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NONLINEAR SYSTEM OF BOUNDARY VALUE PROBLEM INVOLVING ψ-CAPUTO FRACTIONAL DERIVATIVE

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Abstract. In this paper we investigate existence of minimal and maximal solutions to a coupled system of nonlinear boundary value problems involving ψ-Caputo fractional derivative. By establishing a comparison result and monotone iterative technique coupled with method of lower-upper solutions, we establish the existence of solutions for nonlinear system of boundary value problems.

1. Introduction

In the last few decades, theory of fractional differential equations occur frequently in different research areas and engineering, such as physics, chemistry, aerodynamics, biology, fields of control, electromagnetic etc. Several approaches to fractional derivative exist, for example, Riemann-Liouville, Caputo, Hadamard, Caputo- Hadamard (see in [6, 9, 10, 11, 14] and references therein). There are some good methods for studying fractional differential equations such as monotone method, power series method, compositional method and transform method [6, 14] and references therein. Basic theory of fractional differential equations with Riemann-Liouville fractional derivative is well developed in [7, 8, 9]. Some basic theory for fractional differential equations involving the ψ-Caputo fractional differential operator has been discussed by many authors [1, 2, 3, 4, 12]. Juan introduced the generalized fractional Hilfer integral and derivative and studied its fundamental properties [13]

The existence of solutions for boundary value problems of fractional differential equations were studied by many researchers due to the fact that boundary value problems have significant applications in applied sciences. Another important class of fractional differential equations [FDEs] is in the field of system of fractional differential equations with coupled boundary conditions. In [15] Wang et al. discussed existence results for nonlinear system of Riemann-Liouville fractional differential equations. The study of system involving ψ-Caputo fractional differential equations is also important as such system occurs in various problems of applied nature, see [5]. Recently this field much more attracted the attention of researchers towards itself. Many techniques are available in literature among them the monotone iterative techniques coupled with the method of upper and

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lower solutions is interesting and powerful tool applied to constructive existence results for nonlinear systems.

Motivated by aforementioned work, we consider the existence results for following nonlinear system of boundary value problem involving ψ-Caputo fractional derivative by establishing a comparison result and applications of monotone iterative technique coupled with method of lower-upper solutions.

$$\begin{aligned} {}^{C}D^{\mu,\psi}_{c^+}w(r) &= F(r,w(r),z(r)),\\ {}^{C}D^{\mu,\psi}_{c^+}z(r) &= G(r,z(r),w(r)), \quad r\in \mathscr{J}=[c,d],\\ w(c)=c_1^*,\ z(c)=c_2^*, &\qquad w(d)=d_1^*,\ z(d)=d_2^*, \end{aligned} \tag{1.1}$$

where ${}^{C}D^{\mu,\psi}_{c^+}$ is the $\psi-$Caputo fractional derivative of order $0<\mu\leq 1$, $F,G\in C(\mathscr{J}\times\mathbb{R}\times\mathbb{R},\mathbb{R})$, $c_i^*,d_i^*\in\mathbb{R}, i=1,2$ and $c_1^*\leq c_2^*, d_1^*\leq d_2^*$. To the best of our knowledge, this technique has not been applied yet to the nonlinear system of boundary value problems involving ψ-Caputo fractional derivative. Further we provide an example to illustrate our results.

We organize the rest of the paper as follows. In Section 2, the existence and uniqueness of solutions for a linear problem for system of fractional differential equations is discussed and a differential inequality as a comparison principle is established. In Section 3, we prove the existence of minimal and maximal solutions of nonlinear system (1.1) by use of the monotone iterative technique and the method of lower and upper solutions.

2. Preliminaries

In this section, we deduce some preliminary results required in the next section to attain existence and uniqueness results for nonlinear boundary value problems (1.1) involving ψ-Caputo fractional derivative. Let $\mathscr{J}=[c,d]$, where $0\leq c<d<\infty$, be a finite interval and $\psi:\mathscr{J}\to\mathbb{R}$ is an increasing differentiable function such that $\psi'(r)\neq 0$,for all $r\in\mathscr{J}$.

Definition 2.1. [3] The left-sided ψ-Riemann-Liouville fractional integral of order $\mu>0$ for an integrable function $y:\mathscr{J}\to\mathbb{R}$ with respect to function ψ is defined as follows

$$I^{\mu,\psi}_{c^+}y(r)=\frac{1}{\Gamma(\mu)}\int_c^r \psi'(s)[\psi(r)-\psi(s)]^{\mu-1}y(s)\,ds,$$

where $\Gamma(.)$ is the gamma function.

Definition 2.2. [3] Let $n\in N$ and let $\psi,y\in C^n(\mathscr{J},\mathbb{R})$ be two functions. The left-sided ψ-Riemann-Liouville fractional derivative of function y of order $n-1<\mu<n$ with respect to another function ψ is defined by

$$\begin{aligned} D^{\mu,\psi}_{c^+}y(r) &= \left(\frac{1}{\psi'(r)}\frac{d}{dr}\right)^n I^{n-\mu,\psi}_{c^+}y(r)\\ &= \frac{1}{\Gamma(n-\mu)}\left(\frac{1}{\psi'(r)}\frac{d}{dr}\right)^n\int_c^r \psi'(s)[\psi(r)-\psi(s)]^{n-\mu-1}y(s)\,ds, \end{aligned}$$

where $n=[\mu]+1$ and $[\mu]$ denotes the integer part of the real number μ.

Definition 2.3. [3] Let $n \in N$ and let $\psi, y \in C^n(\mathscr{J}, \mathbb{R})$ be two functions. The left-sided ψ-Caputo fractional derivative of y of order $n-1 < \mu < n$ with respect to another function ψ is defined by

$$^{C}D_{c^+}^{\mu,\psi} y(r) = I_{c^+}^{n-\mu,\psi}\left(\frac{1}{\psi'(r)}\frac{d}{dr}\right)^n y(r),$$

where $n = [\mu]+1$ for $\mu \notin \mathbb{N}$, $n = \mu$ for $\mu \in \mathbb{N}$.
To simplify notation, we will use the abbreviated symbol

$$y_\psi^{[n]}(r) = \left(\frac{1}{\psi'(r)}\frac{d}{dr}\right)^n y(r).$$

From the definition, it is clear that

$$^{C}D_{c^+}^{\mu,\psi} y(r) = \begin{cases} \frac{1}{\Gamma(n-\mu)}\int_c^r \psi'(s)[\psi(r)-\psi(s)]^{n-\mu-1} y_\psi^{[n]}(s)\,ds, & \text{if}\, \mu \notin \mathbb{N} \\ y_\psi^{[n]}(r), & \text{if}\, \mu \in \mathbb{N}. \end{cases} \tag{2.1}$$

Note that if $y \in C^n(\mathscr{J}, \mathbb{R})$ the ψ-Caputo fractional derivative of $y(r)$ of order μ is defined in terms of left-sided ψ-Riemann-Liouville fractional derivative as

$$^{C}D_{c^+}^{\mu,\psi} y(r) = D_{c^+}^{\mu,\psi}\left[y(r) - \sum_{k=0}^{n-1}\frac{y_\psi^{[k]}(c)}{k!}[\psi(r)-\psi(c)]^k\right].$$

Lemma 2.4. [12] *The linear boundary value problem for ψ-Caputo fractional differential equation*

$$\begin{aligned} &^{C}D_{c^+}^{\mu,\psi} w(r) + Pw(r) = F(r), \quad r \in \mathscr{J}, \\ &w(c) = c^*, \quad w(d) = d^*, \end{aligned} \tag{2.2}$$

has unique solution

$$\begin{aligned} w(r) = {} & c^* E_{\mu,1}(-P(\psi(r)-\psi(c))^\mu) + \theta E_{\mu,1}(-P(\psi(r)-\psi(c))^\mu) \\ & + \int_0^r \psi'(s)[\psi(r)-\psi(s)]^{\mu-1} E_{\mu,\mu}(-P(\psi(r)-\psi(c))^\mu) F(s)\,ds, \end{aligned} \tag{2.3}$$

$$\text{where} \quad \theta = d^* + \frac{[F(c)-Pw(c)][\psi(d)-\psi(c)]^\mu}{\Gamma(\mu+1)} - \frac{1}{\Gamma(\mu)}\int_c^d \psi'(s)[\psi(d)-\psi(s)]^{\mu-1}[F(s)-Pw(s)]\,ds,$$

and $E_{\mu,\nu}(.)$ is the two-parameter Mittag-Leffler function [6].

Lemma 2.5. *The linear boundary value problem for ψ-Caputo fractional differential equation*

$$\begin{aligned} &^{C}D_{c^+}^{\mu,\psi} w(r) = F_1(r) - Pw(r) - Qz(r),\ r \in \mathscr{J}, \\ &^{C}D_{c^+}^{\mu,\psi} z(r) = F_2(r) - Pz(r) - Qw(r),\ r \in \mathscr{J}, \\ &w(c) = c_1^*,\ z(c) = c_2^*, \quad w(d) = d_1^*,\ z(d) = d_2^*, \end{aligned} \tag{2.4}$$

has unique system of solutions in $C(\mathscr{J}, \mathbb{R})$.

Proof. The proof follows from the fact that the pair (w, z) is a solution of problem (2.4) if and only if $x(r) = w(r) + z(r)$, $y(r) = w(r) - z(r)$ for all $r \in \mathscr{J}$, where x, y solve the problems

$$\begin{aligned}
{}^{C}D^{\mu,\psi}_{c^+}x(r) &= {}^{C}D^{\mu,\psi}_{c^+}w(r) + {}^{C}D^{\mu,\psi}_{c^+}z(r),\\
&= (F_1 + F_2)(r) - (P + Q)\left[w(r) + z(r)\right],\\
&= (F_1 + F_2)(r) - (P + Q)\left[x(r)\right],\\
x(c) &= w(c) + z(c) = c_1^* + c_2^*,\ x(d) = w(d) + z(d) = d_1^* + d_2^* \qquad (2.5)\\
{}^{C}D^{\mu,\psi}_{c^+}y(r) &= {}^{C}D^{\mu,\psi}_{c^+}w(r) - {}^{C}D^{\mu,\psi}_{c^+}z(r)\\
&= (F_1 - F_2)(r) - (P - Q)\left[w(r) - z(r)\right],\\
&= (F_1 - F_2)(r) - (P - Q)\left[y(r)\right],\\
y(c) &= w(c) - z(c) = c_1^* - c_2^*,\ y(d) = w(d) - z(d) = d_1^* - d_2^*. \qquad (2.6)
\end{aligned}$$

Then by Lemma 2.4, both problems (2.5), (2.6) have unique solutions,

$$\begin{aligned}
x(r) &= (c_1^* + c_2^*)E_{\mu,1}(-(P + Q)(\psi(r) - \psi(c))^{\mu}) + \theta_1 E_{\mu,1}(-(P + Q)(\psi(r) - \psi(c))^{\mu})\\
&\quad + \int_0^r \psi'(s)[\psi(r) - \psi(s)]^{\mu-1}E_{\mu,\mu}(-(P + Q)(\psi(r) - \psi(c))^{\mu})(F_1 + F_2)(s)\,ds,\\
\text{where}\quad \theta_1 &= (d_1^* + d_2^*) + \frac{[(F_1 + F_2)(c) - (P + Q)w(c)][\psi(d) - \psi(c)]^{\mu}}{\Gamma(\mu + 1)}\\
&\quad - \frac{1}{\Gamma(\mu)}\int_c^d \psi'(s)[\psi(d) - \psi(s)]^{\mu-1}[(F_1 + F_2)(s) - (P + Q)w(s)]\,ds,\\
\text{and,}\quad y(r) &= (c_1^* - c_2^*)E_{\mu,1}(-(P - Q)(\psi(r) - \psi(c))^{\mu}) + \theta_2 E_{\mu,1}(-(P - Q)(\psi(r) - \psi(c))^{\mu}),\\
&\quad + \int_0^r \psi'(s)[\psi(r) - \psi(s)]^{\mu-1}E_{\mu,\mu}(-(P - Q)(\psi(r) - \psi(c))^{\mu})(F_1 - F_2)(s)\,ds,\\
\text{where}\quad \theta_2 &= (d_1^* - d_2^*) + \frac{[(F_1 - F_2)(c) - (P - Q)w(c)][\psi(d) - \psi(c)]^{\mu}}{\Gamma(\mu + 1)}\\
&\quad - \frac{1}{\Gamma(\mu)}\int_c^d \psi'(s)[\psi(d) - \psi(s)]^{\mu-1}[(F_1 - F_2)(s) - (P - Q)w(s)]\,ds,
\end{aligned}$$

respectively in $C(\mathscr{J}, \mathbb{R})$. In consequence, w and z are unique too.

Lemma 2.6. *(Comparison Result)*[12] *Let $\mu \in (0, 1]$ and $P \in \mathbb{R}$. If $w \in C(\mathscr{J}, \mathbb{R})$ satisfies the following inequalities*

$$\begin{aligned}
{}^{C}D^{\mu,\psi}_{c^+}w(r) &\geq -Pw(r),\ r \in \mathscr{J},\\
w(c) &\geq 0, \quad w(d) \geq 0, \qquad (2.7)
\end{aligned}$$

then $w(r) \geq 0$ for all $r \in \mathscr{J}$.

Lemma 2.7. *(Comparison Result) Let $\mu \in (0,1]$ and $P \in \mathbb{R}$ and $Q \geq 0$ be given. If $w, z \in C(\mathscr{J}, \mathbb{R})$ satisfies the following inequalities*

$$\begin{aligned} {}^cD^{\mu,\psi}_{c^+}w(r) &\geq -Pw(r) + Qz(r),\ r \in \mathscr{J}, \\ {}^cD^{\mu,\psi}_{c^+}z(r) &\geq -Pz(r) + Qw(r),\ r \in \mathscr{J}, \\ w(c) \geq 0, z(c) \geq 0 &\quad w(d) \geq 0, z(d) \geq 0, \end{aligned} \tag{2.8}$$

then $w(r) \geq 0$, $z(r) \geq 0$ for all $r \in \mathscr{J}$.

Proof. Put $x(r) = w(r) + z(r)$ $r \in \mathscr{J}$. Then by (2.8), we have

$$\begin{aligned} {}^CD^{\mu,\psi}_{c^+}x(r) &= {}^CD^{\mu,\psi}_{c^+}w(r) + {}^CD^{\mu,\psi}_{c^+}z(r), \\ &\geq -Pw(r) + Qz(r) - Pz(r) + Qw(r), \\ &= -(P - Q)(w(r) + z(r)) = -(P - Q)x(r), \\ x(c) = w(c) &+ z(c) \geq 0,\ x(d) = w(d) + z(d) \geq 0. \end{aligned}$$

Then by Lemma 2.6, $x(r) \geq 0 \Rightarrow w(r) + z(r) \geq 0$. Next to show that, $w(r) \geq 0$, $z(r) \geq 0$ for all $r \in \mathscr{J}$. By (2.8), we have

$$\begin{aligned} {}^CD^{\mu,\psi}_{c^+}w(r) &\geq -Pw(r) + Qz(r), \\ &\geq -Pw(r) - Qw(r) = -(P + Q)w(r), \\ w(c) &\geq 0, \qquad w(d) \geq 0, \\ {}^CD^{\mu,\psi}_{c^+}z(r) &\geq -Pz(r) + Qw(r), \\ &\geq -Pz(r) - Qz(r) = -(P + Q)z(r), \\ z(c) &\geq 0, \qquad z(d) \geq 0. \end{aligned}$$

Then by Lemma 2.6, it is easy to show that $w(r) \geq 0$, $z(r) \geq 0$ for all $r \in \mathscr{J}$. □

3. Main Results

In this section, we develop monotone iterative scheme and prove the existence of maximal and minimal solutions of the nonlinear BVP (1.1) involving ψ-Caputo fractional derivative. We list the following assumptions for convenience.

(B_1) There exist $w_0(r), z_0(r) \in C(\mathscr{J}, \mathbb{R})$ and $w_0(r) \leq z_0(r)$, such that

$$\begin{aligned} {}^CD^{\mu,\psi}_{c^+}w_0(r) &\leq F(r, w_0(r), z_0(r)), \\ w_0(c) \leq c_1^*,&\ w_0(d) \leq d_1^*, \\ {}^CD^{\mu,\psi}_{c^+}z_0(r) &\geq G(r, z_0(r), w_0(r)), \\ z_0(c) \geq c_2^*,&\ z_0(d) \geq d_2^*. \end{aligned}$$

(B_2) There exist $P \in \mathbb{R}$, $Q \geq 0$ such that

$$F(r, w(r), z(r)) - F(r, w^*(r), z^*(r)) \geq -P(w(r) - w^*(r)) - Q(z(r) - z^*(r)),$$
$$G(r, z(r), w(r)) - G(r, z^*(r), w^*(r)) \geq -P(z(r) - z^*(r)) - Q(w(r) - w^*(r)),$$

where $w_0(r) \leq w^*(r) \leq w(r) \leq z_0(r)$, $w_0(r) \leq z^*(r) \leq z(r) \leq z_0(r)$ and

$$G(r, z(r), w(r)) - F(r, w(r), z(r)) \geq P(w(r) - z(r)) + Q(z(r) - w(r)),$$

with $w_0(r) \leq w(r) \leq z(r) \leq z_0(r)$.

Theorem 3.1. *Suppose that the assumptions (B_1) and (B_2) hold, then there exist monotone iterative sequences $\{w_n\}$, $\{z_n\} \in [w_0, z_0]$ such that $w_n \to w_*$, $z_n \to z_*$ as $(n \to \infty)$ uniformly on $r \in \mathscr{J}$, where $(w_*, z_*) \in [w_0, z_0] \times [w_0, z_0]$ are minimal and maximal solutions of the nonlinear problem (1.1) and satisfy the monotone property*

$$w_0 \leq w_1 \leq \ldots \leq w_n \leq \ldots \leq w_* \leq z_* \leq \ldots \leq z_n \leq \ldots \leq z_1 \leq z_0. \tag{3.1}$$

Proof. For any $w_{n-1}(r), z_{n-1}(r) \in C(\mathscr{J}, \mathbb{R})$, $n \geq 1$, we consider the linear system

$$\begin{aligned}
{}^C D_{c^+}^{\mu,\psi} w_n(r) &= F(r, w_{n-1}(r), z_{n-1}(r)) - P(w_n(r) - w_{n-1}(r)) - Q(z_n(r) - z_{n-1}(r)),\\
{}^C D_{c^+}^{\mu,\psi} z_n(r) &= G(r, z_{n-1}(r), w_{n-1}(r)) - P(z_n(r) - z_{n-1}(r)) - Q(w_n(r) - w_{n-1}(r)),\\
w_n(c) &= c_1^*,\ w_n(d) = d_1^*, \quad z_n(c) = c_2^*,\ z_n(d) = d_2^*.
\end{aligned} \tag{3.2}$$

Then by Lemma 2.5, we know that system (3.2) has unique system of solutions in $C(\mathscr{J}, \mathbb{R})$. Next we show that $\{w_n\}$, $\{z_n\}$ satisfy the property

$$w_{n-1}(r) \leq w_n(r) \leq z_n(r) \leq z_{n-1}(r), n = 1, 2, \ldots.$$

First need to show that $w_0(r) \leq w_1(r) \leq z_1(r) \leq z_0(r)$. Let $\phi(r) = w_1(r) - w_0(r)$, $\sigma(r) = z_0(r) - z_1(r)$. Then from (3.2) and (B_1) we have

$$\begin{aligned}
{}^C D_{c^+}^{\mu,\psi} \phi(r) &= {}^C D_{c^+}^{\mu,\psi} w_1(r) - {}^c D_{c^+}^{\mu,\psi} w_0(r),\\
&\geq F(r, w_0(r), z_0(r)) - P(w_1(r) - w_0(r)) - Q(z_1(r) - z_0(r)) - F(r, w_0(r), z_0(r)),\\
&= -P(w_1(r) - w_0(r)) - Q(z_1(r) - z_0(r)),\\
&= -P\phi(r) + Q\sigma(r),\\
\text{and} \quad \phi(c) &= w_1(c) - w_0(c) \geq c_1^* - c_1^* = 0, \phi(d) = w_1(d) - w_0(d) \geq d_1^* - d_1^* = 0,\\
{}^C D_{c^+}^{\mu,\psi} \sigma(r) &= {}^C D_{c^+}^{\mu,\psi} z_0(r) - {}^C D_{c^+}^{\mu,\psi} z_1(r),\\
&\geq G(r, z_0(r), w_0(r)) - G(r, z_0(r), w_0(r)) + P(z_1(r) - z_0(r)) + Q(w_1(r) - w_0(r)),\\
&= P(z_1(r) - z_0(r)) + Q(w_1(r) - w_0(r)),\\
&= -P\sigma(r) + Q\phi(r),\\
\text{and} \quad \sigma(c) &= z_0(c) - z_1(c) \geq d_2^* - d_2^* = 0, \sigma(d) = z_0(d) - z_1(d) \geq d_2^* - d_2^* = 0.
\end{aligned}$$

Then by Lemma 2.7 we have $\phi(r) \geq 0$, $\sigma(r) \geq 0$ implies that $w_1(r) \geq w_0(r)$, $z_0(r) \geq z_1(r)$, $r \in \mathscr{J}$.

Let $\rho(r) = z_1(r) - w_1(r)$. Then from (3.2) and (B_2) we have

$$\begin{aligned}
{}^C D^{\mu,\psi}_{c^+}\rho(r) &= {}^C D^{\mu,\psi}_{c^+} z_1(r) - {}^C D^{\mu,\psi}_{c^+} w_1(r),\\
&= G(r, z_0(r), w_0(r)) - P(z_1(r) - z_0(r)) - Q(w_1(r) - w_0(r)) - F(r, w_0(r), z_0(r))\\
&\quad + P(w_1(r) - w_0(r)) + Q(z_1 - z_0),\\
&\geq P(w_0(r) - z_0(r)) + Q(z_0(r) - w_0(r)) - P(z_1(r) - z_0(r)) - Q(w_1(r) - w_0(r))\\
&\quad + P(w_1(r) - w_0(r)) + Q(z_1(r) - z_0(r)),\\
&= -(P - Q)(z_1(r) - w_1(r)) = -(P - Q)\rho(r),\\
\text{and}\quad \rho(c) &= z_1(c) - w_1(c) = c_2^* - c_1^* \geq 0, \quad \rho(d) = z_1(d) - w_1(d) = d_2^* - d_1^* \geq 0.
\end{aligned}$$

Then by Lemma 2.6 we have $\rho(r) \geq 0$ implies that $z_1(r) \geq w_1(r)$, $r \in \mathscr{J}$. Hence we have $w_0(r) \leq w_1(r) \leq z_1(r) \leq z_0(r)$, $r \in \mathscr{J}$. Now we assume that $w_{i-1}(r) \leq w_i(r) \leq z_i(r) \leq z_{i-1}(r)$ for some $i \geq 1$. We claim that $w_i(r) \leq w_{i+1}(r) \leq z_{i+1}(r) \leq z_i(r)$ for some $r \in \mathscr{J}$. Set $\phi(r) = w_{i+1}(r) - w_i(r)$, $\sigma(r) = z_i(r) - z_{i+1}(r)$, $\rho(r) = z_{i+1}(r) - w_{i+1}(r)$. By (3.2) and (B_2) we have

$$\begin{aligned}
{}^C D^{\mu,\psi}_{c^+}\phi(r) &= {}^C D^{\mu,\psi}_{c^+} w_{i+1}(r) - {}^C D^{\mu,\psi}_{c^+} w_i(r),\\
&\geq F(r, w_i(r), z_i(r)) - P(w_{i+1}(r) - w_i(r)) - Q(z_{i+1}(r) - z_i(r)) - F(r, w_i(r), z_i(r)),\\
&= -P(w_{i+1}(r) - w_i(r)) - Q(z_{i+1}(r) - z_i(r)) = -P\phi(r) + Q\sigma(r),\\
\text{and}\quad \phi(c) &= w_{i+1}(c) - w_i(c) = c_1^* - c_1^* = 0, \quad \phi(d) = w_{i+1}(d) - w_i(d) = d_1^* - d_1^* = 0.\\
{}^C D^{\mu,\psi}_{c^+}\sigma(r) &= {}^C D^{\mu,\psi}_{c^+} z_i - {}^C D^{\mu,\psi}_{c^+} z_{i+1},\\
&\geq G(r, z_0(r), w_0(r)) - G(r, z_i(r), w_i(r)) + P(z_{i+1}(r) - z_i(r)) + Q(w_{i+1}(r) - w_i(r)),\\
&= P(z_{i+1}(r) - z_i(r)) + Q(w_{i+1}(r) - w_i(r)) = -P\sigma(r) + Q\phi(r),\\
\text{and}\quad \sigma(c) &= z_i(c) - z_{i+1}(c) = c_2^* - c_2^* = 0, \quad \sigma(d) = z_i(d) - z_{i+1}(d) = d_2^* - d_2^* = 0.\\
{}^C D^{\mu,\psi}_{c^+}\rho(r) &= {}^C D^{\mu,\psi}_{c^+} z_{i+1}(r) - {}^C D^{\mu,\psi}_{c^+} w_{i+1}(r),\\
&= G(r, z_i(r), w_i(r)) - P(z_{i+1}(r) - z_i(r)) - Q(w_{i+1}(r) - w_i(r)) - F(r, w_i(r), z_i(r))\\
&\quad + P(w_{i+1}(r) - w_i(r)) + Q(z_{i+1}(r) - z_i(r)),\\
&\geq P(w_i(r) - z_i(r)) + Q(z_i(r) - w_i(r)) - P(z_{i+1}(r) - z_i(r)) - Q(w_{i+1}(r) - w_i(r))\\
&\quad + P(w_{i+1}(r) - w_i(r)) + Q(z_{i+1}(r) - z_i(r)),\\
&= -P(z_{i+1}(r) - w_{i+1}(r)) + Q(z_{i+1}(r) - w_{i+1}(r)),\\
&= -(P - Q)(z_{i+1}(r) - w_{i+1}(r)) = -(P - Q)\rho(r),\\
\text{and}\quad \rho(c) &= z_{i+1}(c) - w_{i+1}(c) = c_2^* - c_1^* \geq 0, \quad \rho(d) = z_{i+1}(d) - w_{i+1}(d) = d_2^* - d_1^* \geq 0.
\end{aligned}$$

Then by Lemma 2.6 and 2.7, we have $\phi(r) \geq 0$, $\sigma(r) \geq 0$, $\rho(r) \geq 0$ implies that $w_i \leq w_{i+1} \leq z_{i+1} \leq z_i$ for some $r \in \mathscr{J}$. Hence from the principle of mathematical induction, we have

$$w_0 \leq w_1 \leq \ldots \leq w_n \leq \ldots \leq z_n \leq \ldots \leq z_1 \leq z_0.$$

Thus the sequences $\{w_n(r)\}$ and $\{z_n(r)\}$ are uniformly bounded and convergent. Hence pointwise limit exists and are given by

$$\lim_{n\to\infty} w_n(r) = w_*(r) \quad \text{and} \quad \lim_{n\to\infty} z_n(r) = z_*(r),$$

uniformly on $r \in J$ and the limit functions w_*, z_* satisfy BVP (1.1).

Moreover, $w_*, z_* \in [w_0, z_0]$. Taking the limits in (3.2), we know that (w_*, z_*) is a system of solutions of (1.1) in $[w_0, z_0] \times [w_0, z_0]$. Moreover, (3.1) is true. Finally, we prove that (1.1) has minimal and maximal solutions. Let $(w, z) \in [w_0, z_0] \times [w_0, z_0]$ is any system of solutions of (1.1). Assume that for some $n \in N$, $w_n \leq w, z \leq z_n$ for some $r \in \mathscr{J}$.
Set $\phi(r) = w(r) - w_{n+1}(r)$, $\sigma(r) = z_{n+1}(r) - z(r)$. Then by (3.2) and (B_2) we have

$$\begin{aligned}
{}^C D^{\mu,\psi}_{c^+}\phi(r) &= {}^C D^{\mu,\psi}_{c^+} w(r) - {}^C D^{\mu,\psi}_{c^+} w_{n+1}(r),\\
&= F(r, w(r), z(r)) - F(r, w_n(r), z_n(r)) + P(w_{n+1}(r) - w_n(r)) + Q(z_{n+1}(r) - z_n(r)),\\
&\geq -P(w(r) - w_n(r)) - Q(z(r) - z_n(r)) + P(w_{n+1}(r) - w_n(r)) + Q(z_{n+1}(r) - z_n(r)),\\
&= -P(w(r) - w_n(r)) + Q(z_n(r) - z(r)) = -P\phi(r) + Q\sigma(r),\\
\text{and} \quad \phi(c) &= w(c) - w_{n+1}(c) = c_1^* - c_1^* = 0, \quad \phi(d) = w(d) - w_{n+1}(d) = d_1^* - d_1^* = 0.\\
{}^C D^{\mu,\psi}_{c^+}\sigma(r) &= {}^C D^{\mu,\psi}_{c^+} z_{n+1}(r) - {}^C D^{\mu,\psi}_{c^+} z(r),\\
&= G(r, z_n(r), w_n(r)) - P(z_{n+1}(r) - z_n(r)) - Q(w_{n+1}(r) - w_n(r)) - G(r, z(r), w(r)),\\
&\geq -P(z_n(r) - z(r)) - Q(w_n(r) - w(r)) - P(z_{n+1}(r) - z_n(r)) - Q(w_{n+1}(r) - w_n(r)),\\
&= -P(z_{n+1}(r) - z(r)) + Q(w(r) - w_{n+1}(r)) = -P\sigma(r) + Q\phi(r),\\
\text{and} \quad \sigma(c) &= z_{n+1}(c) - z(c) = c_2^* - c_2^* = 0, \quad \sigma(d) = z_{n+1}(d) - z(d) = d_2^* - d_2^* = 0.
\end{aligned}$$

Then by Lemma 2.7, $w_{n+1}(r) \leq w(r), z(r) \leq z_{n+1}(r)$ for some $r \in \mathscr{J}$.
Now taking the limits as $n \to \infty$, we have $w_* \leq w, z \leq z_*$. Hence (w_*, z_*) is an minimal and maximal solutions of system (1.1) in $[w_0, z_0] \times [w_0, z_0]$. This completes the proof. □

Example 3.2. Consider the following coupled system of boundary values problem:

$$\begin{aligned}
{}^c D^{1/2,r}_{0+} w(r) &= -w^3(r) + 1 + z(r), \quad r \in J = [0, 1],\\
{}^c D^{1/2,r}_{0+} z(r) &= -z^3(r) + 1 + w(r), \quad r \in J = [0, 1],\\
w(0) = z(0) &= 0, \quad w(1) = z(1) = \frac{3}{4}.
\end{aligned} \tag{3.3}$$

Here $\mu = \frac{1}{2}$, $n = [\frac{1}{2}] + 1 = 1$ $c = 0$, $d = 1$, $c_1^* = c_2^* = 0$, $d_1^* = d_2^* = \frac{3}{4}$, $\psi(r) = r$.
Obviously $F(r, w, z) = -w^3(r) + 1 + z(r)$, $\quad G(r, z, w) = -z^3(r) + 1 + w(r)$.
Now taking $w_0(r) = 0$ and $z_0(r) = r$. Also ${w_0}^{[1]}_{\psi} = 0$ and ${z_0}^{[1]}_{\psi} = \frac{z_0'(r)}{\psi'(r)} = 1$ then,
${}^c D^{1/2,r}_{0+} w_0(r) = 0 \leq 1 + r = F(r, w_0, z_0)$, and
${}^c D^{1/2,r}_{0+} z_0(r) = \frac{1}{\Gamma(\frac{1}{2})} \displaystyle\int_0^r (r - s)^{-1/2}\, ds = \frac{2}{\sqrt{\pi}} r^{1/2} \geq -r^3 + 1 = G(r, z_0, w_0)$.
Also $w_0 \leq z_0$. Hence condition (B_1) of Theorem 3.1 holds. On the other hand, it is easy to show that for $M = 3$ and $N = 0$ condition (B_2) holds. Thus, all conditions of Theorem 3.1 are satisfied. Hence, the nonlinear system (3.3) has the extremal solution $(w_*, z_*) \in [w_0, z_0] \times [w_0, z_0]$, which can be obtained by taking

limits from the iterative sequences, for $n \geq 1$,

$$w_n(r) = \theta_1 E_{\frac{1}{2},1}(-3r^{\frac{1}{2}}) + \int_0^r (r-s)^{-\frac{1}{2}} E_{\frac{1}{2},\frac{1}{2}}(-3r^{\frac{1}{2}})(-w_{n-1}^3(s) + 1 + z_{n-1}(s) + 3w_{n-1})\, ds,$$

$$\text{where} \quad \theta_1 = \frac{3}{4} - \frac{1}{\Gamma(1/2)} \int_0^1 [1-s]^{-\frac{1}{2}} [-w_{n-1}^3(s) + 1 + z_{n-1}(s) - 3w_{n-1}(s)]\, ds,$$

$$z_n(r) = \theta_2 E_{\frac{1}{2},1}(-3r^{\frac{1}{2}}) + \int_0^r (r-s)^{-\frac{1}{2}} E_{\frac{1}{2},\frac{1}{2}}(-3r^{\frac{1}{2}})(-z_{n-1}^3(s) + 1 + w_{n-1}(s) + 3z_{n-1})\, ds,$$

$$\text{where} \quad \theta_2 = \frac{3}{4} - \frac{1}{\Gamma(1/2)} \int_0^1 [1-s]^{-\frac{1}{2}} [-z_{n-1}^3(s) + 1 + w_{n-1}(s) - 3z_{n-1}(s)]\, ds.$$

4. Conclusion

Comparison result for nonlinear system involving ψ-Caputo fractional derivative is established that plays vital role in the construction of monotone method. Existence of minimal and maximal solutions to a coupled system of nonlinear boundary value problems involving ψ-Caputo fractional derivative is investigated. An example is given to validate obtained results. Different type of fractional derivatives such as generalized Hilfer integral and derivative etc. are being used to study qualitative properties of solutions of ψ-Caputo fractional differential equations through fixed point, Green's function theory etc. Monotone method coupled with lower-upper solutions is a powerful and constructive approach to obtain existence of solutions of nonlinear problem. Using this method we can establish existence results for the problem under investigation.

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