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CHARACTERISTIC SUBGROUPS OF A FINITE ABELIAN P-GROUP $\mathbb{Z}_p \times \mathbb{Z}_{p^n}$

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Abstract. In this paper, we study the following points:- (i) list all characteristic subgroups of a finite abelian 2-group $\mathbb{Z}_2 \times \mathbb{Z}_{2^n}$ when $n \in \mathbb{Z}^+$ and $n \geq 2$ (ii) list all characteristic subgroups of a finite abelian p-group $\mathbb{Z}_p \times \mathbb{Z}_{p^n}$ when $n \in \mathbb{Z}^+$ and $n \geq 2$ when p is an odd prime (iii) lattices of characteristic subgroups of a finite abelian 2-group $\mathbb{Z}_2 \times \mathbb{Z}_{2^n}$ when $n \in \mathbb{Z}^+$ and $n \geq 2$ and (iv) Lattices of characteristic subgroups of a finite abelian p-group $\mathbb{Z}_p \times \mathbb{Z}_{p^n}$ when $n \in \mathbb{Z}^+$ and $n \geq 2$ when p is odd prime.

1. Introduction

A subgroup N of a group G is called a characteristic subgroup if $\phi(N) = N$ for all automorphisms ϕ of G. This term was first used by *Frobenius* in 1895. In 1939, Baer [2] considered the following question "When do two groups have isomorphic subgroups lattices?" Since this is a very difficult problem.In 2011,Brent L. Kerby and Emma Rode [3] consider the related question "Lattices of characteristic subgroups of $\mathbb{Z}_p \times \mathbb{Z}_{p^2} \times \mathbb{Z}_{p^4}$ isomorphic to lattices of characteristic subgroups of $\mathbb{Z}_{p^2} \times \mathbb{Z}_{p^5}$ for any prime p".In 2017, Amit Sehgal and Manjeet Jakhar [7] consider the related question "Lattices of characteristic subgroups of $\mathbb{Z}_n \times \mathbb{Z}_n$ isomorphic to lattices of characteristic subgroups of $\mathbb{Z}_n$". We will now consider the problem of lattices of characteristic subgroups of a finite abelian p-group $\mathbb{Z}_p \times \mathbb{Z}_{p^n}$ when $n \in \mathbb{Z}^+$ and $n \geq 2$ when p prime.

2. List of subgroups of group $\mathbb{Z}_p \times \mathbb{Z}_{p^n}$

We know that group $\mathbb{Z}_p \times \mathbb{Z}_{p^n} = \{x^i y^j | x^p = y^{p^n} = e, xy = yx, i = 0, 1, \cdots, p-1, j = 0, 1, \cdots, p^{n-1}\}$ is an abelian group of order p^{n+1}. So converse of Lagrange's theorem is true for this group. So possible order of subgroup are $1, p, p^2, \cdots, p^{n+1}$. There is only one subgroup of order 1 which is $\{e\}$ and there exist only one subgroup of order p^{n+1} which is group itself. Here, all subgroups are subgroups from abelian group, so they must be abelian. Also group of order p must cyclic subgroup because order of the group is prime. Now we have to search for cyclic subgroups whose order are $p, p^2, \cdots, p^{n+1}$ and abelian subgroups which are not cyclic of order $p^2, p^3, \ldots p^{n+1}$

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2.1. List of cyclic subgroups of order p from group $\mathbb{Z}_p \times \mathbb{Z}_{p^n}$. We count elements of order p in $\mathbb{Z}_p \times \mathbb{Z}_{p^n}$ as ((no. of elements of order p from $\mathbb{Z}_p$)$\times$ (no. of elements order 1 or p from $\mathbb{Z}_{p^n}$)+ (no. of elements order 1 from $\mathbb{Z}_p$)$\times$ (no. of elements order p from $\mathbb{Z}_{p^n}$)) $= (p-1) \times p + 1(p-1) = p^2 - 1$. Hence, the number of cyclic subgroups of order p are $\frac{p^2-1}{\phi(p)} = p+1$. The list of these $p+1$ subgroups is $< xy^{jp^{n-1}} >$ where $j = 1, 2, \cdots, p$ and $< y^{p^{n-1}} >$

2.2. List of cyclic subgroups of order p^k from group $\mathbb{Z}_p \times \mathbb{Z}_{p^n}$ where $2 \leq k \leq n$. We count elements of order p^k in $\mathbb{Z}_p \times \mathbb{Z}_{p^n}$ as ((no. of elements of order 1 or p from $\mathbb{Z}_p$)$\times$ (no. of elements order p^k from $\mathbb{Z}_{p^n}$) $= p \times p^{k-1}(p-1) = p^{k+1} - p^k$. Hence, the number of cyclic subgroups of order p^k are $\frac{p^{k+1}-p^k}{\phi(p^k)} = p$. The list of these p subgroups is $< x^j y^{p^{n-k}} >$ where $j = 1, 2, \cdots, p$ and $k = 2, 3, \cdots, n$

2.3. List of abelian subgroups which are not cyclic subgroups of order p^k from group $\mathbb{Z}_p \times \mathbb{Z}_{p^n}$ where $2 \leq k \leq n$. By fundamental theorem of finite abelian group, we know that number of non-isomorphic abelian groups of order p^k are $p(k)$, out of which only one group is cyclic. So abelian group have $p(k) - 1$ subgroups which are not cyclic.

Theorem 2.1. [5] *Number of internal direct product of cyclic subgroups of two cyclic subgroups of order p is $p^2 + p$ from group $\mathbb{Z}_p \times \mathbb{Z}_{p^n}$*

Theorem 2.2. [5] *Number of internal direct product of cyclic subgroup of order p^{k_1} with subgroup isomorphic to $\mathbb{Z}_{p^{k_2}} \times \mathbb{Z}_{p^{k_3}} \times \cdots \times \mathbb{Z}_{p^{k_s}}$ is 0 from group $\mathbb{Z}_p \times \mathbb{Z}_{p^n}$ where $s \geq 3$, $k_i \geq 1$ and $\sum_{i=1}^{s} k_i \leq (n+1)$. In other words, group $\mathbb{Z}_p \times \mathbb{Z}_{p^n}$ does not possess any subgroup of rank more than two.*

Proof. If A is any cyclic subgroups of order p^{k_1} from $\mathbb{Z}_p \times \mathbb{Z}_{p^n}$ and B is a subgroup isomorphic to $\mathbb{Z}_{p^{k_2}} \times \mathbb{Z}_{p^{k_3}} \times \cdots \times \mathbb{Z}_{p^{k_s}}$ where $s \geq 3$ from $\mathbb{Z}_p \times \mathbb{Z}_{p^n}$, then $A \bigcap B \neq \{e\}$ because there are only p+1 cyclic subgroups of order p in $\mathbb{Z}_p \times \mathbb{Z}_{p^n}$ and A contains a unique subgroup of order p and B contains at-least $p+1$ cyclic subgroups of order p. So cyclic subgroup of order p must contain in B if B is possible. So, internal direct product of cyclic subgroup of order p^{k_1} with subgroup isomorphic to $\mathbb{Z}_{p^{k_2}} \times \mathbb{Z}_{p^{k_3}} \times \cdots \times \mathbb{Z}_{p^{k_s}}$ from $\mathbb{Z}_p \times \mathbb{Z}_{p^n}$ is not possible.
Hence, we conclude that the abelian p-group $\mathbb{Z}_p \times \mathbb{Z}_{p^n}$ does not possess any subgroup of rank three or more. □

Theorem 2.3. *Number of internal direct product of cyclic subgroup of order p^{k_1} with cyclic subgroup of order p^{k_2} with $k_1, k_2 > 1$ is 0 from group $\mathbb{Z}_p \times \mathbb{Z}_{p^n}$.*

Proof. We know that $< x^j y^{p^{n-k}} >$ with $j = 1, 2, \cdots, p$ are only cyclic subgroups of order p^k where $k = 2, 3, \cdots, n$. Further, all these subgroups contains only one subgroup of order p which is $< y^{p^{n-1}} >$. Hence, internal direct product between cyclic subgroup of order p^{k_1} with cyclic subgroup of order p^{k_2} with $k_1, k_2 > 1$ is not possible because intersection of these subgroups is not identity.
From last two theorems, we conclude that the only possible non-cyclic subgroups of group $\mathbb{Z}_p \times \mathbb{Z}_{p^n}$ are isomorphic to $\mathbb{Z}_p \times \mathbb{Z}_{p^k}$ where $k = 1, 2, \cdots, n$ □

Theorem 2.4. [5] *Number of internal direct product of cyclic subgroup of order p with subgroup isomorphic to* Z_{p^k} *where* $1 < k \geq n$ *is* p^2 *from group* $\mathbb{Z}_p \times \mathbb{Z}_{p^n}$

Theorem 2.5. [5] *Number of internal direct product of cyclic subgroup of order p with subgroup isomorphic to* Z_{p^2} *is* p^2 *from group* $\mathbb{Z}_p \times \mathbb{Z}_{p^2}$

Theorem 2.6. [5] *Number of subgroups from group* $\mathbb{Z}_p \times \mathbb{Z}_{p^n}$ *which are isomorphic to* $\mathbb{Z}_p \times \mathbb{Z}_p$ *is 1.*

Further, only subgroup which is isomorphic to $Z_p \times Z_p$ is $< x, y^{p^{n-1}} >$.

Theorem 2.7. [5] *Number of subgroups from group* $\mathbb{Z}_p \times \mathbb{Z}_{p^n}$ *which are isomorphic to* $\mathbb{Z}_p \times \mathbb{Z}_{p^k}$ *where* $1 < k \leq n$ *is 1.*

Proof. Number of subgroups from group $\mathbb{Z}_p \times \mathbb{Z}_{p^n}$ which are isomorphic to $\mathbb{Z}_p \times \mathbb{Z}_{p^k}$ where $1 < k \leq n$ is $N = \frac{C}{D}$ (say) where C and D are defined as follows: C= Number of internal direct product of two cyclic subgroups of $\mathbb{Z}_p \times \mathbb{Z}_{p^n}$ which are isomorphic to $\mathbb{Z}_p \times \mathbb{Z}_{p^k}$ and D= Number of internal direct product of two cyclic subgroups of group $\mathbb{Z}_p \times \mathbb{Z}_{p^k}$ which are isomorphic to $\mathbb{Z}_p \times \mathbb{Z}_{p^k}$. Therefore, we get $N = \frac{C}{D} = \frac{p^2}{p^2} = 1$.

Further, only subgroup which is isomorphic to $\mathbb{Z}_p \times \mathbb{Z}_{p^k}$ is $< x, y^{p^{n-k}} >$ for all $k = 2, 3, \cdots, n$.

Finally list of subgroups of group $\mathbb{Z}_p \times \mathbb{Z}_{p^n}$ given below:-
(i) $H_1 =< e >\cong \mathbb{Z}_1$
(ii) $H_{j+1} =< xy^{jp^{n-1}} >\cong \mathbb{Z}_p$ where $j = 1, 2, \cdots, p$
(iii) $H_{p+2} =< y^{p^{n-1}} >\cong \mathbb{Z}_p$
(iv) $H_{kp+2+k} =< x, y^{p^{n-k}} >\cong \mathbb{Z}_p \times \mathbb{Z}_{p^k}$ where $k = 1, 2, \cdots, n$
(v) $H_{j+(k-1)p+1+k} =< x^j y^{p^{n-k}} >\cong \mathbb{Z}_{p^k}$ with $j = 1, 2, \cdots, p$ and $k = 2, 3, \cdots, n$

Hence, we get the list of $np + n + 2$ number of subgroups of group $\mathbb{Z}_p \times \mathbb{Z}_{p^n}$ which is same as result $\tau(p^n)\tau(p)\phi(1) + \tau(p^{n-1})\tau(1)\phi(p)$ in [1, 4].

3. List of Automorphisms of Group $\mathbb{Z}_p \times \mathbb{Z}_{p^n}$ where $n \geq 2$

We know that $\mathbb{Z}_p \times \mathbb{Z}_{p^n} = \{x^i y^j | x^p = y^{p^n} = e, xy = yx, i = 0, 1, \cdots, p-1, j = 0, 1, \cdots, p^{n-1}\}$ is an abelian group of order p^{n+1}. This group is generated by two elements x and y where order of x is p and y is p^n.

We map y into an element of order p^n and elements of order p^n are obtained from product of elements of order 1 or p from $\mathbb{Z}_p$ (say α) and elements of order p^n from $\mathbb{Z}_{p^n}$ (say β). Assume $\alpha = x^{i_1}$ with $o(\alpha) = 1$ or p and $\beta = y^{j_1}$ with $o(\beta) = p^n$. So, there is no condition on i_1 and $(j_1, p^n) = 1$. So $y \mapsto x^{i_1} y^{j_1} \Rightarrow y^{p^{n-1}} \mapsto y^{j_1 p^{n-1}}$ .So, image of y depends upon values of i_1 and j_1, here possibilities for i_1 and j_1 are p and $p^{n-1}(p-1)$ respectively. Hence, the possibilities for y are $p^n(p-1)$

So every element of order p of the type $y^{j_1 p^{n-1}}$ from $\mathbb{Z}_p \times \mathbb{Z}_{p^n}$ is already mapped. So x maps into an element of order p from $\mathbb{Z}_p \times \mathbb{Z}_{p^n}$ other than elements of the type

$y^{j_1 p^{n-1}}$. Hence, x maps to $x^{i_2} y^{j_2 p^{n-1}}$ where $(i_2, p) = 1$ and there is no condition on j_2. So, map of x depends upon values of i_2 and j_2. Here possibilities for i_2 and j_2 are $p-1$ and p respectively. Hence, the possibilities for y are $p(p-1)$.

Finally, we define an automorphism $f : \mathbb{Z}_p \times \mathbb{Z}_{p^n} \mapsto \mathbb{Z}_p \times \mathbb{Z}_{p^n}$ as $f(y) = x^{i_1} y^{j_1}$ and $f(x) = x^{i_2} y^{j_2 p^{n-1}}$ where $i_1, i_2, j_2 = 1, 2, \cdots, p$ with $(i_2, p) = 1$ and $j_1 = 1, 2, \cdots, p^n$ with $(j_1, p^n) = 1$. Hence, the total number of automorphisms are $p^{n+1}(p-1)^2$ which is same as result in [6].

4. List of the Characteristic Subgroups of Group $\mathbb{Z}_p \times \mathbb{Z}_{p^n}$

Theorem 4.1. *Let* $\mathbb{Z}_p \times \mathbb{Z}_{p^n} = \{x^i y^j | x^p = y^{p^n} = e, xy = yx, i = 0, 1, \cdots, p-1, j = 0, 1, \cdots, p^{n-1}\}$*, then list of characteristic subgroups of group* $\mathbb{Z}_p \times \mathbb{Z}_{p^n}$ *when* $n \geq 2$ *given below:-*
(i) $H_1 =< e >\cong \mathbb{Z}_1$
(ii) $H_{p+2} =< y^{p^{n-1}} >\cong \mathbb{Z}_p$
(iii) $H_{kp+1+k} =< y^{p^{n-k}} >\cong \mathbb{Z}_{p^k}$ $k = 2, 3, \cdots, n-1$
(iv) $H_{kp+2+k} =< x, y^{p^{n-k}} >\cong \mathbb{Z}_p \times \mathbb{Z}_{p^k}$ *where* $k = 1, 2, \cdots, n$
(v) (only for case when p is even prime) $H_{(k-1)p+2+k} =< xy^{p^{n-k}} >\cong \mathbb{Z}_{p^k}$ *with* $k = 2, 3, \cdots, n-1$

Proof. Case 1:- Subgroups H_1, H_{kp+2+k} where $k = 1, 2, \cdots, n$

Out of $np+n+2$ subgroups, $n+1$ subgroups namely $H_1 =< e >, H_{kp+2+k} =< x, y^{p^{n-k}} >$ where $i = 1, 2, \cdots, n$ have property that they are not isomorphic to any other subgroups of the group $\mathbb{Z}_p \times \mathbb{Z}_{p^n}$. Hence, image of these subgroups cannot be changed with any of the group automorphisms of the group $\mathbb{Z}_p \times \mathbb{Z}_{p^n}$, so they are characteristic subgroups of $\mathbb{Z}_p \times \mathbb{Z}_{p^n}$.

Case 2:- Subgroups $H_{j+1} =< xy^{jp^{n-1}}$ where $j = 1, 2, \cdots, p$

Subgroups $H_{j+1} =< xy^{jp^{n-1}} \cong \mathbb{Z}_p$ with $j = 1, 2, \cdots, p$ are not characteristic subgroups of $\mathbb{Z}_p \times \mathbb{Z}_{p^n}$ if we choose automorphism $f(x) = xy^{p^{n-1}}$ and $f(y) = y$ then $f(xy^{jp^{n-1}}) = f(xy^{(j+1)p^{n-1}} \notin H_{j+1}$ because $j \not\equiv j+1(mod\ p)$

Case 3:- Subgroups $H_{p+2} =< y^{p^{n-1}} >$

Subgroups $H_{p+2} =< y^{p^{n-1}} >\cong \mathbb{Z}_p$ is a characteristic subgroups of $\mathbb{Z}_p \times \mathbb{Z}_{p^n}$ because we can choose any automorphism, then $f(y^{p^{n-1}}) = y^{j_1 p^{n-1}} \in H_{p+2}$, hence we get $f(H_{p+2}) = H_{p+2}$.

Case 4:- Subgroups $H_{j+(k-1)p+k+1} =< x^j y^{p^{n-k}} \cong \mathbb{Z}_{p^k}$ with $j = 1, 2, \cdots, p-1$ and $k = 2, 3, \cdots, n-1$ when p is odd prime

Finally, we define an automorphism $f : \mathbb{Z}_p \times \mathbb{Z}_{p^n} \mapsto \mathbb{Z}_p \times \mathbb{Z}_{p^n}$ as $f(y) = x^{i_1}y^{j_1}$ and $f(x) = x^{i_2}y^{j_2p^{n-1}}$ where $i_1, i_2, j_2 = 1, 2, , p$ with $(i_2, p) = 1$ and $j_1 = 1, 2, \cdots, p^n$ with $(j_1, p^n) = 1$.

Value of $i_2 = 1, 2, \cdots, p-1$, then $(i_2, p) = 1$ and i_2 has at-least two values. Hence there exists at-least one i_2 such that $i_2 \not\equiv j_1$ $(mod\ p)$
$\implies ji_2 \not\equiv jj_1$ (mod p).
$f(x^jy^{p^{n-k}}) = x^{i_2j}y^{j_2jp^{n-1}}y^{j_1p^{n-k}}$
$= x^{i_2j}y^{p^{n-k}(p^{k-1}j_2j+j_1)} = x^{i_2j+p^{k-1}j_2j^2}y^{p^{n-k}(p^{k-1}j_2j+j_1)}$
$= (x^j)^{i_2+p^{k-1}j_2j}(y^{p^{n-k}})^{p^{k-1}j_2j+j_1} \neq (x^jy^{p^{n-k}})^{p^{k-1}j_2j+j_1}$ because $ji_2 \not\equiv jj_1$ (mod p).Hence $f((x^jy^{p^{n-k}})) \notin H_{j+(k-1)p+k+1}$ for any $j = 1, 2, \cdots, p-1$ and $k = 2, 3, \cdots, n-1$. So, subgroups $H_{j+(k-1)p+k+1}$ is not characteristic subgroup of $\mathbb{Z}_p \times \mathbb{Z}_{p^n}$.

Case 5:-Subgroups $H_{j+(n-1)p+1+n} =< x^jy^1 > \cong \mathbb{Z}_{p^k}$ with $j = 1, 2, \cdots, p$

Finally, we define an automorphism $f : \mathbb{Z}_p \times \mathbb{Z}_{p^n} \mapsto \mathbb{Z}_p \times \mathbb{Z}_{p^n}$ as $f(y) = xy$ and $f(x) = x$, so $f(x^jy) = x^jxy = x^{j+1}y \neq x^jy$ because $j \not\equiv j+1$ (mod p).Hence $f(x^jy) \notin H_{j+(n-1)p+n+1}$ for any $j = 1, 2, \cdots, p$.So, subgroups $H_{j+(n-1)p+n+1}$ is not characteristic subgroup of $\mathbb{Z}_p \times \mathbb{Z}_{p^n}$.

Case 6:-Subgroups $H_{3k} =< x^jy^{2^{n-k}} > \cong Z_{2^k}$ with $k = 2, 3, \cdots, n-1$

Finally, we define an automorphism $f : \mathbb{Z}_2 \times \mathbb{Z}_{2^n} \mapsto \mathbb{Z}_2 \times \mathbb{Z}_{2^n}$ as $f(y) = x^{i_1}y^{j_1}$ and $f(x) = xy^{j_22^{n-1}}$ where $i_1, j_2 = 1, 2$ and $(j_1, 2^n) = 1$. Subgroups $H_{3k} =< x^jy^{2^{n-k}} > \cong Z_{2^k}$ is a characteristic subgroups of $\mathbb{Z}_2 \times \mathbb{Z}_{2^n}$ because we can choose any automorphism and map then $yx^{i_1}y^{j_1} \implies y^{2^{n-k}} \mapsto y^{j_12^{n-k}}$ where $(j_1, 2^n) = 1$ and $f(x) = xy^{j_22^{n-1}}$. On basis of value j_1, we say that $x^{j_1} = x$. So, we get $f(xy^{2^{n-k}}) = xy^{2^{n-1}j_2}y^{2^{n-k}j_1} = xy^{2^{n-k}(j_22^{k-1}+j_1)} = x^{(j_22^{k-1}+j_1)}y^{2^{n-k}(j_22^{k-1}+j_1)} = (xy^{2^{n-k}})^{(j_22^{k-1}+j_1)} \in H_{3k}$. Hence, we get $f(H_{3k}) = H_{3k}$.

Case 7:-Subgroups $H_{(p+1)k+1} =< y^{p^{n-k}} > \cong \mathbb{Z}_{p^k}$ with $k = 2, 3, \cdots, n-1$

Subgroups $H_{p+(k-1)p+k+1} =< y^{p^{n-k}} > \cong \mathbb{Z}_{p^k}$ is a characteristic subgroup of $\mathbb{Z}_p \times \mathbb{Z}_{p^n}$ because we can choose any automorphism, and then $y \mapsto x^{i_1}y^{j_1} \implies y^{p^{n-k}} \mapsto y^{j_1p^{n-k}}$. So we get $f(H_{(p+1)k+1}) = H_{(p+1)k+1}$. □

5. Lattice of Characteristic Subgroups of group $\mathbb{Z}_p \times \mathbb{Z}_{p^n}$ where p is prime

Now we write two already known results which are very useful to form characteristic subgroup lattice for group $\mathbb{Z}_p \times \mathbb{Z}_{p^n}$ where p may be even or odd prime.

Theorem 5.1. [8] *Characteristic property is transitive. That is, if N is characteristic subgroup of K and K is characteristic subgroup of G, then N is characteristic subgroup of G.*

Theorem 5.2. *Characteristic subgroup of a group $\mathbb{Z}_p \times \mathbb{Z}_p$ are $\tau(p) = 2$ and which are $\{e\}$ and group itself.*

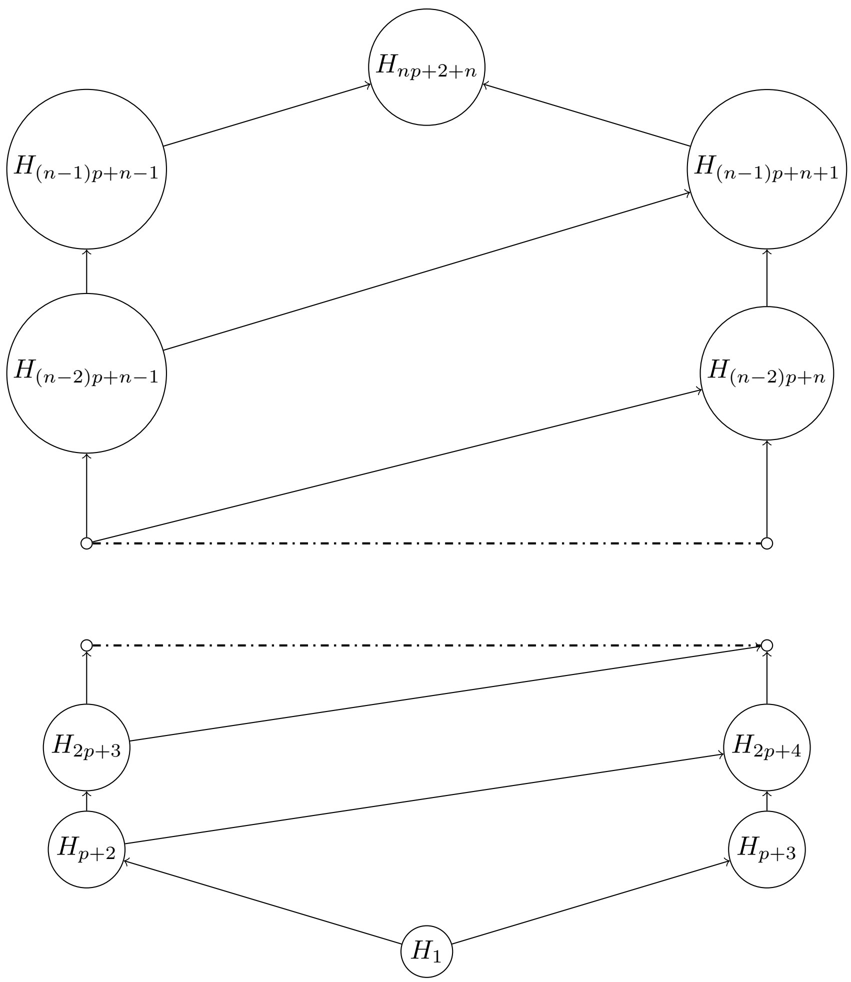

Fig-1 Lattices of characteristic subgroups when p is odd prime

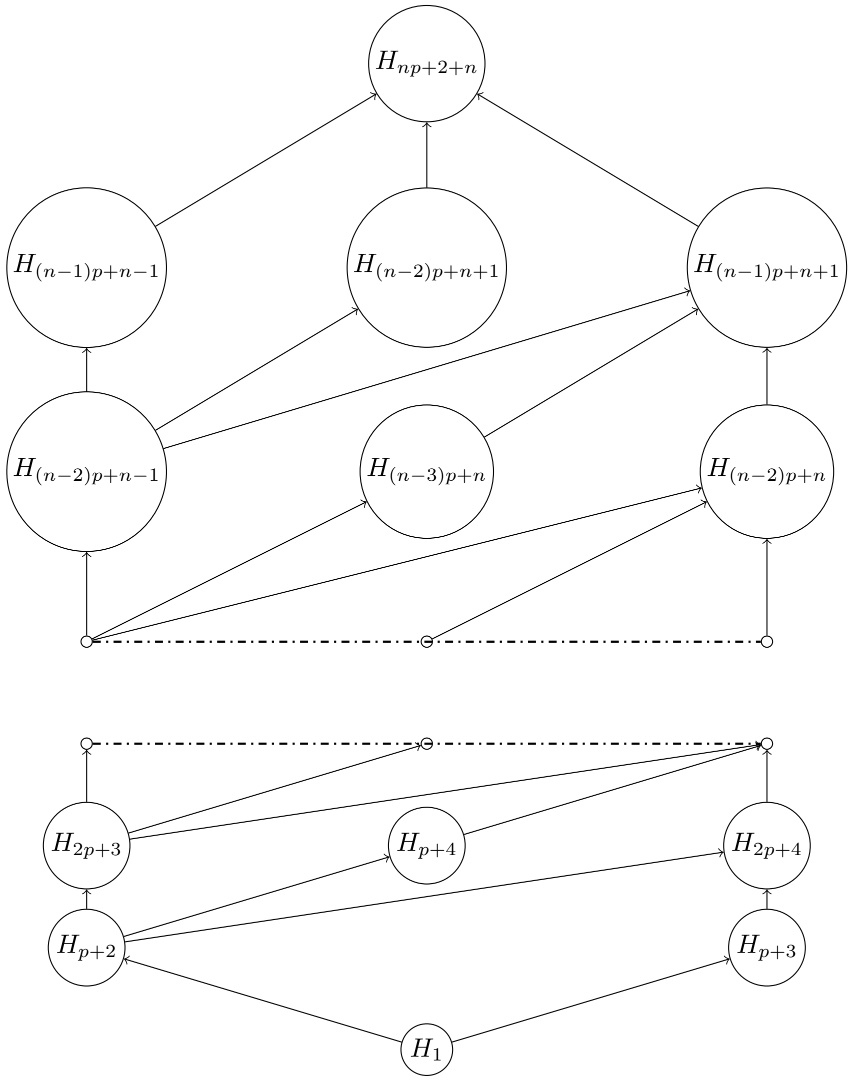

Fig-2 Lattices of characteristic subgroups when p is even prime

6. Further Research

References

1. Admasu, F.S. and Sehgal, A.(2021). Counting subgroups of fixed order in finite abelian groups.*Journal of Discrete Mathematical Sciences and Cryptography*, 24(1), 263-276.
2. Baer, R., (1939). The Significance of the System of Subgroups for the Structure of the Group. *American Journal of Mathematics*, 61(1), 1-44.
3. Kerby , B.L. and Rode , E. (2011). Characteristic subgroups of finite abelian groups *Communications in Algebra*, 39(4), 1315 – 1343 .

4. Sehgal, A.and Kumar, Y.,(2013). On Number of Subgroups of finite Abelian Group $\mathbb{Z}_m \otimes \mathbb{Z}_n$. *International Journal of Algebra*, 7(19), 915-923.
5. Sehgal, A., Sehgal, S. and Sharma, P.K.,(2015). The number of subgroups of a finite abelian p-group of rank two. *Journal for Algebra and Number theory Academia*, 5(1), 23-31.
6. Sehgal, A., Sehgal, S. and Sharma, P.K.,(2015). The number of automorphism of a finite abelian group of rank two. *Journal of Discrete Mathematical Sciences and Cryptography*, 19(1), 163-171.
7. Sehgal, A. and Jakhar, M.,(2017). Characteristic Subgroups of a finite Abelian Group $Z_n \times Z_n$. *Annals of Pure and Applied Mathematics*, 14(1), 119-123.
8. Gallian, J. A., *Contemporary Abstract Algebra*, Norosa, 1999.

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