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CONTROLLABILITY OF NON-INSTANTANEOUS IMPULSIVE EQUATIONS WITH NON-LOCAL CONDITIONS AND UNBOUNDED DELAY

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Abstract. For several years we have been proving the following conjecture: For a control system, impulses, delays and non-local conditions are intrinsic phenomena that under certain conditions do not change the controllability of a control system. That is, if we consider these elements as disturbances of the system, it turns out that the controllability is robust under these influences not taken into account in many mathematical models that represent extremely important problems in real life. Therefore, in this work we will prove the exact controllability of a semilinear control system governed by a delayed differential equation with infinite delay, non-local conditions and non-instantaneous impulses. To achieve this goal, we must choose a natural space that includes these new elements and satisfies the axiomatic theory proposed by Hale and Kato to study the behavior of differential equations with unbounded delay. Then, the controllability problem of the system is transformed into a fixed point problem, for which we will apply Rothe's Fixed Point Theorem.

1. Introduction

For a control system we believe impulses, delays and non-local conditions are intrinsic phenomena that, under certain conditions, do not destroy the controllability of the system. In other word, if we consider the impulses, delays and non-local conditions as perturbations of the system, it turns out that the controllability is robust under these influences not taken into account in many mathematical models that represent important problems in real life. Therefore, in this work we will prove the exact controllability of the following semilinear control system governed by a delayed differential equation with infinite delay, non-local conditions and non-instantaneous impulses

$$\begin{cases} z'(t) = \mathcal{A}(t)z(t) + \mathcal{B}(t)u(t) + \mathcal{F}(t, z_t, u(t)) & t \in I_k, \quad k = 0, ..., N \\ z(t) = \mathcal{J}_k(t, z(t_k^-)) & t \in J_k, \quad k = 1, ..., N \\ z(s) = \varphi(s) - h(z_{\pi_1}, \dots, z_{\pi_q})(s) & s \in (-\infty, 0]. \end{cases} \tag{1.1}$$

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where $I_0 = (0, t_1]$, $I_k = (s_k, t_{k+1}]$, $J_k = (t_k, s_k]$, $0 = s_0 < t_1 < s_1 < t_2 < s_2 < \cdots < s_N < t_{N+1} = \tau$, for a fixed τ. The control $u \in \mathcal{PW}_u$ and $0 \leq \pi_1 < \pi_2 < \cdots < \pi_q < \tau$. Also $h : \mathfrak{B}^q \to \mathfrak{B}$, $\varphi : \mathbb{R}_- \longrightarrow \mathbb{R}^n$, $\quad \varphi \in \mathfrak{B}$, $\mathfrak{B}$ is the phase space to be specified later. $\mathcal{F} : \mathbb{R}_+ \times \mathfrak{B} \times \mathbb{R}^m \longrightarrow \mathbb{R}^n$ is a smooth enough function and for $k = 1, 2, 3, \cdots, N$ we have that $\mathcal{J}_k : J_k \times \mathbb{R}^n \longrightarrow \mathbb{R}^n$ are continuous and represents the impulsive effect in the system (1.1), i.e., we are considering that the system can have abrupt changes that stay there for an interval of time. These alterations in state might be due to certain external factors, which cannot be well described by pure ordinary differential equations, (see, for instance, [11] and reference therein). $A(t) \in \mathbb{R}^{n\times n}$. For this type of problems the phase space for initial functions plays an important role in the study of both qualitative and quantitative theory, for more details, in case without impulses and non local conditions, we refer to Hale and Kato [7], Hino et al [8] and Shin [18, 19]. The function $z_t(\theta) = z(t + \theta)$ for $\theta \in (-\infty, 0]$ illustrate the history of the state up to the time t, and also remembers much of the historical past of φ, carrying part of the present to the past. A similar study was carried out by Muslim Malik, et al in [17], but only with non-instantaneous impulses; the delay and the nonlocal conditions were not considered; of course, some ideas were taken from this paper. Here, we consider the impulses, infinite delay, and non-local conditions simultaneously.

2. Preliminaries

First, and before we start to introduce necessary concepts, let's start defining the spaces where this problem will be set. We shall define the function space $\mathcal{PW}_u = \mathcal{PW}_u(([0, \tau]; \mathbb{R}^m)$ as follows:

$$\mathcal{PW}_u = \{u : [0, \tau] \to \mathbb{R}^m : u \text{ is bounded and } u \in \mathcal{C}(I; \mathbb{R}^m)\},$$

where $I = \bigcup_{i=0}^{N}(s_i, t_{i+1}]$, endowed with the norm

$$\|u\|_0 = \sup_{t\in[0,\tau]} \|u(t)\|_{\mathbb{R}^m}.$$

And now, let's define $\mathcal{PW} = \mathcal{PW}((-\infty, 0]; \mathbb{R}^n)$ as the normalized piecewise continuous functions, which can be written as follows:

$$\mathcal{PW} = \left\{\varphi : (-\infty, 0] \longrightarrow \mathbb{R}^n : \varphi\Big|_{[a,0]} \text{is a piecewise continuous function,} \quad \forall a < 0\right\}.$$

Using some ideas from [16], we consider a function $g : \mathbb{R} \to \mathbb{R}_+$ such that

$$g(0) = 1, \quad g(-\infty) = +\infty, \quad g \quad \text{is decreasing.}$$

Remark 2.1. A particular function g is $g(s) = \exp(-as)$, with $a > 0$.

Now, we define the following functions space

$$C_g = \left\{z \in \mathcal{PW} : \sup_{s\leq 0} \frac{\|z(s)\|}{g(s)} < \infty\right\}.$$

We can see that it is actually a Banach space (see [2]).

Lemma 2.2. *The space C_g equipped with the norm*

$$\|z\|_g = \sup_{s\leq 0} \frac{\|z(s)\|}{g(s)}, \quad z \in C_g,$$

is a Banach space.

Our phase space will be

$$\mathfrak{B} := C_g,$$

equipped with the norm

$$\|z\|_{\mathfrak{B}} := \|z\|_g.$$

For more details about it, one can see [1, 7, 15, 16].
Thus, $\mathfrak{B}$ will be a Banach space of functions mapping $(-\infty, 0]$ into $\mathbb{R}^n$ endowed with a norm $\|\cdot\|_{\mathfrak{B}}$.

Now, we shall consider the following larger space $\mathcal{PW}_{g\tau} := \mathcal{PW}_{g\tau}((-\infty,\tau];\mathbb{R}^n)$ defined by

$$\mathcal{PW}_{g\tau} = \Big\{ z : (-\infty,\tau] \to \mathbb{R}^n : z\Big|_{\mathbb{R}_-} \in \mathfrak{B} \text{ and } z\Big|_{(0,\tau]} \text{ is a continuous except at } t_k, \\ k = 1,2,...,N \text{ where side limits } z(t_k^+), z(t_k^-) \text{ exist and } z(t_k^+) = z(t_k) \}.$$

From Lemma 2.2, we have the following result,

Lemma 2.3. *$\mathcal{PW}_{g\tau}$ is a Banach space endowed with the norm*

$$\|z\| = \left\|z|_{\mathbb{R}_-}\right\|_{\mathfrak{B}} + \left\|z|_I\right\|_\infty,$$

where $\left\|z|_I\right\|_\infty = \sup\limits_{t\in I=(0,\tau]} \|z(t)\|$.

Now, let us denote by

$$\mathfrak{B}^q = \mathfrak{B} \times \mathfrak{B} \times ... \times \mathfrak{B} = \prod_{i=1}^{q} \mathfrak{B}.$$

i.e.,

$$z = (z_1, ..., z_q)^T \in \mathfrak{B}^q,$$

and the norm in the space $\mathfrak{B}^q$ is given by

$$\|y\|_q = \sum_{i=1}^{q} \|y_i\|_{\mathfrak{B}}.$$

Also, we consider the following hypotheses for the development of the main proof:

a) The nonlinear function $\mathcal{F} : \mathbb{R}_+ \times \mathfrak{B} \times \mathbb{R}^m \longrightarrow \mathbb{R}^n$ satisfies

$$\|\mathcal{F}(t,\nu,u)\|_{\mathbb{R}^n} \leq a_0 \|\nu\|_{\mathfrak{B}}^{\alpha_0} + b_0 \|u\|_{\mathbb{R}^m}^{\beta_0} + c_0, \quad t \in (0,\tau], \nu \in \mathfrak{B}, u \in \mathbb{R}^m.$$

b) The non instantaneous impulses function, $\mathcal{J}_k \in \mathcal{C}((t_k, s_k] \times \mathbb{R}^n; \mathbb{R}^n)$ for all $k = 1,2,3,\ldots,N$ and satisfies

$$\|\mathcal{J}_k(t,z)\|_{\mathbb{R}^n} \leq a_k \|z\|_{\mathbb{R}^n}^{\alpha_k} + c_k,$$

and

$$\| \mathcal{J}_k(s,z) - \mathcal{J}_k(t,w)\| \leq d_k (|s-t| + \|z-w\|).$$

c) The function associated with the non local condition $h : \mathfrak{B}^q :\longrightarrow \mathfrak{B}$ satisfies, for $z, w \in \mathfrak{B}^q$, the following estimate
$$\|h(z)\| \leq c\,\|z\|^{\eta}_{\mathfrak{B}^q}\,,$$
and
$$\|h(z) - h(w)\| \leq d_q\,\|z - w\|_{\mathfrak{B}^q}\,,$$
where $\eta, \alpha_k, \beta_0 \in [0, 1)$, $a_k, b_0, c_k, d_k, c, d_q$ are positive constants for $k = 0, 1, \cdots, N$.

The main goal of this work is to prove the exact controllability of system (1.1); that is to say, the ability to steer the system from an initial state to a final state in finite time. In other words, the system (1.1) is said to be exactly controllable on $[0, \tau]$ if for every $\varphi \in \mathfrak{B}$, and $z^1 \in \mathbb{R}^n$, there exists a control $u \in \mathcal{PW}_u$ such that the corresponding solution $z(\cdot)$ satisfies:
$$z(0) = \varphi(0) - h(z_{\pi_1}, \dots, z_{\pi_q})(0) \quad \text{and} \quad z(\tau) = z^1.$$
According to [1, 2], under the conditions assumed on the non linear terms involving in the system (1.1), the solution of this problem is given by
$$z(t) = \begin{cases} \mathcal{U}(t,0)[\phi(0) - h(z_{\pi_1}, \dots, z_{\pi_q})(0)] + \displaystyle\int_0^t \mathcal{U}(t,s)\mathcal{F}(s, z_s, u(t))ds & \\ + \displaystyle\int_0^t \mathcal{U}(t,s)B(s)u(s)ds, & t \in I_0, \\ \mathcal{U}(t,s_k)\mathcal{J}_k(t, z(t_k^-)) + \displaystyle\int_{s_k}^t \mathcal{U}(t,s)\mathcal{F}(s, z_s, u(t))ds & \\ + \displaystyle\int_{s_k}^t \mathcal{U}(t,s)\mathcal{B}(s)u(s)ds, & t \in I_k, \\ \mathcal{J}_k(t, z(t_k^-)), & t \in J_k, \\ \phi(t) - h(z_{\pi_1}, \dots, z_{\pi_q})(t), & t \in \mathbb{R}^-. \end{cases} \tag{2.1}$$
In order to continuous with the preliminaries results, we consider now the linear control system associated with (1.1)
$$z'(t) = \mathcal{A}(t)z(t) + \mathcal{B}(t)u(t), \tag{2.2}$$
which is assumed to be controllable in any interval $[\alpha, \beta] \subseteq [0, \tau]$.

Let's also define the evolution operator or the transition matrix corresponding to this finite dimensional linear system, which is given by $\mathcal{U}(t, s) = \Phi(t)\Phi^{-1}(s)$, where Φ is the fundamental matrix of the linear system $z'(t) = \mathcal{A}(t)z(t)$, and consider the following bound
$$M = \sup_{s,t\in[0,\tau]} \|U(t, s)\|\,.$$
Due to previous studies, it is already known that the controllability of the linear system (2.2) is equivalent to the subjectivity of the controllability operator given by:
$$\mathcal{G} : L^2([0, \tau]; \mathbb{R}^m) \to \mathbb{R}^n,$$

$$\mathcal{G}u = \int_0^\tau \mathcal{U}(\tau, s)\mathcal{B}(s)u(s)ds, \tag{2.3}$$

whose corresponding adjoint operator $\mathcal{G}^* : \mathbb{R}^n \to L^2([0,\tau];\mathbb{R}^m)$ is easily calculated

$$(\mathcal{G}^* z)(s) = \mathcal{B}^*(s)\mathcal{U}^*(\tau, s)z, \quad s \in [0,\tau]. \tag{2.4}$$

And a control $u \in L^2([0,\tau];\mathbb{R}^m)$ steering the system (2.2) from initial state z^0 to a final state z^1 on $[0,\tau]$ is given by the following formula:

$$u(t) = \mathcal{B}^*(t)\mathcal{U}^*(\tau, t)(\mathcal{W}_{[0,\tau]})^{-1}(z^1 - \mathcal{U}(\tau, 0)z^0), \quad t \in [0,\tau], \tag{2.5}$$

where $\mathcal{W}_{[0,\tau]} : \mathbb{R}^n \to \mathbb{R}^n$ is the Controllability Gramian Operator, matrix in this case, defined as

$$\mathcal{W}_{[0,\tau]}z = \mathcal{G}\mathcal{G}^* z = \int_0^\tau \mathcal{U}(\tau, s)\mathcal{B}(s)\mathcal{B}^*(s)\mathcal{U}^*(\tau, s)z \, ds. \tag{2.6}$$

In the same way, the linear system (2.2) is controllable on $[\alpha, \beta] \subseteq [0,\tau]$ if, and only if, the controllability operator given by

$$\mathcal{G}_{\alpha\beta}u = \int_\alpha^\beta \mathcal{U}(\beta, s)\mathcal{B}(s)u(s)ds, \tag{2.7}$$

is surjective. i.e., The Gramian Operator $\mathcal{W}\alpha\beta$ given by

$$\mathcal{G}_{\alpha\beta}\mathcal{G}^*_{\alpha\beta}z = \mathcal{W}_{[\alpha,\beta]}z = \int_\alpha^\beta \mathcal{U}(\beta, s)\mathcal{B}(s)\mathcal{B}^*(s)\mathcal{U}^*(\beta, s)zds, \tag{2.8}$$

is invertible. For the aforementioned matrix, there exist $\delta_\alpha > 0$ such that $\left\|\mathcal{W}^{-1}_{[\alpha,\beta]}\right\| < \frac{1}{\delta_\alpha}$, and a control u steering the linear system (2.2) from z^α to z^β on $[\alpha, \beta]$ is given by

$$u(t) = \mathcal{B}^*(t)\mathcal{U}^*(\beta, t)(\mathcal{W}_{[\alpha,\beta]})^{-1}(z^\beta - \mathcal{U}(\beta, \alpha)z^\alpha), \quad t \in [\alpha, \beta]. \tag{2.9}$$

Remark 2.4. When we study the exact controllability of finite-dimensional linear systems with controls in L^2-spaces, we must bear in mind that the system is controllable if, and only if, it is controllable with controls in any dense subspace of L^2. Therefore, the system (2.2) is controllable with controls on L^2 if, and only if, it is controllable with controls on $\mathcal{PW}_u$(see [14]). Given our main problem, and in order to study the exact controllability of it, we will use the following Lemma and theorem that will be clue pieces in the development of the proof of it.

Lemma 2.5. *For all function $z \in \mathcal{PW}_{g\tau}$ the following estimate holds for all $s \in [0,\tau]$:*

$$\|z_s\|_{\mathfrak{B}} \leq \|z\|_{\mathcal{PW}_{g\tau}}.$$

Theorem 2.6. *(Rothe's Fixed Theorem, see* [9]*) Let E be a Banach space. Let $B \subset E$ be a closed convex subset such that the zero of E is contained in the interior of B. Let $\Psi : B \to E$ be a continuous mapping with $\Psi(B)$ relatively compact in E and $\Psi(\partial B) \subset B$. Then there is a point $x^* \in B$ such that $\Psi(x^*) = x^*$.*

3. Main Result

This section is devoted to the main result of this paper, i.e., the proof of the exact controllability of system (1.1) with infinite delay, non instantaneous impulses and nonlocal. In doing so, we define the following operators to transform our problem into a fixed point problem:

$$\begin{aligned} \mathcal{R}_1 : \mathcal{PW}_{g\tau} \times \mathcal{PW}_u &\longrightarrow \mathcal{PW}_{g\tau} \\ (z,u)(t) &\longmapsto y(t) = \mathcal{R}_1(z,u)(t), \end{aligned}$$

$$\begin{aligned} \mathcal{R}_2 : \mathcal{PW}_{g\tau} \times \mathcal{PW}_u &\longrightarrow \mathcal{PW}_u \\ (z,u)(t) &\longmapsto v(t) := \mathcal{R}_2(z,u)(t), \end{aligned}$$

which, for arbitrary states $z^{t_{k+1}}, with \quad k = 0, 1, 2, \ldots, N$, are given by:

$$y(t) = \begin{cases} \phi(t) - h(z_{\pi_1}, \ldots, z_{\pi_q})(t), & t \in \mathbb{R}^- \\ \mathcal{U}(t,0)[\phi(0) - h(z_{\pi_1}, \ldots, z_{\pi_q})(0)] + \displaystyle\int_0^{t_1} \mathcal{U}(t,s)\mathcal{F}(s, z_s, u(t))ds & \\ \quad + \displaystyle\int_0^{t_1} \mathcal{U}(t,s)B(s)(\Upsilon_0 \mathfrak{L}_0(z,u))(s)ds, & t \in I_0 \\ \mathcal{U}(t,s_k)\mathcal{J}_k(t, z(t_k^-)) + \displaystyle\int_{s_k}^{t} \mathcal{U}(t,s)\mathcal{F}(s, z_s, u(t))ds & \\ \quad + \displaystyle\int_{s_k}^{t} \mathcal{U}(t,s)\mathcal{B}(s)(\Upsilon_k \mathfrak{L}_k(z,u))(s)ds, & t \in I_k \\ \mathcal{J}_k(t, z(t_k^-)), & t \in J_k \end{cases} \tag{3.1}$$

and

$$v(t) = \begin{cases} \Upsilon_k \mathfrak{L}_k(z,u) := \mathcal{B}^*(t)\mathcal{U}^*(t_{k+1},t)(\mathcal{W}_{[s_k,t_{k+1}]})^{-1}\mathfrak{L}_k(z,u)(t), & t \in (s_k, t_{k+1}], \\ 0, & t \in (t_k, s_k), \end{cases} \tag{3.2}$$

where,

$$\mathfrak{L}_k(z,u) = z^{t_{k+1}} - \mathcal{U}(t_{k+1}, s_k)\mathcal{J}_k(s_k, z(t_k^-)) - \int_{s_k}^{t_{k+1}} \mathcal{U}(t_{k+1}, s)\mathcal{F}(s, z_s, u(s))ds, \tag{3.3}$$

and

$$\mathcal{W}_{[s_k,t_{k+1}]}z = \int_{s_k}^{t_{k+1}} \mathcal{U}(t_{k+1}, s)\mathcal{B}(s)\mathcal{B}^*(s)\mathcal{U}^*(t_{k+1}, s)z ds, \tag{3.4}$$

with $\delta_k > 0$, such that for each k, we have $\left\|\mathcal{W}_{[s_k,t_{k+1}]}\right\| < \frac{1}{\delta_k}$.

Now, using the foregoing two operators, we define the operator $\mathcal{R}$ as follows:

$$\begin{aligned} \mathcal{R} : \mathcal{PW}_{g\tau} \times \mathcal{PW}_u &\longrightarrow \mathcal{PW}_{g\tau} \times \mathcal{PW}_u \\ (z(t), u(t)) &\longmapsto \mathcal{R}(z,u) = (\mathcal{R}_1(z,u)(t), \mathcal{R}_2(z,u)(t)). \end{aligned}$$

The following remark describes the properties of $\mathcal{R}$ and it can be trivially show from the definition of it.

Remark 3.1. The semi-linear system with non-instantaneous impulses, infinite delay, and nonlocal conditions (1.1) is controllable on $[0,\tau]$, if and only if, for all initial state $\varphi \in \mathcal{PW}$ and a final state z^1 the operator $\mathcal{R}$ has a fixed point. i.e., there exist (z,u) in the domain of $\mathcal{R}$ satisfying

$$\mathcal{R}(z,u) = (z,u).$$

Theorem 3.2. *Under the conditions a)-c), the system* (1.1) *is controllable. Therefore, it is enough to prove that the operator* $\mathcal{R}$ *defined above has a fixed point. Moreover, given* $\varphi \in \mathfrak{B}$, $z^1 \in \mathbb{R}^n$ *and arbitrary points* $z^{t_{k+1}} \in \mathbb{R}^n$, $\quad k = 0,1,2,\dots,N$ *there exists a control* $u \in \mathcal{PW}_u$ *such that the corresponding solution* $z(\cdot)$ *of* (1.1) *satisfies:*

$$z(0) - h(z_{\pi_1},\dots,z_{\pi_q})(0) = \varphi(0), \qquad z(t_{k+1}) = z^{t_{k+1}}, \quad k = 0,1,2,\dots,N$$

where

$$z(t_{N+1}) = z^{t_{N+1}} = z^1.$$

In adittion, for all $t \in (s_k, t_{k+1}]$ *and* $k = 0,1,2,\dots,N$

$$u(t) = \mathcal{B}^*(t)\mathcal{U}^*(t_{k+1},t)(\mathcal{W}_{(s_k,t_{k+1}]})^{-1}\mathfrak{L}_k(z,u),$$

with $\mathfrak{L}_k(z,u)$ *as in* (3.3).

The prove of this theorem will be given by steps.

Step 1 The operator $\mathcal{R}$ is continuous.

In fact, consider hypotheses b) and c), lemma 2.5 and the following cases:

i) $t \in (0,t_1]$

$$\|\mathcal{R}_1(z,u)(t) - \mathcal{R}_1(w,v)(t)\| \leq \hat{C}_0 \|z-w\| + \hat{D}_0 \sup_{s\in(0,t_1]} \|\mathcal{F}(s,z_s,u(s)) - \mathcal{F}(s,w_s,v(s))\|$$

ii) $t \in (t_k, s_k]$

$$\|\mathcal{R}_1(z,u)(t) - \mathcal{R}_1(w,v)(t)\| \leq d_0 \|z-w\|$$

iii) $t \in (s_k, t_{k+1}]$

$$\|\mathcal{R}_1(z,u)(t) - \mathcal{R}_1(w,v)(t)\| \leq \hat{C}_k \|z-w\| + \hat{D}_k \sup_{s\in(s_k,t_{k+1}]} \|\mathcal{F}(s,z_s,u(s)) - \mathcal{F}(s,w_s,v(s))\|$$

iv) $t \in (-\infty,0]$

$$\|\mathcal{R}_1(z,u)(t) - \mathcal{R}_1(w,v)(t)\| \leq d_q q \|w-z\|$$

where,

$$\hat{C}_k = C_k[1+\hat{K}_k], \quad \hat{D}_k = D_k[1+\hat{D}], \quad \hat{K}_k = \frac{(t_{k+1}-s_k)\|B\|^2 M^2}{\delta_k}$$

$$C_0 = Md_q q \qquad ,C_k = Md_k, \qquad D_k = M(t_{k+1}-s_k)$$

Then, because of the continuity of $\mathcal{F}$, $\mathcal{J}_k$, h, we get that $\mathcal{R}_1$ is continuous. Aditionally, $\mathcal{R}_2$ is continuous since $\mathcal{B}$, $\mathcal{U}$, $\mathfrak{L}_k$, and $\mathcal{W}_{[s_k,t_{k+1}]}$ are also continuous. Using the foregoing estimates we obtain, as consequence, that the operator $\mathcal{R}$ is continuous. Note that in the interval $(-\infty, 0]$ we get right bound of $\mathcal{R}_1$ from the hypothesis (c), and the operator $\mathcal{R}_2$ is zero there.

Step 2 The operator $\mathcal{R}$ maps bounded sets into equicontinuous sets. In fact, we notice that

$$\begin{aligned}\|\mathcal{R}(z,u)(t_2)-\mathcal{R}(z,u)(t_1)\| &= \|\mathcal{R}_1(z,u)(t_2)-\mathcal{R}_1(z,u)(t_1)\|\\ &+ \|\mathcal{R}_2(z,u)(t_2)-\mathcal{R}_2(z,u)(t_1)\|\end{aligned}$$

Now, let $D \subset \mathcal{PW}_{g\tau}$ be a bounded set, and recall that $\mathcal{R}(D) = \mathcal{R}_1(D)\times\mathcal{R}_2(D)$. Then, for $\mathcal{R}_1$, we consider hypothesis b) and the following cases:

i) $l_1, l_2 \in (0, t_1]$ such that $0 < l_1 < l_2 \le t_1$

$$\begin{aligned}\|\mathcal{R}_1(z,u)(l_2)-\mathcal{R}_1(z,u)(l_1)\| &\le \Big\|\mathcal{U}(l_2,0)\{\varphi(0)-h(z_{\pi_1},\ldots,z_{\pi_q})(0)\}\\ &+ \int_0^{l_2}\mathcal{U}(l_2,s)\mathcal{B}(s)(\Upsilon\mathfrak{L}(z,u))(s)ds\\ &+ \int_0^{l_2}\mathcal{U}(l_2,s)\mathcal{F}(s,z_s,u(s))ds\\ &- \mathcal{U}(l_1,0)\{\varphi(0)-h(z_{\pi_1},\ldots,z_{\pi_q})(0)\}\\ &+ \int_0^{l_1}\mathcal{U}(l_1,s)\mathcal{B}(s)(\Upsilon\mathfrak{L}(z,u))(s)ds\\ &+ \int_0^{l_1}\mathcal{U}(l_1,s)\mathcal{F}(s,z_s,u(s))ds\Big\|\\ &\le \|\mathcal{U}(l_2,0)-\mathcal{U}(l_1,0)\|\,\big\|\varphi(0)-h(z_{\pi_1},\ldots,z_{\pi_q})(0)\big\|\\ &+ \int_0^{l_1}\|\mathcal{U}(l_2,s)-\mathcal{U}(l_1,s)\|\,\|\mathcal{B}(s)(\Upsilon_0\mathfrak{L}_0(z,u))(s)\|\,ds\\ &+ \int_{l_1}^{l_2}\|\mathcal{U}(l_2,s)\|\,\|\mathcal{B}(s)(\Upsilon_0\mathfrak{L}_0(z,u))(s)\|\,ds\\ &+ \int_0^{l_1}\|\mathcal{U}(l_2,s)-\mathcal{U}(l_1,s)\|\,\|\mathcal{F}(s,z_s,u(s))\|\,ds\\ &+ \int_{l_1}^{l_2}\|\mathcal{U}(l_2,s)\|\,\|\mathcal{F}(s,z_s,u(s))\|\,ds.\end{aligned}$$

ii) $l_1, l_2 \in (t_k, s_k]$ such that $t_k < l_1 < l_2 \le s_k$

$$\begin{aligned}\|\mathcal{R}_1(z,u)(l_2)-\mathcal{R}_1(z,u)(l_1)\| &= \big\|\mathcal{J}_k(l_2,z(t_k^-))-\mathcal{J}_k(l_1,z(t_k^-)\big\|\\ &\le d_0|l_2-l_1|.\end{aligned}$$

iii) $l_1, l_2 \in (s_k, t_{k+1}]$ such that $s_k < l_1 < l_2 \le t_{k+1}$.

$$\begin{aligned}
\|\mathcal{R}_1(z,u)(l_2)-\mathcal{R}_1(z,u)(l_1)\| &\leq \Big\|\mathcal{U}(l_2,s_k)\mathcal{J}_k(s_k,z(t_k^-))\\
&+ \int_{s_k}^{l_2}\mathcal{U}(l_2,s)\mathcal{B}(s)(\Upsilon_k\mathfrak{L}_k(z,u))(s)ds\\
&+ \int_{s_k}^{l_2}\mathcal{U}(l_2,s)\mathcal{F}(s,z_s,u(s))ds-\mathcal{U}(l_1,s_k)\mathcal{J}_k(s_k,z(t_k^-))\\
&+ \int_{s_k}^{l_1}\mathcal{U}(l_1,s)\mathcal{B}(s)(\Upsilon_k\mathfrak{L}_k(z,u))(s)ds\\
&+ \int_{s_k}^{l_1}\mathcal{U}(l_1,s)\mathcal{F}(s,z_s,u(s))ds\Bigg\|\\
&\leq \|\mathcal{U}(l_2,s_k)-\mathcal{U}(l_1,s_k)\|\left\|\mathcal{J}_k(s_k,z(t_k^-))\right\|\\
&+ \int_{s_k}^{l_1}\|\mathcal{U}(l_2,s)-\mathcal{U}(l_1,s)\|\,\|\mathcal{B}(s)(\Upsilon_k\mathfrak{L}_k(z,u))(s)\|\,ds\\
&+ \int_{l_1}^{l_2}\|\mathcal{U}(l_2,s)\|\,\|\mathcal{B}(s)(\Upsilon_k\mathfrak{L}_k(z,u))(s)\|\,ds\\
&+ \int_{s_k}^{l_1}\|\mathcal{U}(l_2,s)-\mathcal{U}(l_1,s)\|\,\|\mathcal{F}(s,z_s,u(s))\|\,ds\\
&+ \int_{l_1}^{l_2}\|\mathcal{U}(l_2,s)\|\,\|\mathcal{F}(s,z_s,u(s))\|\,ds,
\end{aligned}$$

and

$$\begin{aligned}
\|\mathcal{R}_2(z,u)(l_2)-\mathcal{R}_2(z,u)(l_1)\| \leq \|&\mathcal{U}(t_{k+1},l_2)\mathcal{B}(l_2)\\
&-\mathcal{U}(t_{k+1},l_1)\mathcal{B}(l_1)\|\left\|(\mathcal{W}_{[s_k,t_{k+1}]})^{-1}\mathfrak{L}_k(z,u)\right\|.
\end{aligned}$$

iv) $l_1,l_2\in(-\infty,0]$ such that $-\infty\leq l_1\leq l_2\leq 0$, then we get

$$\begin{aligned}
\|\mathcal{R}_1(z,u)(l_2)-\mathcal{R}_1(z,u)(l_1)\| &= \big\|\varphi(l_2)-h(z_{\pi_1},\dots,z_{\pi_q})(l_2)\\
&\quad -\varphi(l_1)+h(z_{\pi_1},\dots,z_{\pi_q})(l_1)\big\|\\
&\leq \|\varphi(l_2)-\varphi(l_1)\|\,\big\|h(z_{\pi_1},\dots,z_{\pi_q})(l_2)\\
&\quad -h(z_{\pi_1},\dots,z_{\pi_q})(l_1)\big\|.
\end{aligned}$$

By the continuity of the evolution operator $\mathcal{U}$ and $\mathcal{W}_{[s_k,t_{k+1}]}$, the boundedness of h on D, with l_2 and l_1 close enough, and i), ii), iii), iv), the equicontinuity of the sets $\mathcal{R}_1(D)$ and $\mathcal{R}_2(D)$ is obtained, which at the same time implies the equicontinuity of $\mathcal{R}(D)$.

Step 3 For any bounded subset $D\subset\mathcal{PW}_{g\tau}\times\mathcal{PW}_u$, $\mathcal{R}(D)$ is relatively compact.
In fact, let D be a bounded subset of $\mathcal{PW}_{g\tau}\times\mathcal{PW}_u$. By the continuity of $\mathcal{F}$, $\mathfrak{L}$,

and $\mathcal{J}_k$, it follows that

$$\|\mathcal{F}(\cdot,z,u)\|_0 \leq \sup_{s\in(0,\tau]} \|\mathcal{F}(s,z_s,u(s))\|, \quad \|\mathcal{W}^{-1}_{[s_k,t_{k+1}]}\mathfrak{L}_k\| \leq T_k,$$
$$\|\mathcal{J}_k\| \leq T_{k+N+1}, \quad k=1,2,\dots,N, \quad \forall (z,u)\in D,$$

where $\|\mathcal{J}_k\| = \sup_{t\in(0,\tau]}\{\|\mathcal{J}_k(t,z(t_k^-))\|_{\mathbb{R}^n}, T_1,\cdots,T_{2N+1}\in\mathbb{R}$. Therefore, $\mathcal{S}(D)$ is uniformly bounded.
Now, we consider a sequence $\{\psi_i = (y_i,v_i) : i=1,2,\dots,\}$ in $\mathcal{S}(D)$. Since $\{v_i : i=1,2,\dots\}$ is contained in $\mathcal{S}_2(D)\subset \mathcal{PW}_u$ and $\mathcal{S}_2(D)$ is an uniformly bounded and equicontinuous family, by Arzelà-Ascoli Theorem, we can assume, without loss of generality, that $\{v_i : i=1,2,\dots,\}$ converges.
On the other hand, since $\{y_i : i=1,2,\dots,\}$ is contained in
$\mathcal{S}_1(D)\subset\mathcal{PW}_{g\tau}((-\infty,\tau];\mathbb{R}^n)$, then $y_i\Big|_{(-\infty,-\tau_q]} = \phi - h(\phi_{\Pi_1},\phi_{\Pi_2},\dots,\phi_{\Pi_q})$, $i = 1,2,\dots,$.
Taking into account that $\{y_i : i=1,2,\dots,\}$ is bounded and equicontinuous in $[0,t_1]$, we can apply Arzelà-Ascoli Theorem to ensure the existence of a subsequence $\{y_i^1 : i=1,2,\dots,\}$ of $\{y_i : i=1,2,\dots,\}$, which is uniformly convergent on $[0,t_1]$. Now, consider the sequence $\{\phi_i^1 : i=1,2,\dots,\}$ on the interval $[t_1,t_2]$. On this interval the sequence $\{y_k^1 : i=1,2,\dots,\}$ is uniformly bounded and equicontinuous, and for the same reason, it has a subsequence $\{y_i^2 : i=1,2,\dots,\}$ uniformly convergent on $[0,t_2]$. In this way, for the intervals $[t_2,t_3]$, $[t_3,t_4]$, $\dots$, $[t_N,\tau]$, we see that the sequence $\{\phi_i^{N+1} : i=1,2,\dots,\}$ converges uniformly on the interval $[0,\tau]$.
Besides, in the interval $[-\Pi_q,0]$ the function y_i is piecewise continuous, then repeating the foregoing process we can assume that the subsequence $\{\psi_i^{N+1} = (y_i^{N+1},v_i^{N+1}) : i=1,2,\dots,\}$ converges uniformly on $(-\infty,\tau]$. This means that $\overline{\mathcal{S}(D)}$ is compact, i.e., $\mathcal{S}(D)$ is relatively compact.

Step 4.

The following limit holds

$$\lim_{\|(z,u)\|\to\infty} \frac{\|\mathcal{R}(z,u)\|}{\|(z,u)\|} = 0,$$

where $\|\cdot\|$ is the norm in the space $\mathcal{PW}_{g\tau}\times\mathcal{PW}_u$. In fact, first we have to make the following estimates:

a) For $t\in(0,t_1]$, we have that

$$\|\mathfrak{L}_0(z,u)\| \leq \left\|z^{t_1}\right\| + M\|\varphi\| + Mcq\|z\|^{\eta} + Mt_1[a_0q\|z\|^{\alpha_0} + b_0\|u\|^{\beta_0}_{\mathbb{R}^m} + c_0],$$

which implies that $\mathcal{R}_2$ and $\mathcal{R}_1$ are bounded by:

$$\|\mathcal{R}_2(z,u)(t)\| \leq \frac{M\|\mathcal{B}\|}{\delta_0}\left\|z^{t_1}\right\| + \frac{\|\mathcal{B}\|M^2}{\delta_0}\|\varphi\| + \frac{\|\mathcal{B}\|M}{\delta_0}cq\|z\|^{\eta}$$
$$+ \frac{\|\mathcal{B}\|M^2t_1}{\delta_0}[a_0q\|z\|^{\alpha_0} + b_0\|u\|^{\beta_0}_{\mathbb{R}^m} + c_0]$$

and

$$\|\mathcal{R}_1(z,u)(t)\| \leq M\|\varphi\| + Mcq\|z\|^{\eta} + Mt_1[a_0 q\|z\|^{\alpha_0} + b_0\|u\|_{\mathbb{R}^m}^{\beta_0} + c_0] + M\|\mathcal{B}\| t_1 \times \left\{ \frac{M\|\mathcal{B}\|}{\delta_0}\left\|z^{t_1}\right\| + \frac{\|\mathcal{B}\| M^2}{\delta_0}\|\varphi\| + \frac{\|\mathcal{B}\| M}{\delta_0} cq\|z\|^{\eta} + \frac{\|\mathcal{B}\| M^2 t_1}{\delta_0}[a_0 q\|z\|^{\alpha_0} + b_0\|u\|_{\mathbb{R}^m}^{\beta_0} + c_0] \right\}$$

From the above inequality, we get an estimate for $\mathcal{R}$

$$\|\mathcal{R}(z,u)\| = \|\mathcal{R}_1(z,u)\| + \|\mathcal{R}_2(z,u)\| \leq E_0\|\varphi\| + H_0\left\|z^{t_1}\right\| + D_0\|z\|^{\eta} + F_0[a_0 q\|z\|^{\alpha_0} + b_0\|u\|_{\mathbb{R}^m}^{\beta_0} + c_0].$$

Hence

$$\frac{\|\mathcal{R}(z,u)(t)\|}{\|(z,u)\|} \leq \frac{E_0\|\varphi\| + H_0\|z^{t_1}\|}{\|z\| + \|u\|} + D_0\|z\|^{\eta-1} + F_0\left[a_0 q\|z\|^{\alpha_0-1} + b_0\|u\|_{\mathbb{R}^m}^{\beta_0-1} + \frac{c_0}{\|z\| + \|u\|}\right]. \tag{3.5}$$

b) For $t \in (s_k, t_{k+1}]$, we get the following bound

$$\|\mathfrak{L}_k(z,u)\| \leq \left\|z^{t_{k+1}}\right\| + M[a_k\|z\|^{\alpha_k} + c_k] + M(t_{k+1} - s_k)[a_0 q\|z\|^{\alpha_0} + b_0\|u\|_{\mathbb{R}^m}^{\beta_0} + c_0].$$

Consequently, the operator $\mathcal{R}_2$ satisfies

$$\|\mathcal{R}_2(z,u)(t)\| \leq \frac{M\|\mathcal{B}\|}{\delta_k}\left\|z^{t_{k+1}}\right\| + \frac{\|\mathcal{B}\| M^2}{\delta_k}[a_k\|z\|^{\alpha_k} + c_k] + \frac{\|\mathcal{B}\| M^2(t_{k+1} - s_k)}{\delta_k}[a_0 q\|z\|^{\alpha_0} + b_0\|u\|_{\mathbb{R}^m}^{\beta_0} + c_0]$$

and, for operator for the operator $\mathcal{R}_1$, we obtain

$$\|\mathcal{R}_1(z,u)(t)\| \leq M[a_k\|z\|^{\alpha_i} + c_i] + Mt_{k+1}\|\mathcal{B}\|\left[\tfrac{M\|\mathcal{B}\|}{\delta_k}\|z^{t_{k+1}}\| + \tfrac{\|\mathcal{B}\| M^2}{\delta_k}[a_k\|z\|^{\alpha_k} + c_k] + \tfrac{\|\mathcal{B}\| M^2(t_{k+1}-s_k)}{\delta_k}[a_0 q\|z\|^{\alpha_0} + b_0\|u\|_{\mathbb{R}^m}^{\beta_0} + c_0]\right] + M(t_{k+1} - s_k)[a_k\|z\|^{\alpha_k} + c_k].$$

Thus, dividing by the norm of the space, we can get the following estimate

$$\frac{\|\mathcal{R}(z,u)(t)\|}{\|(z,u)\|} \leq E_k\left[a_k\|z\|^{\alpha_k-1} + \frac{c_k}{\|z\| + \|u\|}\right] + \frac{H_k\|z^{t_{k+1}}\|}{\|z\| + \|u\|} + F_k\left[a_0 q\|z\|^{\alpha_0-1} + b_0\|u\|_{\mathbb{R}^m}^{\beta_0-1} + \frac{c_0}{\|z\| + \|u\|}\right]. \tag{3.6}$$

c) For $t \in (t_k, s_k]$, we have that

$$\|\mathcal{R}_1(z,u)(t)\| \leq a_k\|z\|^{\alpha_k} + c_k,$$

consequently,

$$\frac{\|\mathcal{R}(z,u)(t)\|}{\|(z,u)\|} \leq a_k \|z\|^{\alpha_k - 1} + \frac{c_i}{\|z\| + \|u\|}, \tag{3.7}$$

where,

$$D_0 = \left[Mcq + \frac{\|\mathcal{B}\|^2 M^2 t_1 cq}{\delta_0} + \frac{\|\mathcal{B}\| Mcq}{\delta_0} \right]$$

,

$$E_k = \left[M + \frac{(t_{k+1} - s_k) \|B\|^2 M^3}{\delta_k} + \frac{\|\mathcal{B}\| M^2}{\delta_k} \right],$$

$$H_k = \left[\frac{(t_{k+1} - s_k) \|B\|^2 M^2}{\delta_k} + \frac{\|\mathcal{B}\| M}{\delta_k} \right],$$

and

$$F_k = \left[\frac{(t_{k+1} - s_k)^2 \|\mathcal{B}\|^2 M^3}{\delta_k} + M(t_{k+1} - s_k) + \frac{(t_{k+1} - s_k) \|\mathcal{B}\| M^2}{\delta_k} \right].$$

Hence, considering the hypotheses (a)-(c), Lemma 2.5 and $0 < \alpha_k < 1$, $0 < \beta_0 < 1$, $k = 0, 1, \ldots, N$, $0 < \eta < 1$, it follows from (3.5), (3.6) and (3.7) that

$$\lim_{\|(z,u)\| \to \infty} \frac{\|\mathcal{R}(z,u)\|}{\|(z,u)\|} = 0.$$

Now, we are ready to prove that operator $\mathcal{R}$ has fixed point. In fact, for a fixed $0 < \rho < 1$, there exists $r > 0$ big enough, such that

$$\|\mathcal{R}(z,u)\| \leq \rho \|(z,u)\|, \quad \text{for all} \quad \|(z,u)\| \geq r.$$

In particularly, if $\|(z,u)\| = r$, then $\|\mathcal{R}(z,u)\| \leq \rho r < r$. Consequently,

$$\mathcal{R}(\partial B(0,r)) \subset B(0,r).$$

So, applying Rothe's Fixed Point Theorem 2.6, we conclude that the operator $\mathcal{R}$ has a fixed point $(z,u) \in \mathcal{PW}_{g\tau} \times \mathcal{PW}_u$. i.e., $\mathcal{R}(z,u) = (z,u)$, which prove the controllability of system (1.1).

Moreover, from the definition of the operator $\mathcal{R}$ and the proof of the above theorem, we have obtained that, for $\varphi \in \mathfrak{B}$, $z^1 \in \mathbb{R}^n$ and arbitrary points $z^{t_{k+1}} \in \mathbb{R}^n$, $\quad k = 0, 1, 2, \ldots, N$, there exists a control $u \in \mathcal{PW}_u$ such that

$$u(t) = \mathcal{B}^*(t)\mathcal{U}^*(t_{k+1}, t)(\mathcal{W}_{(s_k, t_{k+1}]})^{-1}\mathfrak{L}_k(z,u)$$

for $t \in (s_k, t_{k+1}]$, $k = 1, 2, \cdots, N$. Replacing u into the solution (2.1), and evaluating it at $t = 0, t_1, \cdots t_{k+1}$, we obtain that:

$$z(0) + h(z_{\pi_1}, \ldots, z_{\pi_q})(0) = \varphi(0),$$

$$
\begin{aligned}
z(t_1) &= \mathcal{U}(t_1,0)[\varphi(0)-h(z_{\pi_1},\ldots,z_{\pi_q})(0)]+\int_0^{t_1}\mathcal{U}(t_1,s)\mathcal{F}(s,z_s,u(s))ds\\
&\quad+\int_0^{t_1}\mathcal{U}(t_1,s)\mathcal{B}(s)\mathcal{B}^*(s)\mathcal{U}^*(t_1,s)(\mathcal{W}_{(0,t_1]})^{-1}\{z_{t_1}-\mathcal{U}(t_1,0)[\varphi(0)\\
&\quad-h(z_{\pi_1},\ldots,z_{\pi_q})(0)]-\int_0^{t_1}\mathcal{U}(t_1,v)\mathcal{F}(v,z_v,u(v))dv]\}ds\\
&= \mathcal{U}(t_1,0)[\varphi(0)-h(z_{\pi_1},\ldots,z_{\pi_q})(0)]+\int_0^{t_1}\mathcal{U}(t_1,s)\mathcal{F}(s,z_s,u(s))ds\\
&\quad+(\mathcal{W}_{(0,t_1]})(\mathcal{W}_{(0,t_1]})^{-1}\{z_{t_1}-\mathcal{U}(t_1,0)[\varphi(0)-h(z_{\pi_1},\ldots,z_{\pi_q})(0)]\\
&\quad-\int_0^{t_1}\mathcal{U}(t_1,v)\mathcal{F}(v,z_v,u(v))dv\}:=z^{t_1}
\end{aligned}
$$

$$
\begin{aligned}
z(t_{k+1}) &= \mathcal{U}(t_{k+1},s_k)\mathcal{J}_k(s_k,z(t_k^-))+\int_{s_k}^{t_{k+1}}\mathcal{U}(t_{k+1},s)\mathcal{F}(s,z_s,u(s))ds\\
&\quad+\int_{s_k}^{t_{k+1}}\mathcal{U}(t_{k+1},s)\mathcal{B}(s)\mathcal{B}^*(s)\mathcal{U}^*(t_{k+1},s)(\mathcal{W}_{(s_k,t_{k+1}]})^{-1}\{z_{t_{k+1}}\\
&\quad-\mathcal{U}(t_{k+1},s_i)\mathcal{J}_k(s_k,z(t_k^-))-\int_{s_k}^{t_{k+1}}\mathcal{U}(t_1,v)\mathcal{F}(v,z_v,u(v))dv\}ds\\
&= \mathcal{U}(t_{k+1},s_k)\mathcal{J}_k(s_k,z(t_k^-))+\int_{s_k}^{t_{k+1}}\mathcal{U}(t_{k+1},s)\mathcal{F}(s,z_s,u(s))ds\\
&\quad+(\mathcal{W}_{(s_k,t_{k+1}]})(\mathcal{W}_{(s_k,t_{k+1}]})^{-1}\{z^{t_{k+1}}-\mathcal{U}(t_{k+1},s_k)\mathcal{J}_k(s_k,z(t_k^-))\\
&\quad-\int_{s_k}^{t_{k+1}}\mathcal{U}(t_{k+1},v)\mathcal{F}(v,z_v,u(v))dv\}:=z^{t_{k+1}}.
\end{aligned}
$$

Observe that, if $k=N$, then $z(t_{N+1})=z^{t_{N+1}}=z^1$, and since $t_{N+1}=\tau$, we get that $z(\tau)=z^1$. This complete the proof.

4. Final Remark

In this work we have proved the exact controllability of a control system governed by a retarded differential equation with infinite delay, non-local conditions and non-instantaneous impulses, which confirmers that impulses, delays and non-local conditions are intrinsic phenomena that under certain conditions do not destroy the controllability of a system if some conditions are assumed. That is, if we consider these elements as disturbances of the system, it turns out that the controllability is robust under these influences not taken into account in many mathematical models that represent extremely important problems in real life. To achieve this goal, we must choose a natural space that includes these new elements and satisfies the axiomatic theory proposed by Hale and Kato to study the behavior of differential equations with unbounded delay. Then, the controllability problem of the system is transformed into a fixed point problem, for which we apply Rothe's Fixed Point Theorem. Also, the ideas presented here, can be used to study the controllability of infinite dimensional systems in Hilbert spaces where the dynamical is given by the infinitesimal generator A of a compact semigroup $\{T(t)\}_{t\geq 0}$, in this case we only get approximate controllability of the system.

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