Journal of Mathematical Control Science and Applications
Vol. 7 No. 2 (July-December, 2021)

Submitted: 15th June 2021 *Revised: 11th July 2021* *Accepted: 19th August 2021*

EXISTENCE OF INVARIANT CURVES FOR THE EQUATION OF THE MICROTRON

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Abstract. Initially, an analysis of the phase change and energy equations of the "microtron" is carried out, determining its fixed points and their classification, from which the existence of an elliptical fixed point is determined. Using the theorems KAM, Moser's Twist and normal forms, the existence of invariant curves around the point is determined fixed elliptical.
Keywords: Microtron, invariant curves, theorem KAM, Moser's Twist theorem, theorem of normal forms.

1. Introduction

Race-**t**rack **m**icrotron (RTM), is a cyclic particle accelerator for low to intermediate energies, combining properties of **lin**ear **ac**celerator (LINAC) and circular machine, the microtron is sometimes called "electron cyclotron" [SM78, R.E84a, Lid94]. For applications in which a modest beam power at a relatively high beam energy is required, this type of particle accelerators allows to get pulsed in such way that, higher energy continuous beams can be emitted, optimizing cost in a very effective way, within machine optimal dimensions. Main features of the longitudinal beam dynamics in RTMs, in particular small width of the region of stable phase oscillations and nonlinear resonances, are essentially the same as in case of the classical microtron [A.A56, R.E84b, VMVV86].

The phase change and energy equations of the "microtron" can be modeled by the equations:

$$\begin{cases} \delta\phi_{n+1} = \delta\phi_n + \frac{2\pi\nu}{\Delta W}\delta W_n, \\ \delta W_{n+1} = \Delta W_0 \cos(\phi_s + \delta\phi_{n+1}) + \delta W_n - \Delta W, \end{cases} \tag{1.1}$$

where $\Delta W = 2$ and $\nu = 1$ are physical constants and $\Delta W_0 = \frac{\Delta W}{\cos(\phi_s)}$.

The recurrence relations (1.1) suggest to model the dynamics as iterations of a map which can be written as follows:

$$F(x, y) = F_{\phi_s}(x, y) = (f_1(x, y), f_2(x, y)),$$

where

2000 *Mathematics Subject Classification.* Primary 37-XX; Secondary 37D99.

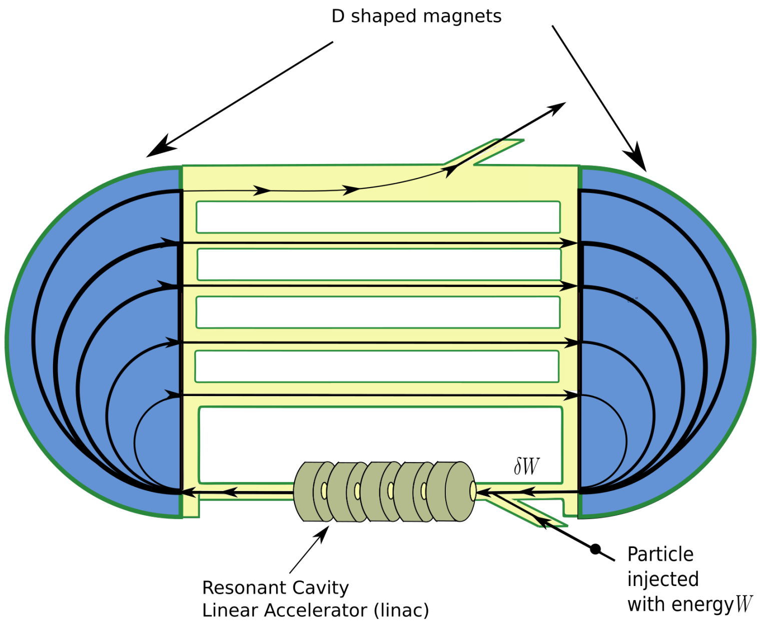

FIGURE 1. Microtron scheme.

$$\begin{aligned} f_1(x,y) &= x+\pi y, \\ f_2(x,y) &= \Delta W_0\cos(\phi_s + f_1(x,y)) + y - 2, \end{aligned} \tag{1.2}$$

with which $(\delta\phi_{n+1}, \delta W_{n+1}) = F_{\phi_s}(\delta\phi_n, \delta W_n)$.

The stability region associated with the map (1.2) (called acceptance) is the set of initial conditions at which the particles in the microtron obtain a maximum energy gain, without escaping to infinity.

2. Normal form

Some properties of F_{ϕ_s}

(1) $F_{-\phi_s}(2\phi_s + x, y) = F_{\phi_s}(x,y) + (2\phi_s, 0)$.
(2) $F_{\phi_s}(x,y) = F_{\phi_s+2\pi}(x,y)$.
(3) $F_{\phi_s}(x+2\pi, y) = F_{\phi_s}(x,y) + (2\pi, 0)$.
(4) There exists $R : \mathbb{R}^2 \to \mathbb{R}^2$, $R(x,y) = (x+\pi y, -y)$, such that $F_{\phi_s}^{-1} = R \circ F_{\phi_s} \circ R$ and $R \circ R = Id$.
(5) $\text{Fix}\{F_{\phi_s}\} = \{(0+2n\pi, 0); (-2\phi_s + 2n\pi, 0)\}$, with $n \in \mathbb{Z}$.

Property 3 shows that it is sufficient to know the phase portrait at $[0, 2\pi] \times \mathbb{R}$. In the property 4, R is not the unique such that $R \circ F_{\phi_s}^{-1} = F_{\phi_s} \circ R$ and $R \circ R = Id$, since $R_2 = F_{\phi_s} \circ R$ also verifies it.

It can be seen that, when $\phi_s \in (-\arctan(2/\pi), 0)$, the origin is a hyperbolic fixed point of the map F and, in addition, $(-2\phi_s, 0)$ is an elliptical fixed point.

For some parameters, the intersection between the stable and unstable manifold of the origin is transversal. Moreover, by Fontich-Simó results [FS90] if ϕ_s is sufficiently small then the angle at its first intersection is exponentially small.

We will apply the Kolmogorov-Arnold-Moser (KAM) theory to garantee the existence of invariant curves enclosing the stability region of the elliptic point. For this the map F must fulfill some technical conditions. One first step is to choose new variables to transform F into $Id + G_h$, that is a perturbation of the identity. With this aim in mind we use the following coordinate change.

$$T(x, y) = \left(\frac{x}{h^2}, \frac{y}{2h^3}\right). \tag{2.1}$$

In the new variables our map can be written as

$$\bar{F} = T \circ F \circ T^{-1}$$

So we obtain:

$$\begin{aligned} x_1 &= x + 2\pi h y, \\ y_1 &= y + \left(\cos(h^2 x_1) + \frac{2}{\pi}\sinh^2(h/2)\sin(h^2 x_1) - 1\right)/h^3, \end{aligned} \tag{2.2}$$

where x_1 and y_1 denote the new variables and $\tan(\phi_s) = -\frac{2}{\pi}\sinh^2(\frac{h}{2})$.

The interesting thing about this variable change is that the new $\bar{F}$ map is close to the identity. Although the fixed points of $\bar{F}$ depend on the h parameter, they remain almost immobile when the h varies, while the two fixed point of the original map F, which depends on the parameter ϕ_s, tends to collapse into a single fixed point when h tends to 0.

Note that $DT(x, y)$, is a constant diagonal matrix so that $|\det(D\bar{F}(x, y))| = 1$, that is, the property that the application is conservative is preserved by the variable change.

Remark 2.1. If p a fixed point of F, then $p^* = T(p)$ is a fixed point of $\bar{F} = T \circ F \circ T^{-1}$.

Thus $\bar{F}$ has the following properties:

(1) We know that the fixed points of F are $(0, 0)$ and $(-2\phi_s, 0)$. Then, by remark 2.1, the fixed points of $\bar{F}$ are:

$$P_1 = (0, 0) \text{ and } P_2 = (-2\phi_s/h^2, 0),$$

where

$$\tan(\phi_s) = -\frac{2}{\pi}\sinh^2(\frac{h}{2}). \tag{2.3}$$

and $\tan(\phi_s) \in (-\frac{2}{\pi}, 0)$.

The maximum value h^* for $h > 0$, attained by

$$\sinh(h^*/2) = 1,$$

where

$$h^* = \log(3 + \sqrt{8}).$$

(2) The eigenvalues of

$$M_1 = D\bar{F}(P_1) = \begin{bmatrix} 1 & 2\pi h \\ \frac{2}{\pi h}\sinh^2(\frac{h}{2}) & 4\sinh^2(\frac{h}{2}) + 1 \end{bmatrix}.$$

are $e^{\pm h}$, therefore P_1 is a hyperbolic fixed point of $\bar{F}$, for all $h \neq 0$.

Although when we put P_2 into the differential we have:

$$M_2 = D\bar{F}(P_2) = \begin{bmatrix} 1 & 2\pi h \\ -\frac{2}{\pi h}\sinh^2(\frac{h}{2}) & -4\sinh^2(\frac{h}{2}) + 1 \end{bmatrix}.$$

Since $\sinh^2(x)$ is an increasing function for all $x > 0$ we have:

$$\sinh^2\left(\frac{h}{2}\right) < \sinh^2\left(\frac{h^*}{2}\right) = 1 \text{ for } h \in (0, h^*),$$

thus $(4\sinh^2(\frac{h}{2}) - 2)^2 - 4 < 0$ and therefore M_2 matrix has eigenvalues $e^{\pm\theta i}$. We conclude that, in this range of parameters, P_2 is an elliptical fixed point of $\bar{F}$. Moreover, M_2 is linearly equivalent to a rotation:

$$M_2 \cong \begin{bmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{bmatrix} = Rot(\theta).$$

We can establish a functional relation between h and θ as follows:

$$\cos(\theta) = 1 - 2\sinh^2(\frac{h}{2}), \tag{2.4}$$

which is obtained from the characteristic polynomial of M_2:

$$\lambda^2 - 2(1 - 2\sinh^2(h/2))\lambda + 1.$$

The equation (2.4) is equivalent to:

$$\sinh^2(h/2) = \sin^2(\theta/2) \text{ or } \cosh(h) + \cos(\theta) = 2, \tag{2.5}$$

Also note:

$$\tan(\phi_s) = \frac{\cos(\theta) - 1}{\pi}.$$

(3) As we had seen $\bar{F}$ is conservative. Now using the properties 1 and 2 we have that:

$$(P^{-1} \circ \bar{F}_2 \circ P)(x, y)^t = \begin{bmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} + \mathcal{O}_2(x, y), \tag{2.6}$$

where ^{t} means transposed, and

$$\bar{F}_2(x, y) = \bar{F}(x - 2\phi_s/h^2, y)^t - \begin{bmatrix} -2\phi_s/h^2 \\ 0 \end{bmatrix}, \tag{2.7}$$

translate the fixed point $(-2\phi_s/h^2, 0)$ of $\bar{F}$ to the origin, so that $\bar{F}_2(0,0) = (0,0)$.
Calling $\bar{\bar{F}}(x,y)^t$ to the left member of (2.6) we have:

$$\bar{\bar{F}}(x,y)^t = \begin{bmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} + \mathcal{O}_2(x,y). \tag{2.8}$$

Explicitly $\bar{\bar{F}}(x,y) = (x_1, y_1)$ can be written as:

$$y_1 = \sin(\theta)x + \cos(\theta)y \tag{2.9}$$

$$x_1 = -\frac{h^3\cos(\phi_s)\left[(\cos(\theta)-2)\,y_1 + y\right] + \cos(\phi_s + 2\pi h^3 y_1)}{h^3\sin(\theta)\cos(\phi_s)} \tag{2.10}$$

$$+\frac{1}{h^3\sin(\theta)}. \tag{2.11}$$

Moreover, x_1 have a Taylor series expansion:

$$x_1 \sim \cos(\theta)x - \sin(\theta)y + \sum_{n=2}(-1)^{k(n)}a_{k(n)}(2\pi)^n h^{3n-3}y_1^n. \tag{2.12}$$

Note that the application (2.7) can also be written as follows:

$$\tilde{F} = (\tilde{x}_1, \tilde{y}_1), \tag{2.13}$$

with:

$$\tilde{x}_1 = \frac{(1-e^{\theta i})\tilde{x} + \frac{1}{2}(e^{-\theta i}-1)^2\tilde{y} - \frac{1}{2h^3} + COS\_NEW(\tilde{x},\tilde{y})}{\sin(\theta)}i, \tag{2.14}$$

$$\tilde{y}_1 = \frac{-\frac{1}{2}(e^{\theta i}-1)^2\tilde{x} - (1-e^{-\theta i})\tilde{y} + \frac{1}{2h^3} - COS\_NEW(\tilde{x},\tilde{y})}{\sin(\theta)}i \tag{2.15}$$

where:

$$COS\_NEW(x,y) := \frac{\cos\left(\phi_s + 2h^3\pi(xe^{\theta i} + ye^{-\theta i})\right)}{2h^3\cos(\phi_s)}.$$

With a variable change defined by:

$$PN(x,y) = \begin{bmatrix} -2\pi h & -2\pi h \\ 1-e^{\theta i} & 1-e^{-\theta i} \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}.$$

Thus we have that:

$$\tilde{F}(x,y) = (PN^{-1} \circ \bar{F}_2 \circ PN)(x,y)$$

Remark 2.2. Note that

$$\begin{bmatrix} u \\ v \end{bmatrix} = PN(x,y).$$

implies that $v = \bar{u}$.

Remark 2.3. If $\tilde{f}_1(\tilde{x},\tilde{y}) = \tilde{x}_1$ then:

(1)
$$\tilde{y}_1 = \overline{\tilde{f}_1(\bar{\tilde{y}}, \bar{\tilde{x}})}. \tag{2.16}$$

(2) Since $\tilde{y} = \bar{\tilde{x}}$ we have $\tilde{y}_1 = \overline{\tilde{x}_1}$.

(3)
$$\tilde{y}_1 + \tilde{x}_1 = e^{\theta i}\tilde{x} + e^{-\theta i}\tilde{y}. \tag{2.17}$$

(4) $\tilde{y}_1 + \tilde{x}_1 = 2\Re(\tilde{x}_1) \in \mathbb{R}$.

Expanding (2.14) and (2.15) into Taylor series and using (2.16) we have:

$$\begin{aligned} \tilde{f}_1(x,y) &= e^{\theta i}x + \frac{i}{\sin(\theta)}\sum_{n=2}(-1)^{k(n)}\frac{\tilde{a}_{k(n)}(2h^3)^{n-1}\pi^n}{n!}(e^{\theta i}x + e^{-\theta i}y)^n, \\ \tilde{x}_1 &= \tilde{f}_1(\tilde{x},\tilde{y}), \qquad (2.18) \\ \tilde{y}_1 &= \overline{\tilde{x}_1}, \qquad (2.19) \end{aligned}$$

where:
$$\tilde{a}_{k(n)} = \begin{cases} 1, & \text{if } n = 2k(n) \\ \frac{1-\cos(\theta)}{\pi}, & \text{if } n = 2k(n)+1 \end{cases}$$

That is:
$$\begin{aligned} \tilde{f}_1(x,y) &= e^{\theta i}x - \frac{h^3\pi^2}{\sin(\theta)}(e^{\theta i}x + e^{-\theta i}y)^2 i \qquad (2.20) \\ &\quad - \frac{(1-\cos(\theta))(2h^3\pi)^2}{3!\sin(\theta)}(e^{\theta i}x + e^{-\theta i}y)^3 i \\ &\quad + \frac{i}{\sin(\theta)}\sum_{n=4}(-1)^{k(n)}\frac{\tilde{a}_{k(n)}(2h^3)^{n-1}\pi^n}{n!}(e^{\theta i}x + e^{-\theta i}y)^n, \\ \tilde{x}_1 &= \tilde{f}_1(\tilde{x},\tilde{y}), \\ \tilde{y}_1 &= \overline{\tilde{x}_1}. \end{aligned}$$

3. Existence of invariant curves

In this section we will find the invariant curves of the $\tilde{F}$ map, around the origin.

Definition 3.1 (Twist map)**.** An area-preserving twist map, $\mathcal{F} : \mathfrak{A} \to \mathfrak{A}$, then takes the form
$$\mathcal{F} : (\theta, \tau) \mapsto (\theta + \alpha(\tau), \tau)$$
where $\mathrm{d}\alpha/\mathrm{d}\tau = \alpha'(\tau) \neq 0$ for $\tau \in [a,b]$ and the set $\mathfrak{A} = \{(\theta,\tau) \mid a \leqslant \tau \leqslant b, 0 \leqslant \theta < 2\pi\}$, is called an annulus.

Theorem 3.2 (KAM)**.** *Let $\mathcal{F} : \mathfrak{A} \to \mathfrak{A}$ be, a twist map and a small perturbation $\varepsilon\mathcal{G}(\theta, r)$ of class C^k, with $k > 3$, in the annulus $\mathfrak{A}$. Then there are simple closed invariant curves close to the circles of constant radius and the measure of the complementary of their union is small when the perturbation is small. The orbit of any point on one of these invariant curves is dense under the iterations of the $\mathcal{G}$. (see* [AKN88, Chapter 5: Perturbation Theory for Integrable Systems, Section 3: The KAM Theory, Theorem 19]*,* [AP90]*).*

Remark 3.3. The map $\tilde{F}$ defined in (2.13), is not a perturbation of a map Twist, for this reason we cannot directly use the KAM theorem 3.2.

Theorem 3.4 (Birkhoff's Normal Forms)**.** *Let* $G:\mathbb{R}^2\to\mathbb{R}^2$,

$$G\begin{pmatrix} u \\ v \end{pmatrix} = \begin{bmatrix} \cos(\alpha) & -\sin(\alpha) \\ \sin(\alpha) & \cos(\alpha) \end{bmatrix} \begin{bmatrix} u \\ v \end{bmatrix} + \mathcal{O}_2\left(u,v\right), \tag{3.1}$$

area-preserving.

For every $n\in\mathbb{N}$, $p\in\mathbb{Z}$ *and* $\forall k\in\{1,\dots,2(n+1)\}$ *where* $\alpha\neq 2\pi\frac{p}{k}$ *and* p *and* k *are coprimes there exists a variable change*

$$T_n(x,y) = \left(x + \sum_{i+j=2}^{2n+1} a_{ij}x^iy^j, y + \sum_{i+j=2}^{2n+1} b_{ij}x^iy^j \right),$$

which transforms the G *into a map in:*

$$\begin{aligned} \bar{G}_n\begin{pmatrix} x \\ y \end{pmatrix} &= T_n^{-1}GT_n\begin{pmatrix} x \\ y \end{pmatrix} \\ &= \begin{bmatrix} \cos(\beta_n) & -\sin(\beta_n) \\ \sin(\beta_n) & \cos(\beta_n) \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} + \mathcal{O}_{2n+2}\left(x,y\right), \end{aligned} \tag{3.2}$$

with

$$\beta_n = \beta_n(x,y) = \alpha + \sum_{i=1}^{n} \alpha_i (x^2+y^2)^i, \tag{3.3}$$

Remark 3.5. If we omit the terms $\mathcal{O}_{2n+2}\left(x,y\right)$, in (3.2) then we have

$$\bar{G}_n^1\begin{pmatrix} x \\ y \end{pmatrix} = \begin{bmatrix} \cos(\beta_n) & -\sin(\beta_n) \\ \sin(\beta_n) & \cos(\beta_n) \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}.$$

$\bar{G}_n^1$ is a Twist map as long as any of the α_k coefficients are non-zero, since $\frac{\mathrm{d}\beta_n}{\mathrm{d}r}\neq 0$, where $r^2=x^2+y^2$, now applying the KAM theorem to $\bar{G}_n$, we can guarantee the existence of invariant curves close to circles in a neighborhood of the origin.

Theorem 3.6 (Moser)**.** *Let*

$$G\begin{pmatrix} x \\ y \end{pmatrix} = \begin{bmatrix} \cos(\alpha) & -\sin(\alpha) \\ \sin(\alpha) & \cos(\alpha) \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} + \mathcal{O}_2\left(x,y\right),$$

area-preserving, if for some $n\in\mathbb{N}$, *and* $\forall p\in\mathbb{Z}$ *and* $k\in\{1,\dots,2(n+1)\}$, *coprime such that:* $\alpha\neq 2\pi\frac{p}{k}$, *and* $\alpha_1=\cdots=\alpha_{n-1}=0$, $\alpha_n\neq 0$, *in the normal form* (3.2), (3.3)*. Then there are invariant curves close to circles in a sufficiently small neighborhood of the origin. Furthermore, the orbit of a point is dense on the corresponding invariant curve.*

The following lemma will be used to construct an algorithm which provides a suitable coordinate change of variable to find the Birkhoff normal form of our map, in order to apply twist theorem.

Lemma 3.7. *Let* $F:\mathbb{R}^n\to\mathbb{R}^n$ *be a diffeomorphism such that* $F(0)=0$, *with Taylor expansion* $F(X)=F_1(X)+F_2(X)+\cdots+F_k(X)+\mathcal{O}_{k+1}(X)$, *up to order* r, *where $F_r \in \mathcal{H}_r$ the set of homogeneous polynomials of order r, and $F_1(X) = AX$. Suppose $F(X) = AX + F_r(X) + \mathcal{O}_{r+1}(X)$, and let*

$$X = Y + K_r(Y) = K(Y),$$

be the variable change defined by K_r. Then we define

$$\mathcal{L}(K_r(x)) = K_r(AX) - AK_r(X), \quad K_r \in \mathcal{H}_r \tag{3.4}$$

The map $\mathcal{L}$, is a linear operator that act on the linear space of homogeneous polynomials of order r. If $K_r(X)$ is a solution for the equation $\mathcal{L}(K_r(x)) = F_r(X)$, then the corresponding variable change, $X = K(Y)$, annihilate the terms of order r.

Proof. Let $F(X) = AX + F_r(X) + \mathcal{O}_{r+1}(X)$, with $r \geq 2$, consider the following coordinate change

$$K(Y) = X = Y + K_r(Y) \text{ with } K_r \in \mathcal{H}_r. \tag{3.5}$$

$K_{r(Y)}$ indicates that we have a homogeneous polynomial of order r, as in F_r. From now on, as long as we have a subscript r, this will indicate that we have a homogeneous polynomial of order r.

Since K transforms F into

$$\tilde{F}(Y) = (K^{-1} \circ F \circ K)(Y)$$

and

$$K^{-1}(X) = X - K_r(X) + \mathcal{O}_{r+1}(X), \tag{3.6}$$

then we get

$$\begin{aligned}
\tilde{F}(Y) &= K^{-1}(F(Y + K_r(Y))) \\
&= K^{-1}\left(A(Y + K_r(Y)) + F_r(Y + K_r(Y))\right) + \mathcal{O}_{r+1}(X) \\
&= K^{-1}\left(AY + AK_r(Y) + F_r(Y)\right) + \mathcal{O}_{r+1}(X) \\
&= AY + AK_r(Y) + F_r(Y) - K_r(AY + AK_r(Y) + F_r(Y)) + \mathcal{O}_{r+1}(X) \\
&= AY + AK_r(Y) + F_r(Y) - K_r(AY) + \mathcal{O}_{r+1}(X).
\end{aligned}$$

It is wanted to annihilate the lower order terms in the new variable,

$$\tilde{F}(Y) = AY + \tilde{F}_{r+1}(Y) + \mathcal{O}_{r+2}(Y).$$

For this you should have to:

$$AK_r(Y) + F_r(Y) - K_r(AY) = 0 \Rightarrow F_r(Y) = K_r(AY) - AK_r(Y). \tag{3.7}$$

The equation (3.7) is known as the homological equation. □

It is of interest to know under what conditions equation (3.7) can be solved. For this, remember that $\mathcal{L}$ is a linear operator that acts on the linear space of homogeneous polynomials of order r, $\mathcal{H}_r$. If the linear part of F is assumed to be diagonalizable and

$$A = \begin{bmatrix} \lambda_1 & 0 & \dots & 0 \\ 0 & \lambda_2 & 0 & \vdots \\ \vdots & & \ddots & \vdots \\ 0 & \dots & 0 & \lambda_n \end{bmatrix},$$

then, $\boldsymbol{x}^{\boldsymbol{m}} e_j$ are the eigenvectors of $\mathcal{L}$ with the respective eigenvalues $\boldsymbol{\lambda}^{\boldsymbol{m}} - \lambda_j$, $j = 1, \dots, n$.

Therefore the coefficients of $K_r(x)$ are given by:

$$K_{\boldsymbol{m},j} = \frac{F_{\boldsymbol{m},j}}{\boldsymbol{\lambda}^{\boldsymbol{m}} - \lambda_j}, \tag{3.8}$$

with $j = 1, \dots, n$.

The notation used here means the following:

- $\boldsymbol{\lambda} = (\lambda_1, \dots, \lambda_n)$.
- $\boldsymbol{m} = (m_1, \dots, m_n)$, with $\sum_{j=1}^n m_j = r$ where, $m_j \in \mathbb{N} \cup \{0\}$.
- $\boldsymbol{\lambda}^{\boldsymbol{m}} = \lambda_1^{m_1} \lambda_2^{m_2} \dots \lambda_n^{m_n}$, with $\sum_{j=1}^n m_j = r$ where, $m_j \in \mathbb{N} \cup \{0\}$.
- $\boldsymbol{x}^{\boldsymbol{m}} = x_1^{m_1} x_2^{m_2} \dots x_n^{m_n}$, with $\sum_{j=1}^n m_j = r$ where, $m_j \in \mathbb{N} \cup \{0\}$.
- $F_{\boldsymbol{m},j}$ is the coefficient of $\boldsymbol{x}^{\boldsymbol{m}}$ in F_r for the coordinate j -th.
- $K_{\boldsymbol{m},j}$ is the coefficient of $\boldsymbol{x}^{\boldsymbol{m}}$ in K_r for the coordinate j -th.

Remark 3.8. Whenever $\boldsymbol{\lambda}^{\boldsymbol{m}} - \lambda_j = 0$ it is said that the eigenvalues $\lambda_1, \dots, \lambda_n$ have a resonance. The existence of resonances is also called the "small denominator problem" because of the equation (3.8). Most of KAM theory is based on the idea that a certain Newton's algorithm can be used to solve small denominator problems. This is what will be done next for the microton model.

Remark 3.9. If $r = 2$ on the equation (3.5), then we get

$$K(Y) = X = Y + K_2(Y), \tag{3.9}$$

and if we assume the equation (3.6) then:

$$K^{-1}(X) = X + h_2(X) + h_3(X) + \mathcal{O}_4(X). \tag{3.10}$$

The composition between K and K^{-1} is

$$\begin{aligned} Y &= (K \circ K^{-1})(Y) \\ &= K(Y + h_2(Y) + h_3(Y) + \mathcal{O}_4(Y)) \\ &= Y + h_2(Y) + h_3(Y) + K_2(Y + h_2(Y) + h_3(Y)) + \mathcal{O}_4(Y), \end{aligned}$$

therefore

$$0 = h_2(Y) + h_3(y) + K_2(Y + h_2(Y) + h_3(Y)) + \mathcal{O}_4(Y). \tag{3.11}$$

Now applying the Taylor expansion to $K_2(Y + h_2(Y) + h_3(Y))$ we have:

$$\begin{aligned} K_2(Y + h_2(Y) + h_3(Y)) &= K_2(Y) + DK_2(Y)\,(h_2(Y) + h_3(Y)) + \mathcal{O}_4(Y) \\ &= K_2(Y) + DK_2(Y) h_2(Y) + \mathcal{O}_4(Y). \end{aligned}$$

Thus we have the equation (3.11)

$$h_2(Y) = -K_2(Y) \tag{3.12}$$

$$h_3(Y) = -DK_2(Y) h_2(Y) = DK_2(Y) K_2(Y) \tag{3.13}$$

Now that we have all the tools, let's begin by applying the lema 3.7 to our map $\tilde{F}$ (see (2.20)) in which we have $r = 2$ and $n = 2$. For this, define:

$$\tilde{f}(x, y) = -\frac{i}{\sin(\theta)} h^3 \pi^2 (e^{\theta i} x + e^{-\theta i} y)^2,$$

therefore

$$F_2(x,y) = (\tilde{f}(x,y), \overline{\tilde{f}(x,y)}),$$

and according to the notation, we get that

$$-\tilde{F}_{(2,0),1} = \tilde{F}_{(2,0),2} = \frac{\pi^2 h^3}{\sin(\theta)} e^{2\theta i} i,$$

$$-\tilde{F}_{(1,1),1} = \tilde{F}_{(1,1),2} = 2\frac{\pi^2 h^3}{\sin(\theta)} i,$$

$$-\tilde{F}_{(0,2),1} = \tilde{F}_{(0,2),2} = \frac{\pi^2 h^3}{\sin(\theta)} e^{-2\theta i} i,$$

By other hand, we have $\boldsymbol{\lambda^m} = \lambda_1^{m_1}\lambda_2^{m_2} = \lambda^{m_1 - m_2}$, but $m_1 + m_2 = 2$ then $\boldsymbol{\lambda^m} = \lambda^{2m_1 - 2}$, where $\lambda = \lambda_1 = \lambda_2^{-1} = e^{\theta i}$. Thus we have the following tables:

m_1	m_2	$\lambda_1^{m_1}\lambda_2^{m_2}$	$\boldsymbol{\lambda^m} - \lambda_1$	resonance
2	0	λ^2	$\lambda(\lambda - 1) = e^{2\theta i} - e^{\theta i}$	$\theta = \pi$
1	1	1	$1 - \lambda = 1 - e^{\theta i}$	$\theta = 2\pi$
0	2	λ^{-2}	$\lambda^{-2}(1 - \lambda^3) = e^{-2\theta i} - e^{\theta i}$	$\theta = 2\pi/3$

m_1	m_2	$\lambda_1^{m_1}\lambda_2^{m_2}$	$\boldsymbol{\lambda^m} - \lambda_2$	resonance
2	0	λ^2	$\lambda^{-1}(\lambda^3 - 1) = e^{2\theta i} - e^{-\theta i}$	$\theta = 2\pi/3$
1	1	1	$\lambda^{-1}(\lambda - 1) = 1 - e^{-\theta i}$	$\theta = 2\pi$
0	2	λ^{-2}	$\lambda^{-2}(1 - \lambda^3) = e^{-2\theta i} - e^{-\theta i}$	$\theta = 2\pi/3$

Therefore

$$K_2(y_1, y_2) = \frac{\pi^2 h^3}{sin(\theta)} i \begin{bmatrix} -\frac{e^{2\theta i}}{e^{2\theta i} - e^{\theta i}} y_1^2 - \frac{2}{1 - e^{\theta i}} y_1 y_2 - \frac{e^{-2\theta i}}{e^{-2\theta i} - e^{\theta i}} y_2^2 \\ \\ \frac{e^{2\theta i}}{e^{2\theta i} - e^{-\theta i}} y_1^2 + \frac{2}{1 - e^{-\theta i}} y_1 y_2 + \frac{e^{-2\theta i}}{e^{-2\theta i} - e^{-\theta i}} y_2^2 \end{bmatrix}, \tag{3.14}$$

and using (3.6) we have:

$$K^{-1}(x_1, x_2) = (x_1, x_2) - K_2(x_1, x_2) + \mathcal{O}_3(x_1, x_2). \tag{3.15}$$

However, we want to determine the inverse $K^{-1}(x_1, x_2)$, up to order 3, to see the resonant terms in equation (2.13), for this reason equation (3.13) will be used. For this, DK_2 must first be found.

$$DK_2 = \begin{pmatrix} [DK_2]_{(1,1)} & [DK_2]_{(1,2)} \\ \\ \overline{[DK_2]_{(1,2)}} & \overline{[DK_2]_{(1,1)}} \end{pmatrix}, \tag{3.16}$$

where

$$[DK_2]_{(1,1)} = -\frac{2\pi^2 h^3}{sin(\theta)} i \left(\frac{x}{1 - e^{-\theta i}} + \frac{y}{1 - e^{\theta i}} \right) = \overline{[DK_2]_{(2,2)}},$$

and

$$[DK_2]_{(1,2)} = -\frac{2\pi^2 h^3}{sin(\theta)} i \left(\frac{y}{1 - e^{3\theta i}} + \frac{x}{1 - e^{\theta i}} \right) = \overline{[DK_2]_{(2,1)}}.$$

Thus

$$h_3(x,y) = DK_2(x,y) K_2(x,y).$$

Therefore the development of order 3 for equation (2.13), together with the composition of $K^{-1}(X) \sim X + h_2(X) + h_3(X)$ and $K(Y) = X = Y + K_2(Y)$ we obtain that:

$$\tilde{F}_{new}(x,y) = (K^{-1} \circ \tilde{F} \circ K)(x,y) = \begin{bmatrix} e^{\theta i}x \\ e^{-\theta i}y \end{bmatrix} + \begin{bmatrix} h_3^1(x,y) \\ h_3^2(x,y) \end{bmatrix} + \mathcal{O}_4(x,y).$$

We denote the homogeneous polynomials, $\begin{bmatrix} h_3^1(x,y) \\ h_3^2(x,y) \end{bmatrix}$ by $H_3(x,y)$.

Now we are going to repeat the procedure to leave only the resonant terms of order 3. Therefore we have $\boldsymbol{\lambda^m} = \lambda_1^{m_1}\lambda_2^{m_2} = \lambda^{m_1-m_2} = \lambda^{2m_1-3}$, where $\lambda = \lambda_1 = \lambda_2^{-1} = e^{\theta i}$. Thus we have the following tables:

m_1	m_2	$\lambda_1^{m_1}\lambda_2^{m_2}$	$\boldsymbol{\lambda^m} - \lambda_1$	resonance
3	0	λ^3	$\lambda(\lambda^2-1) = e^{3\theta i} - e^{\theta i}$	$\theta = \pi$
2	1	λ	0	$\forall\, \theta$
1	2	λ^{-1}	$\lambda^{-1}(1-\lambda^2) = e^{-\theta i} - e^{\theta i}$	$\theta = \pi$
0	3	λ^{-3}	$\lambda^{-3}(1-\lambda^4) = e^{-3\theta i} - e^{\theta i}$	$\theta = \pi/2$

m_1	m_2	$\lambda_1^{m_1}\lambda_2^{m_2}$	$\boldsymbol{\lambda^m} - \lambda_2$	resonance
3	0	λ^3	$\lambda^{-1}(\lambda^4-1) = e^{3\theta i} - e^{-\theta i}$	$\theta = \pi/2$
2	1	λ	$\lambda^{-1}(\lambda^2-1) = e^{\theta i} - e^{-\theta i}$	$\theta = \pi$
1	2	λ^{-1}	0	$\forall\, \theta$
0	3	λ^{-3}	$\lambda^{-3}(1-\lambda^2) = e^{-3\theta i} - e^{-\theta i}$	$\theta = \pi$

From this tables we get that

$$K_3 = \begin{bmatrix} h^1_{3,(3,0)}\frac{x^3}{e^{3\theta i}-e^{\theta i}} + \mathrm{C}_1x^2y + h^1_{3,(1,2)}\frac{xy^2}{e^{-\theta i}-e^{\theta i}} + h^1_{3,(0,3)}\frac{y^3}{e^{-3\theta i}-e^{\theta i}} \\ h^2_{3,(3,0)}\frac{x^3}{e^{3\theta i}-e^{-\theta i}} + h^2_{3,(2,1)}\frac{x^2y}{e^{\theta i}-e^{-\theta i}} + \mathrm{C}_2xy^2 + h^2_{3,(0,3)}\frac{y^3}{e^{-3\theta i}-e^{-\theta i}} \end{bmatrix} \tag{3.17}$$

where C_1 y C_2 can take arbitrary real values.
Now making

$$\begin{aligned} \tilde{K} &= I + K_3, \\ \tilde{K}^{-1} &= I - K_3 + \mathcal{O}_4, \end{aligned}$$

we have

$$\tilde{F}_{new2}(x,y) = (\tilde{K}^{-1} \circ \tilde{F}_{new} \circ \tilde{K})(x,y) = \begin{bmatrix} \lambda x \\ \lambda^{-1}y \end{bmatrix} + \begin{bmatrix} r(\lambda)x^2y \\ -\frac{r(\lambda)}{\lambda^2}xy^2 \end{bmatrix} + \mathcal{O}_4(x,y),$$

where

$$\begin{aligned} r(\lambda) &= -2h^6\pi^2\lambda\frac{p(\lambda)}{q(\lambda)}, \\ p(\lambda) &= \lambda^6 - 3\lambda^5 + (3+8\pi^2)\lambda^4 + (4\pi^2-2)\lambda^3 + \\ &\quad (3+8\pi^2)\lambda^2 - 3\lambda + 1, \\ q(\lambda) &= (\lambda-1)^2(\lambda^3-1)(\lambda+1), \\ \lambda &= e^{\theta i}. \end{aligned}$$

Remark 3.10.

(1) The functions $p(\lambda)$ and $q(\lambda)$, do not depend on the values C_1 and C_2.
(2) $\bar{\lambda} = \lambda^{-1}$.
(3) $\overline{r(\lambda)} = -\frac{r(\lambda)}{\lambda^2}$.
(4) $\overline{\left(\frac{r(\lambda)}{\lambda}\right)\ i} = -\frac{r(\lambda)}{\lambda} i \in \mathbb{R}$.

Now if we make

$$\alpha(\lambda) = \frac{r(\lambda)}{\lambda i}, \tag{3.18}$$

Moreover, using using (2.5), we can see $\alpha(\lambda)$ explicitly:

$$\alpha(\lambda(\theta)) = 2\pi^2 h^6 \frac{2\cos(\theta)^3 - 3\cos(\theta)^2 + 4\pi^2\cos(\theta) + 1 + \pi^2}{(2\cos(\theta)+1)(\cos(\theta)-1)\sin(\theta)}, \tag{3.19}$$

and

$$h = \operatorname{arccosh}(2 - \cos(\theta)).$$

Using (3.18), we have that:

$$\tilde{F}_{new2}(x,y) = \begin{bmatrix} e^{(\theta+\alpha(\lambda)xy)i}x \\ e^{(-\theta-\alpha(\lambda)xy)i}y \end{bmatrix} + \mathcal{O}_4(x,y). \tag{3.20}$$

where $x, y \in \mathbb{C}$. Now, note that if we apply the following change of variable

$$P(u,v) = \frac{1}{2}\begin{bmatrix} -i & 1 \\ i & 1 \end{bmatrix}\begin{bmatrix} u \\ v \end{bmatrix},$$

we obtain that

$$\begin{aligned} \tilde{F}_{new3}(u,v) &= P^{-1} \circ \tilde{F}_{new2} \circ P(u,v) \\ &= \begin{bmatrix} \cos(\beta) & -\sin(\beta) \\ \sin(\beta) & \cos(\beta) \end{bmatrix}\begin{bmatrix} u \\ v \end{bmatrix} + \mathcal{O}_4(u,v), \end{aligned} \tag{3.21}$$

where

$$\beta = \theta + \alpha(\lambda)(u^2 + v^2).$$

and $u, v \in \mathbb{R}$.

The above result is clearly indicated in Birkhoff's theorem.

Now let's see under what conditions the area-preserving map (2.8) fulfills the hypotheses of Moser's theorem 3.6, in order to determine the existence of invariant curves.

For $n = 1$, we have the equivalent to $\alpha_1 \neq 0$ and $\theta \notin \{2\pi\mathbb{Z}, \frac{2\pi}{2}\mathbb{Z}, \frac{2\pi}{3}\mathbb{Z}, \frac{2\pi}{4}\mathbb{Z}\} = \{\frac{2\pi}{3}\mathbb{Z}, \frac{\pi}{2}\mathbb{Z}\}$, since $\theta \in (0,\pi)$ we have $\theta \neq \frac{2}{3}\pi, \frac{1}{2}\pi$, note that the condition $\alpha_1 \neq 0$ indicate that $p(\lambda(\theta)) \neq 0$ in the equation (3.18), for this reason we have to find when $p(\lambda(\theta))$ is zero. According to this, we can first use a graphical method in which we see when $f(\theta) = 2\cos(\theta)^3 - 3\cos(\theta)^2 + 4\pi^2\cos(\theta) + 1 + \pi^2$, intersects the axis X and then we get the intersection using Newton, see Figure 2.

Note that there are two intersections, one of them is $\theta_0 \approx 1.8429983434$, which is in $(0,\pi)$, we can also verify that $\frac{\theta_0}{\pi} \notin \mathbb{Q}$, using continued fractions. So for

$\theta = \theta_0$, we can do more steps in a normal way until $\alpha_k \neq 0$, and thus we have the conditions of Moser's theorem 3.6 which guarantee the existence of invariant curves around the elliptical fixed point.

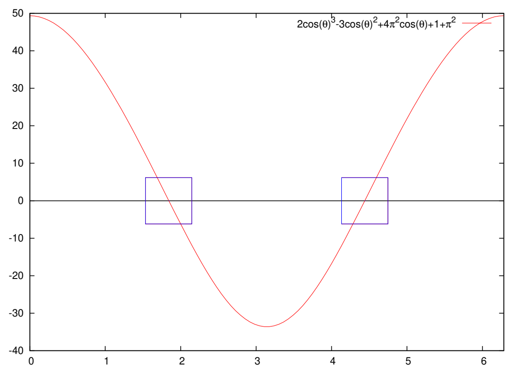

FIGURE 2. Corresponding graphics $f(\theta)$.

4. Conclusions

The KAM theorem guarantees the existence of invariant curves. This theorem cannot be applied directly, which is why the theorem must be used in normal form with which we can make some changes to transform our equation, under certain conditions in the necessary hypotheses to be able to apply Moser's Twist theorem.

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