p^{\alpha_i}$, $1 \leq i \leq k$, defined $P$ invariant, mutually orthogonal distribution which we denote by $D_p^0$, $D_p^1$, and $D_p^{\alpha_i}$, $1 \leq i \leq k$, respectively. The tangent bundle of $M$ has the following decomposition

$$TM = D^0 \bigoplus D^1 \bigoplus D^{\alpha_1} \bigoplus ...... \bigoplus D^{\alpha_k}$$

if $k = 0$ then $M$ is a semi-invariant submanifold. Also, if $k = 0$, and $D_p^0(D_p^1)$ is trivial, then $M$ is an invariant (anti-invariant) submanifold of $\overline{M}$.

## 3. Skew semi-invariant submanifold

We denote by $\overline{\nabla}$ the Levi-Civita connection on $\overline{M}$ with respect to $g$. Let $M$ be a Riemannian manifold isometrically immersed in $\overline{M}$ and $g$ is the Riemannian metric induced on $M$, for $p \in M$ and tangent vector $X_p \in T_pM$, we write

$$JX_p = PX_p + QX_p, \tag{3.1}$$

where $PX_p \in T_pM$ is tangent to $M$ and $QX_p \in T_p^{\perp}M$ is normal to $M$.
For any two vectors $X_p, Y_p \in T_pM$, we have

$$g(JX_p, Y_p) = g(PX_p, Y_p),$$

which implies that

$$g(JX_p, Y_p) = g(X_p, PY_p).$$

So, $P$ and $P^2$ are all symmetric operators on the tangent space $T_pM$. If $\alpha(p)$ is the eigen value of $P^2$ at $p \in M$. Since $P^2$ is a composition of an isometry and a projection, hence $\alpha(p) \in [0, 1]$.
For each $p \in M$, we set $D_p^{\alpha} = ker(P^2 - \alpha(p)I)$, where I is the identity transformation on $T_pM$, and $\alpha(p)$ is an eigen value of $P^2$ at $p \in M$, obviously, we have

$$D_p^0 = kerP, D_p^1 = kerQ.$$

$D_p^1$ is the maximal $J$ invariant subspace of $T_pM$ and $D_p^0$ is the maximal $J$ anti-invariant subspace of $T_pM$.
If $\alpha_1(p), ........, \alpha_k(p)$ are all eigen values of $P^2$ at $p$, then $T_pM$ can be decomposed as the direct sum of the mutually orthogonal eigenspaces, that is

$$T_pM = D_p^{\alpha_1} \bigoplus D_p^{\alpha_2} \bigoplus ...... \bigoplus D_p^{\alpha_k}.$$

For $N \in T^{\perp}M$, we write

$$JN = tN + fN, \tag{3.2}$$

where $tN \in TM$, and $fN \in T^{\perp}M$.
We denote by $\nabla$ the induced connection in $M$, we have formulas by Gauss and Weingarten

$$\overline{\nabla}_X Y = \nabla_X Y + h(X, Y), \tag{3.3}$$

$$\overline{\nabla}_X N = -A_N X + \nabla_X^{\perp} N \tag{3.4}$$

for all vectors $X, Y \in TM$ and $N \in T^{\perp}M$.
We have

$$g(h(X, Y), N) = g(A_N X, Y). \tag{3.5}$$

From (3.2), (3.3) and (3.4) we get

$$J(\overline{\nabla}_X Y) = \overline{\nabla}_X JY - Y\overline{\nabla}_X J$$
$$J(\overline{\nabla}_X Y) = \overline{\nabla}_X JY.$$

From (3.3) and (3.1), we have

$$J(\nabla_X Y + h(X,Y)) = \overline{\nabla}_X(PX + QY).$$

From (3.3) and (3.4), we obtain

$$J(\nabla_X Y) + J(h(X,Y)) = \nabla_X PY + h(X,PY) - A_{QY}X + \nabla^{\perp}_X QY.$$

From (3.1) and (3.2), we obtain

$$P(\nabla_X Y)+Q(\nabla_X Y)+th(X,Y)+)fh(X,Y) = \nabla_X PY+h(X,PY)-A_{QY}X+\nabla^{\perp}_X QY. \tag{3.6}$$

for all $X, Y \in TM$.
Comparing tangential and normal components, we have

$$P\nabla_X Y = \nabla_X PY - th(X,Y) - A_{QY}Y, \tag{3.7}$$
$$Q(\nabla_X Y) = h(X,PY) + \nabla^{\perp}_X QY - fh(X,Y). \tag{3.8}$$

From (3.7) and (3.8),

$$P[X,Y] = \nabla_X PY - \nabla_Y PX + A_{QX}Y - A_{QY}X, \tag{3.9}$$
$$Q[X,Y] = h(X,PY) - h(X,PY) + \nabla^{\perp}_X QY - \nabla^{\perp}_Y QX. \tag{3.10}$$

For above, we must have the next lemma

**Lemma 3.1.** *Let $M$ be a skew semi-invariant submanifold of a golden Riemannian manifold $\overline{M}$, then*
*(i) The distribution $D^0$ is integrable if and only if $A_{JX}Y = A_{JY}X$ for all $X, Y \in D^0$.*
*(ii) The distribution $D^1$ is integrable if and only if $h(X,JY) = h(JX,Y)$ for all $X, Y \in D^1$.*

We define the covariant derivative of $P$ and $Q$ in a manner as follows:

$$(\nabla_X P)Y = \nabla_X PY - P\nabla_X Y \tag{3.11}$$
$$(\nabla_X Q)Y = \nabla^{\perp}_X QY - Q\nabla_X Y \tag{3.12}$$

for all $X, Y \in TM$.

$$P\nabla_X Y = \nabla_X PY - th(X,Y) - A_{QY}X.$$

Using (3.11) in (3.7), we get

$$(\nabla_X P)Y = th(X,Y) + A_{QY}X. \tag{3.13}$$

From equality (3.8) and (3.12), we have

$$(\nabla_X Q)Y = fh(X,Y) - h(X,PY). \tag{3.14}$$

Let $D^1$ and $D^2$ be two distributions defined on a manifold $\overline{M}$. We say that $D^1$ is parallel with respect to $D^2$ if for all $X \in D^2$ and $Y \in D^1$ if $\nabla_X Y = 0$.
Let $M$ be a submanifold of $\overline{M}$. A distribution $D$ on $M$ is said to be totally geodesic if for all $X, Y \in D$ we have $h(X,Y) = 0$. In this case we say also that $M$ is totally

geodesic. For two distributions $D^1$ and $D^2$ defined on $M$, we say that $M$ is $D^1 - D^2$ mixed totally geodesic if for all $X \in D^1$ and $Y \in D^2$ we have $h(X,Y) = 0$.

**Proposition 3.2.** *Let $M$ be a skew semi-invariant submanifold of a golden Riemannian manifold $\overline{M}$ and $\overline{\nabla} J = 0$, for any distribution $D^\alpha$, if $A_N PX = PA_N X$ for all $X \in D^\alpha$ and $N \in T^\perp M$, then $M$ is $D^\alpha - D^\beta$ mixed totally geodesic, where $\alpha \neq \beta$.*

*Proof.* From the assumption we have

$$P^2 A_N X - \alpha A_N X = 0$$

which implies that $A_N X \in D^\alpha$. So for all $Y \in D^\beta$, $N \in T^\perp M$, $\alpha \neq \beta$ we have

$$g(A_N X, Y) = 0$$

$$g(h(X,Y), N) = 0,$$

that is

$$h(X,Y) = 0.$$

Hence $M$ is $D^\alpha - D^\beta$ mixed totally geodesic. □

From (3.1) and (3.2),

$$JX_p = PX_p + QX_p$$

for all $X_p \in TM$.

$$JN = tN + fN$$

for all $N \in T^\perp M$.

$$J^2 X_p = J(JX_p) = J(PX_p + QX_p),$$

$$(J+I)X_p = P^2 X_p + QPX_p + t(QX_p) + fQX_p$$

$$PX_p + QX_p + X_p = P^2 X_p + QPX_p + tQX_p + fQX_p.$$

Comparing tangent and normal components, we get

$$PX_p + X_p = P^2 X_p + tQX_p, \tag{3.15}$$

$$fQX_p = QX_p - QPX_p. \tag{3.16}$$

Similarly,

$$J^2 N = J(tN + fN)$$

$$N + tN + fN = PtN + QtN + tfN + f^2 N.$$

Comparing tangent and normal components, we get

$$tN = PtN + tfN, \tag{3.17}$$

$$QtN = N + fN - f^2 N, \tag{3.18}$$

for all $X_p \in T_pM, N \in T_p^\perp M$. Furthermore, for $X_p \in D_p^{\alpha_i}$, $\alpha \in \{\alpha_1, \alpha_2, ....., \alpha_k\}$, we have

$$f^2 QX_p = (\alpha_p + 1)QX_p$$

Also, if $X_p \in D_p^0$ then it is clear that $f^2 QX_p = 0$. Thus if $X_p$ is an eigen value of $P^2$ corresponding to the eigen value $\alpha p$, $QX_p$ is an eigen vector of $f^2$ with the eigen value $\alpha(p) + 1$. From (3.18), if $fN = 0$ then

$$QtN = (1 - f^2)N.$$

$(\alpha(p)+1)$ is an eigen value of $f^2$ if and only if $-\alpha(p)$ is an eigen value of Qt. Since $Qt$ and $f^2$ are symmetric operators on the normal bundle $T^{\perp}M$, their eigen spaces are orthogonal. The dimension of the eigen space of $Qt$ corresponding to eigen value $-\alpha(p)$ is equal to the dimension of $D^{\alpha}$. Consequently we have,

**Lemma 3.3.** *Let $M$ be a submanifold of a golden Riemannian manifold $\overline{M}$. $M$ is a skew semi-invariant sumbanifold if and only if the eigen values of $Qt$ are constant and the eigen spaces of $Qt$ have constant dimension.*

**Theorem 3.4.** *Let $M$ be a submanifold of a golden Riemannian manifold $\overline{M}$ and structure $J$ is parallel to Levi-Civita connection i.e$\overline{\nabla}J=0$, if $\nabla P=0$, then $M$ is a skew semi-invariant submanifold. Furthermore, each of the $P$ invariant distribution $D^0$, $D^1$ and $D^{\alpha_i}$, $1\leq i\leq k$, is parallel.*

*Proof.* Suppose $p\in M$, for any $Y_p\in D_p^{\alpha_i}$ and any vector field $X\in TM$, let $Y$ be a parallel translation of $Y_p$ along the integral curve of $X$, since $(\nabla_X P)Y=0$ , we have

$$\nabla_X(P^2-\alpha(p)Y)=P^2\nabla_X Y-\alpha_p\nabla_X Y=0.$$

Since $P^2Y-\alpha(p)Y=0$ at $p$, it is identical on $M$. Thus, eigen values of $P^2$ are constant.
Moreover, parallel translation of $T_pM$ along any curve is an isometry which preserve each $D^{\alpha}$, thus the dimension of each $D^{\alpha}$ is constant and $M$ is a skew semi-invariant submanifold.
Now, if $Y$ is any vector field in $D^{\alpha}$, we have

$$P^2Y=\alpha Y,$$

where $\alpha$ is constant.

$$P^2\nabla_X Y=\alpha\nabla_X Y,$$

which implies that $D^{\alpha}$ is parallel. □

**Proposition 3.5.** *Let $M$ be a skew semi-invariant submanifold of a golden Riemannian $\overline{M}$, if $\nabla Q=0$, then $M$ is $D^{\alpha}-D^{\beta}$ mixed totally geodesic for all $\alpha\neq\beta$. Moreover, if $X\in D^{\alpha}$ then either $h(X,Y)=0$ or $h(X,Y)$ is an eigen vector of $f^2$ with eigen value $\alpha$.*

*Proof.* From equality (3.14)

$$fh(X,Y)=h(X,PY)$$

for each $X,Y\in TM$. $ifY\in D^{\alpha}$

$$\Rightarrow f^2h(X,Y)=fh(X,P^2Y)$$
$$\Rightarrow f^2h(X,Y)=\alpha h(X,Y),$$

if $Y\in D^{\beta}$

$$\Rightarrow f^2h(X,Y)=\alpha h(X,Y)$$
$$\Rightarrow (f^2-\alpha)h(X,Y)=0$$
$$\Rightarrow h(X,Y)=0.$$

Hence, $M$ is $D^{\alpha}-D^{\beta}$ mixed totally geodesic.

**Theorem 3.6.** *Let $M$ be a submanifold of a golden Riemannian manifold $\overline{M}$, if $\nabla Q = 0$, then $M$ is skew semi-invariant submanifold.*

*Proof.* Let $p$ be a point in $M$ and vector $X_p \in D_p^\alpha$, $\alpha \neq 1$.
Let $N_p = QX_p$ then by equality (3.17), $N_p$ is an eigen vector of $Qt$ with eigen value $-\alpha(p)$.
Now let $Y \in TM$ and $N$ be the translation of $N_p$ in the normal bundle $T^\perp M$ along an integral curve of $Y$,
we have

$$\nabla_Y^\perp(QtN + \alpha(p)N) = \nabla_Y^\perp QtN + \alpha(p)\nabla_Y^\perp N.$$

From Lemma 3.3, we get

$$\nabla_Y^\perp(QtN + \alpha(p)N) = Qt\nabla_Y^\perp N + \alpha(p)\nabla_Y^\perp N$$

$$\nabla_Y^\perp(QtN + \alpha(p)N) = (Qt + \alpha(p))\nabla_Y^\perp N.$$

Since $(QtN + \alpha(p)N) = 0$ at $p$. Hence $(QtN + \alpha(p)N) = 0$ on $M$.
$Qt$ are constant and eigen spaces of $Qt$ have constant dimension.
Then by Lemma 3.3, $M$ is a skew semi-invariant submanifold. □

Let $\overline{R}$ and $R$ denote the curvature tensor of $\overline{M}$ and $M$ respectively, then the equation of Gauss is given by

$$g(R(X,Y)Z,W) = g(\overline{R}(X,Y)Z,W) + g(h(X,W),h(Y,Z)) - g(h(X,Z),h(Y,W)) \tag{3.19}$$

for $X, Y, Z, W \in TM$.

The sectional curvature of a plane section of $\overline{M}$ determine by two orthogonal unit vector $X, Y \in T\overline{M}$ is given by

$$K_{\overline{M}}(X \wedge Y) = g(\overline{R}(X,Y)Y,X). \tag{3.20}$$

The sectional curvature of a plane section of $M$ determine by two orthogonal unit vectors $X, Y \in TM$ is given by

$$K_M(X \wedge Y) = g(R(X,Y)Y,X) \tag{3.21}$$

for $X, Y \in TM$.
From (3.19), (3.20) and (3.21), we can obtain

$$K_M(X \wedge Y) = K_{\overline{M}}(X \wedge Y) + g(h(X,X),h(Y,Y)) - |h(X,Y)|^2. \tag{3.22}$$

**Proposition 3.7.** *Let $M$ be a skew semi-invariant submanifold of a golden Riemannian manifold $\overline{M}$ and $J$ is parallel to Levi-Civita connection i.e$\overline{\nabla}_X J = 0$. If $\nabla Q = 0$, then for any unit vectors $X \in D^\alpha$ and $Y \in D^\beta$, $\alpha \neq \beta$, we have*

$$K_M(X \wedge Y) = K_{\overline{M}}(X \wedge Y).$$

*Proof.* From Proposition 3.2. Since $X \in D^\alpha, Y \in D^\beta$, then

$$h(X,Y) = 0.$$

By equality (3.22), we have

$$K_M(X \wedge Y) = K_{\overline{M}}(X \wedge Y).$$

**Lemma 3.8.** *Let $M$ be a skew semi-invariant submanifold of a golden Riemannian manifold $\overline{M}$ and $\overline{\nabla}_X J = 0$, then the following are equivalent.*
*(i)* $(\nabla_X Q)Y - (\nabla_Y Q)X = 0$ *for all* $X, Y \in D^\alpha$,
*(ii)* $h(PX, Y) = h(X, PY)$ *for all* $X, Y \in D^\alpha$,
*(iii)* $Q[X, Y] = \nabla^\perp_X QY - \nabla_Y QX$ *for all* $X, Y \in D^\alpha$,
*(iv)* $A_N PY - PA_N Y$ *is perpendicular to* $D^\alpha$ *for all* $Y \in D^\alpha$ *and* $N \in T^\perp M$.

*Proof.* Using equality (3.14), we have

$$(\nabla_X Q)Y = fh(X, Y) - h(X, PY), \tag{3.23}$$

$$(\nabla_Y Q)X = fh(Y, X) - h(Y, PX). \tag{3.24}$$

From (3.23) and (3.24), we get

$$(\nabla_X Q)Y - (\nabla_Y Q)X = h(Y, PX) - h(X, PY).$$

Comparing tangential and normal parts,

$$(\nabla_X Q)Y - (\nabla_Y Q)X = 0,$$

$$h(X, PY) - h(PX, Y) = 0$$

for all $X, Y \in D^\alpha$.
From equality (3.10) and $(ii)$, we get

$$Q[X, Y] = \nabla^\perp_X QY - \nabla^\perp_Y QX$$

Now,

$$g(A_N PY - PA_N Y, Y) = 0,$$

then we get $(iv)$.
$P\alpha$ is commutative if any of the equivalent conditions in the Lemma 3.8 holds. □

For each P invariant $D^\alpha$ we can assume a local orthonormal basis $E^1, \ldots.., E^{n(\alpha)}$. $D^\alpha$ mean curvature vector is defined by $H^\alpha = \sum_{i=1}^{n(\alpha)} h(E^i, E^i)$, then mean curvature vector is defined by

$$H = \frac{1}{n}(H^0 + H^1 + H^{\alpha_1} + \ldots.. + H^{\alpha_k}), n = dim M$$

A skew semi-invariant submafold $M$ of a golden Riemannian manifold $\overline{M}$, where $\nabla J = 0$ is called $D^\alpha$ minimal if $H^\alpha = 0$ and minimal if $H = 0$.
For unit vector $X \in D^\alpha$, $\alpha \neq 0$, define the $\alpha-$ sectional curvature of $\overline{M}$ and $M$ by

$$\overline{H}_\alpha(X) = K_{\overline{M}}(X \wedge Y), H_\alpha(X) = K_M(X \wedge Y)$$

respectively, where $Y = \frac{PX}{\sqrt{\alpha}}$. From (3.24), we get

$$H_\alpha(X) = \overline{H}_\alpha(X) + \frac{1}{\alpha} g(h(X, X), h(PX, PX)) - \frac{1}{\alpha}|h(X, PX)|^2.$$

**Proposition 3.9.** *Let $M$ be a skew semi-invariant submanifold of a golden Riemannian manifold $\overline{M}$, where $\overline{\nabla} J = 0$ if $P$ is $\alpha$ commutative, $\alpha \neq 0$, then*

$$H_\alpha(X) = \overline{H}_\alpha(X) + |h(X, X)|^2 - \frac{1}{\alpha}|h(X, PX)|^2.$$

*Proof.* From the equality (3.22)

$$H_\alpha(X) = \overline{H}_\alpha(X) + g(h(X,X), h(\frac{PX}{\sqrt{\alpha}}, \frac{PX}{\sqrt{\alpha}}) - |h(X, \frac{PX}{\sqrt{\alpha}}|^2$$

$$H_\alpha(X) = \overline{H}_\alpha(X) + \frac{1}{\alpha} g(h(X,X), h(X, P^2X)) - \frac{1}{\alpha}|h(X,PX)|^2.$$

Since $P^2X = \alpha X$, $X \in D^\alpha$,

$$H_\alpha(X) = \overline{H}_\alpha(X) + \frac{1}{\alpha} g(h(X,X), h(X, \alpha X)) - \frac{1}{\alpha}|h(X,PX)|^2$$

$$H_\alpha(X) = \overline{H}_\alpha(X) + |h(X,X)|^2 - \frac{1}{\alpha}|h(X,PX)|^2.$$

□

**Proposition 3.10.** *Let $M$ be a skew semi-invariant submanifold of a golden Riemannian manifold $\overline{M}$ and structure $J$ is parallel to Levi-Civita connection, i.e$\overline{\nabla} J = 0$ then*
*(i) If $H^\alpha$ is perpendicular to $H^\beta$, $\alpha \neq \beta$, then $\rho_{\alpha\beta} \leq \overline{\rho}_{\alpha\beta}$, and the equality holds if and only if $M$ is $D^\alpha - D^\beta$ mixed totally geodesic.*
*(ii) If $M$ is $D^\alpha$ minimal, then $\rho_{\alpha\alpha} \leq \overline{\rho}_{\alpha\alpha}$, and the equality holds if and only if $M$ is $D^\alpha$ totally geodesic.*

*Proof.* Let $(E^1, ...., E^{n(\alpha)})$ and $(E^1, ...., E^{n(\beta)})$ be the local orthonormal basis for $D^\alpha$ and $D^\beta$, respectively. We define $\alpha - \beta$ sectional curvatures of $\overline{M}$ and $M$ by

$$\overline{\rho}_{\alpha\beta} = \sum_{i=1}^{n(\alpha)} \sum_{j=1}^{n(\beta)} K_{\overline{M}}(E^i \wedge E^j),$$

$$\rho_{\alpha\beta} = \sum_{i=1}^{n(\alpha)} \sum_{j=1}^{n(\beta)} K_M(E^i \wedge E^j)$$

respectively.
From (3.22) and for $\alpha \neq \beta$, we get

$$\rho_{\alpha\beta} = \overline{\rho}_{\alpha\beta} + g(H^\alpha, H^\beta) - \sum_{i=1}^{n(\alpha)} \sum_{j=1}^{n(\beta)} |h(E^i, E^j)|^2.$$

If $H^\alpha$ is perpendicular to $H^\beta$ then

$$\rho_{\alpha\beta} = \overline{\rho}_{\alpha\beta} - \sum_{i=1}^{n(\alpha)} \sum_{j=1}^{n(\beta)} |h(E^i, E^j)|^2.$$

$$\Rightarrow \rho_{\alpha\beta} \leq \overline{\rho}_{\alpha\beta}.$$

If $M$ is $D^\alpha - D^\beta$ mixed geodesic then $h(X,Y) = 0$ $\forall X \in D^\alpha, Y \in D^\beta$, then

$$\rho_{\alpha\alpha} = \overline{\rho}_{\alpha\alpha}.$$

which is $(i)$.
Now, for $\alpha = \beta$, since $D^\alpha$ is minimal, then $H^\alpha = 0$. Hence

$$\rho_{\alpha\alpha} = \overline{\rho}_{\alpha\alpha} - \sum_{i=1}^{n(\alpha)} \sum_{j=1}^{n(\beta)} |h(E^i, E^j)|^2$$

$$\rho_{\alpha\alpha} \leq \overline{\rho}_{\alpha\alpha},$$

which is $(ii)$.

□

## 4. Example

**Example 4.1.** Let the golden Riemannian manifold $R^{10} = R^5 \times R^5$ with the usual metric $g$ and structure $J$ satisfies $\overline{\nabla} J = 0$ and $J$ defined by

$$J(\frac{\partial}{\partial x_i}, \frac{\partial}{\partial y_i}) = (\phi\frac{\partial}{\partial x_1}, \phi\frac{\partial}{\partial x_2}, \overline{\phi}\frac{\partial}{\partial x_3}, \overline{\phi}\frac{\partial}{\partial x_4}, \overline{\phi}\frac{\partial}{\partial x_5}, \phi\frac{\partial}{\partial y_1}, \phi\frac{\partial}{\partial y_2}, \phi\frac{\partial}{\partial y_3}, \overline{\phi}\frac{\partial}{\partial y_4}, \overline{\phi}\frac{\partial}{\partial y_5}), \tag{4.1}$$

where $i, j \in \{1, 2, 3, 4, 5\}$

$$J^2(\frac{\partial}{\partial x_i}, \frac{\partial}{\partial y_i}) = (\phi^2\frac{\partial}{\partial x_1}, \phi^2\frac{\partial}{\partial x_2}, \overline{\phi}^2\frac{\partial}{\partial x_3}, \overline{\phi}^2\frac{\partial}{\partial x_4}, \overline{\phi}^2\frac{\partial}{\partial x_5}, \phi^2\frac{\partial}{\partial y_1}, \phi^2\frac{\partial}{\partial y_2}, \phi^2\frac{\partial}{\partial y_3}, \overline{\phi}^2\frac{\partial}{\partial y_4}, \overline{\phi}^2\frac{\partial}{\partial y_5}),$$

$$J^2(\frac{\partial}{\partial x_i}, \frac{\partial}{\partial y_i}) = ((\phi+1)\frac{\partial}{\partial x_1}, (\phi+1)\frac{\partial}{\partial x_2}, (\overline{\phi}+1)\frac{\partial}{\partial x_3}, (\overline{\phi}+1)\frac{\partial}{\partial x_4},$$
$$(\overline{\phi}+1)\frac{\partial}{\partial x_5}, (\phi+1)\frac{\partial}{\partial y_1}, (\phi+1)\frac{\partial}{\partial y_2}, (\phi+1)\frac{\partial}{\partial y_3}, (\overline{\phi}+1)\frac{\partial}{\partial y_4}, (\overline{\phi}+1)\frac{\partial}{\partial y_5}),$$

$$J^2(\frac{\partial}{\partial x_i}, \frac{\partial}{\partial y_i}) = J(\frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2}, \frac{\partial}{\partial x_3}, \frac{\partial}{\partial x_4},$$
$$\frac{\partial}{\partial x_5}, \frac{\partial}{\partial y_1}, \frac{\partial}{\partial y_2}, \frac{\partial}{\partial y_3}, \frac{\partial}{\partial y_4}, \frac{\partial}{\partial y_5}) + (\frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2}, \frac{\partial}{\partial x_3}, \frac{\partial}{\partial x_4},$$
$$\frac{\partial}{\partial x_5}, \frac{\partial}{\partial y_1}, \frac{\partial}{\partial y_2}, \frac{\partial}{\partial y_3}, \frac{\partial}{\partial y_4}, \frac{\partial}{\partial y_5}),$$

$$J^2 = J + I.$$

Let M be a submanifold of $\overline{M} = (R^{10}, g, J)$ given by

$$f(xy, z, u, v) = (x + y, x - y, xcosu, xsinu, z, -z, x, 2y, xcosv, xsinv).$$

We can find the local frame of $TM$ spanned by

$$Z_1 = \frac{\partial}{\partial x_1} + \frac{\partial}{\partial x_2} + cosu\frac{\partial}{\partial x_3} + sinu\frac{\partial}{\partial x_4} + \frac{\partial}{\partial x_5} + cosv\frac{\partial}{\partial y_4} + sinv\frac{\partial}{\partial y_5},$$

$$Z_2 = \frac{\partial}{\partial x_1} - \frac{\partial}{\partial x_2} + \frac{\partial}{\partial y_3}, Z_3 = i\frac{\partial}{\partial y_1} + \frac{\partial}{\partial y_2},$$

$$Z_4 = -xsinu\frac{\partial}{\partial x_3} + xcosu\frac{\partial}{\partial x_4},$$

$$Z_5 = -xsinv\frac{\partial}{\partial y_4} + xcosv\frac{\partial}{\partial y_5},$$

where $\phi = \frac{1+\sqrt{5}}{2}$ is the golden number and $\overline{\phi} = 1 - \phi$.
We have $J(Z_2) = \phi Z_2$, $J(Z_4) = \overline{\phi} Z_4$, $J(Z_5) = \overline{\phi} Z_5$, and

$$J(Z_1) = \phi\frac{\partial}{\partial x_1} + \phi\frac{\partial}{\partial x_2} + \overline{\phi}cosu\frac{\partial}{\partial x_3} + \overline{\phi}sinu\frac{\partial}{\partial x_4} + \overline{\phi}\frac{\partial}{\partial x_5} + \overline{\phi}cosv\frac{\partial}{\partial y_4} + \phi sinv\frac{\partial}{\partial y_5}$$

$$J(Z_3) = i\phi\frac{\partial}{\partial y_1} + \phi\frac{\partial}{\partial y_2}.$$

We can verify that
$\|J(Z_1)\|^2 = 8 - \phi$, $\|J(Z_2)\|^2 = 2(1 + \phi)$, $\|J(Z_3)\|^2 = 0$, $\|J(Z_4)\|^2 = x^2(2 - \phi)$ and $\|J(Z_5)\|^2 = x^2(2 - \phi)$.
Moreover, we have $< J(Z_1), Z_1 >= 3 - \phi$ and $< J((Z_i), Z_j) >= 0$, for any $i \neq j$, where $i, j \in \{1, 2, 3, 4, 5\}$.

$$cos\alpha = \frac{< J(Z_1), Z_1 >}{\|Z_1\|.\|J(Z_1)\|}$$

$$cos\alpha = \frac{3 - \phi}{\sqrt{5(8 - \phi)}}.$$

We define $D^\alpha = span\{Z_1\}$ is a slant distribution with slant angle $\alpha = arccos[\frac{3-\phi}{\sqrt{5(8-\phi)}}]$ and $D^\perp = span\{Z_3\}$ is an anti-invariant distribution. Sklince $J(Z_3)$ is orthogonal to $TM$ and $D^T = span\{Z_2, Z_4, Z_5\}$ since $J(D^T) \subset D^T$ i.e $D^T$ is an invariant distribution with respect to $J$.
Thus, we can conclude that $M$ is a proper skew semi-invariant submanifold of $\overline{M}$.
Let $D^\alpha$ and $D^T$ be slant and invariant distribution on $M$, respectively. Then $M$ is called $(D^\alpha, D^T)-$mixed totally geodesic if $h(Y, X) = 0$, where $Y \in D^\alpha$ and $X \in D^T$.

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