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OUTPUT FEEDBACK BASED NON-FRAGILE FAULT TOLERANT CONTROL FOR NETWORKED SYSTEMS WITH RANDOM TRANSMISSION DELAYS AND PACKET LOSS

A. ARUNKUMAR, HUGO LEIVA*, V. DHANYA, K. P. SRIDHAR, AND A. TRIDANE

ABSTRACT. This paper deals with non-fragile fault-tolerant control for discrete-time networked control systems (NCS) with data packet dropout and transmission delays induced by communication channels. We model the discrete-time NCSs with data packet dropout and transmission delays which are assumed to be randomly time-varying in Bernoulli distributed sequences. This paper's main objective is to design a non-fragile fault tolerant output feedback controller such that for all admissible uncertainties and actuator failure cases, the resulting closed-loop form of considered NCS is robustly asymptotically stable. All the conditions are established in linear matrix inequality (LMI), easily solved using standard numerical software. Finally, we give four numerical examples with their simulation results, where we show the illustrations of our theoretical findings. We also validate and compare our results with existing literature of the proposed non-fragile reliable control scheme.

1. Introduction

The NCS is a control system in which plants, sensors, controllers, and actuators can be interconnected via communication networks [4, 7, 8, 9, 28]. The NCS plays an essential role in computer and networking technologies. It has found successful applications in various research areas such as robotic manipulators, spacecraft, human surveillance systems, vehicle industry, aerospace systems, traffic systems, mobile robots,... just mention some applications. In recent years, NCS has become more significant and has many outstanding advantages of the network architectures, including reduced system wiring, plug-and-play devices, increased system agility, cost-effectiveness, simplicity in installation, maintenance, and high reliability. However, due to the network's advantages, major issues in NCS have been raised due to the effects of network-induced delays and data packet dropouts on the system performance. The network induced is a time-varying delay that affects the accuracy of timing-dependent computing and can degrade the control performance. The data packet dropout leads to complete information of the NCS becomes unavailable. In both cases, the controller or actuator has to decide, with incomplete information, what control signals to output. From these problems,

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many existing control technologies may become infeasible for specific networked control applications. Therefore, the stability analysis and control design of the NCS with a network-induced delay and packet dropout has attracted much attention, and, subsequently, several papers published to investigate this problem, see [20, 22, 23]. Truong and Ahn [21] proposed a robust variable sampling period controller for a network control system with random time delays and packet losses. In [13], Peng et al. studied the problem of output feedback stabilization of control systems, for a discrete-time state-space, with a network-induced delay and packet dropout using Lyapunov's stability theory and LMI approach. Liu and Yang [10] developed a new dynamic output feedback controller design for NCS subject to communication delays and an event-triggered scheme.

The instability and the performance deterioration of NCS due to the time-varying delay were investigated by many authors [12, 17, 27]. Another source of performance degradation in NCS is uncertainty. Thus, the description of NCS inevitably contains modeling errors and changing environments which can also affect the stability and performance of the NCS [15]. Therefore, robust control for NCS subject to time delay and uncertainties is becoming a vital research topic. Particularly with the rapid development of the LMI approach. In fact, there is an increased number of results on various types of systems with time delays and parameter uncertainties [4, 10]. Sakthivel et al. [18] derived the reliable, robust stabilization problem for a class of uncertain Takagi-Sugeno fuzzy systems together with randomly occurring time delays.

In practical systems, the actuators aging, zero shift, electromagnetic interference, and nonlinear amplification frequency in different fields are the sources of actuator faults in the system model [1, 25]. In particular, when the fault occurs in the dynamical system, the conventional controller will become conservative and may not satisfy certain control performance indexes, then the closed-loop system becomes unstable. Therefore, a high degree of fault tolerance control is essential for the systems' overall better performance. We should mention that a reliable control system can automatically accommodate system failures and maintain the overall system stability and acceptable performance when existing some abnormal actuators failures in the system model [18]. Based on Lyapunov function techniques, the problem of a robust, reliable controller for vehicle suspension systems by using an input delay approach in terms of LMI has been reported in [15].

Recently, the non-fragile control issue is considered for some real-time systems. It is shown that without considering the relatively small uncertainties in controller implementation, the robust controller design could even make the closed-loop system unstable. Such controllers are often named "fragile". Therefore, it is necessary and essential that any controllers in the system should be able to tolerate some level of controller gain variations, and we ensure that the closed-loop system maintains the stability and performance level. In this case, the non-fragile control concept is how to design a feedback control that will be insensitive to some error in gains of feedback loop [11, 17, 23, 26]. Very recently, Liu et al. [11] discussed non-fragile H_∞ filter design for a class of continuous time delayed Takagi-Sugeno fuzzy systems with randomly occurring gain variations to the implementation of the filter. Using Lyapunov stability theory, the authors in Zhang et al. [26] addressed non-fragile distributed filtering for fuzzy systems with a multiplicative gain variation.

Sampled-data control theory has many applications in systems and control areas. A periodic clock drives a sampled-data controller, and on each clock edge, it samples its inputs, changes state, and updates its outputs. Thus, with the rapid development of computer hardware, the sampled-data control technology has shown superiority over other control approaches [16]. Peng et al. [12] discussed event-triggered output-feedback H_∞ control for NCS with the time-varying delay with a network-induced delay and packet dropout in the sampling period. Robust variable control for networked control systems based on sampling period techniques has been obtained in Truong and Ahn [21]. However, to the best of the authors' knowledge, no result has been reported on asymptotic stabilization for the uncertain NCS under reliable non-fragile sampled-data control with random delay.

This paper focuses on a new class of non-fragile reliable discrete-time uncertain NCS with time-varying delays. The effect of both variation range and distribution probability of the time delay is taken into account in the proposed approach, which is mainly different from the traditional methods and will lead to less conservative results. Our results take some well-studied models as special cases. We translate the distribution probability of the time delay into parameter matrices of the transferred systems. By implementing novel Lyapunov-Krasovskii functional together with LMI approach, a robust control law is derived which guarantees the asymptotical stability of the NCS with random delays about its equilibrium point for all admissible uncertainties. We formulate our results in terms of LMI, which the MATLAB LMI toolbox can easily verify. Finally, a numerical example with simulation results illustrates the effectiveness and less conservativeness of the obtained results.

2. Model of an NCS with random packet dropout and transmission delays

Due to limited bandwidth, the data packet dropout is unavoidable in NCS. When packet collision occurs, it is better to drop the old packet and transmit a new one rather than repeating the transmission attempt to yield more advantages. We consider the transmission delays induced by the network, besides the data packet dropout problem. ρ_{sc} and ρ_{ca} denote transmission delay in sensor- controller channel and controller-actuator channel, respectively; d_{sc} and d_{ca} denote the number of packet dropouts in the sensor-controller channel and controller-actuator channel, respectively. These four delays can be combined if the feedback controller is static.

The network plant with data packet dropout and transmission delays are shown in Fig. 1, where the plant is described by the following discrete-time networked system model:

$$\begin{aligned} x((k+1)h) &= Ax(kh) + Bu^f(kh), \\ y(kh) &= Hx(kh) \end{aligned} \tag{2.1}$$

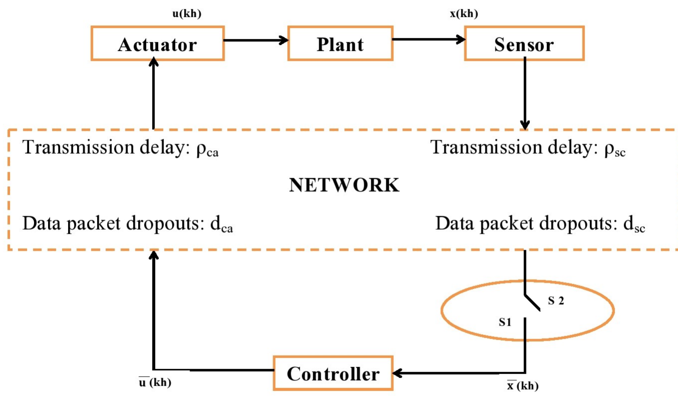


FIGURE 1. An NCS with data packet dropout and transmission delays.

where $x(kh) \in \mathbb{R}^n$ is the state vector; $u^f(kh) \in \mathbb{R}^w$ is the reliable control signal of the NCS; $y(kh) \in \mathbb{R}^m$ is the output vector; A and B are constant matrices; H is a nonsingular matrix with an appropriate dimension; h is a positive constant scalar and denotes sampling period.

The controller and the actuators are event-driven, whereas the sensors are time-driven. i.e., the controller and the actuators act when the new data arrives, whereas the sensors measure the states at each sampling instants. The reliable control input is written as

$$u^f(kh) \quad = \quad Gu(kh) = G\overline{u}(kh - \rho_{ca} - d_{ca}h) \;= G\hat{K}\overline{y}(kh - \rho_{ca} - d_{ca}h) \tag{2.2}$$

where G is the actuator fault matrix; $\widehat{K} = K + \Delta K(k)$, K is the state feedback gain to be designed and $\Delta K(k)$ is a priori norm bounded gain variation. Here, it is assumed that the gain variation $\Delta K(k)$ has the structure $\Delta K(k) = MF(k)N$, where M and N being known constant matrices; $F(k)$ the uncertain parameter matrix satisfying $F^T(k)F(k) \leq I$ [17].

The sampling and the transmission are taken by the sensor in Fig. 1. Thus the delayed sampling value of the state is the output of the network. The network is modeled as a switch. The network packet (containing $x(k_t)$) is transmitted, and the controller uses the updated data if the switch is closed (in position S1), whereas the packet is lost, and the controller uses the old data if the switch is open (in position $S2$). The maximum quantity of packet loss that does not destabilize the closed-loop system for a fixed sampling period. The dynamics of the switch at time k_t can be expressed as follows:

$$\overline{y}(kh) = \begin{cases} y(k_th - \rho_{sc} - d_{sc}h), & if\ NCS\ (2.1)-(2.2)\ with\ no\ packet\ dropout \\ y(k_th - \rho_{sc} - d_{sc}h - h), & if\ NCS\ (2.1)-(2.2)\ with\ one\ packet\ dropout \\ \vdots \\ y(k_th - \rho_{sc} - d_{sc}h - \varsigma(k)h), & if\ NCS\ (2.1)-(2.2)\ with\ \varsigma(k)\ packet\ dropout \end{cases}$$

The quantity of dropped packets is accumulated from the latest time when $\overline{y}(kh)$ has been updated.

Thus the closed-loop nonfragile fault tolerent NCS with transmission delays and network packet loss effects is described as

$$x((k+1)h) = Ax(kh) + BG\hat{K}y(k_th - \rho_{ca} - \rho_{sc} - d_{ca}h - d_{sc}h - \varsigma(k)h). \quad (2.3)$$

For simplicity's sake, we omit h and let $\rho(k) = k - k_th + \rho_{ca} + \rho_{sc} + d_{ca}h + d_{sc}h + \varsigma(k)h$. Then the formulation of the non-fragile reliable NCS is as follows

$$\begin{aligned} x(k+1) &= Ax(k) + BG\hat{K}y(k - \rho(k)), \\ x(k) &= \Phi(k),\ k = -\rho_M, -\rho_M + 1, \cdots, 0 \end{aligned} \quad (2.4)$$

where $\Phi(k)$ is a given initial condition sequence.
Naturally, $\rho(k)$ also satisfies $0 \le \rho_m \le \rho(k) \le \rho_M < \infty$, where ρ_m and ρ_M representing the minimum and maximum network allowable equivalent delays respectively.
Then the closed-loop NCS with input delays,

$$x(k+1) = \bar{A}(k)x(k) + BG\widehat{K}Hx(k - \rho(k)) \quad (2.5)$$

is obtained in consideration of the non-fragile reliable NCS in (2.4) with the norm-bounded parameter uncertainties and the output feedback controller signal, where $\bar{A}(k) = A + \wp A(k)$; $\wp A(k)$ is time varying matrices representing parametric uncertainties, and described by $\wp A(k) = W\wp(k)N_a$ where W and N_a are known constant matrices of appropriate dimensions; $\wp(k)$ is known time varying matrix with Lebesgue measurable elements bounded by $\wp^T(k)\wp(k) \le I$.
Assumption 1: The time delay $\rho(k)$ is bounded, $0 \le \rho_m \le \rho(k) \le \rho_M$, and its probability distribution can be observed. i.e., suppose $\rho(k)$ takes values in $[\rho_m : \rho_0]$ or $(\rho_0 : \rho_M]$ and Prob$\{\rho(k) \in [\rho_m : \rho_0]\} = \delta_0$, where $\rho_m \le \rho_0 < \rho_M$ and $0 \le \delta_0 \le 1$.

Remark 2.1. The binary stochastic variable was first introduced in [14] and then successfully used in [1]. Under **Assumption 1**, we know that δ_0 is dependent on the values of ρ_m, ρ_0 and ρ_M. In addition, Prob$\{\rho(k) \in (\rho_0 : \rho_M]\} = 1 - \delta_0 = \bar{\delta}_0$,

In order to describe the probability distribution of the time varying delay, we define two sets

$$\mathbb{C}_1 = \{k|\rho(k) \in [\rho_m : \rho_0]\} \quad \text{and} \quad \mathbb{C}_2 = \{k|\rho(k) \in (\rho_0 : \rho_M]\} \quad (2.6)$$

where ρ_0 is an integer satisfying $\rho_m \le \rho_0 \le \rho_M$. Define two mapping functions

$$\rho_1(k) = \begin{cases} \rho(k), & k \in \mathbb{C}_1 \\ \rho_m, & else \end{cases} \quad \text{and} \quad \rho_2(k) = \begin{cases} \rho(k), & k \in \mathbb{C}_2 \\ \rho_0, & else \end{cases} \quad (2.7)$$

It follows from (2.6) that $\mathbb{C}_1 \cup \mathbb{C}_2 = \mathbb{Z}_{\geq 0}$ and $\mathbb{C}_1 \cap \mathbb{C}_2 = \Phi$, where Φ is an empty set. It is easy to check that $k \in \mathbb{C}_1$ implies the event $\rho(k) \in [\rho_m, \rho_0]$ occurs and $k \in \mathbb{C}_2$ implies the event $\rho(k) \in (\rho_0, \rho_M]$ occurs.
Defining a stochastic variable as

$$\delta(k) = \begin{cases} 1, & k \in \mathbb{C}_1 \\ 0, & k \in \mathbb{C}_2 \end{cases} \tag{2.8}$$

The nonfragile fault tolerant uncertain NCS (2.5) can be equivalently rewritten as

$$x(k+1) = \bar{A}(k)x(k) + \delta(k)BG\widehat{K}Hx(k-\rho_1(k)) + (1-\delta(k))BG\widehat{K}Hx(k-\rho_2(k)) \tag{2.9}$$

Remark 2.2. Under **Assumption 1** and (2.8), we see that $\delta(k)$ is a Bernoulli distributed white sequence with $\text{Prob}\{\delta(k)=1\} = \mathbb{E}\{\delta(k)\} = \delta_0$ and $\text{Prob}\{\delta(k) = 0\} = \mathbb{E}\{\delta(k)\} = 1-\delta_0$. In addition, we can show that $\mathbb{E}\{\delta(k)-\delta_0\} = 0$ and $\mathbb{E}\{(\delta(k)-\delta_0)^2\} = \delta_0(1-\delta_0)$.

Further, for notation convenience we write $x(k),\ x(k+i),\ x(k-\rho_1(k)),\ x(k-\rho_2(k)),\ \rho(k),\ \rho_1(k),$
$\rho_2(k),\ \delta(k),\ 1-\delta(k)$ as $x_k,\ x_{k+i},\ x_{\rho,1},\ x_{\rho,2},\ \rho_k,\ \rho_{k,1},\ \rho_{k,2},\ \delta_k$ and $\overline{\delta_k}$ respectively. Then (2.9) can be rearranged as,

$$x_{k+1} \;=\; \bar{A}(k)x_k + \delta_k BG\widehat{K}Hx_{\rho,1} + \overline{\delta_k}BG\widehat{K}Hx_{\rho,2} \tag{2.10}$$

Before we give our main results, we need the following lemmas, which we will use to prove the coming theorems.

Lemma 2.3. [2] *For any symmetric constant matrix $Q \in \mathbb{R}^{n\times n}, Q \geq 0$, two scalars ρ_m and ρ_M satisfying $\rho_m \leq \rho_M$, a vector valued function $\eta(k) = x(k+1)-x(k)$, the following sums are well defined and it holds:*

$$\begin{aligned}
\sum_{s=k-\rho_M}^{k-\rho_m-1} \eta_S^T Q \eta_S &\leq \frac{-1}{\rho_M-\rho_m}\left[\sum_{s=k-\rho_M}^{k-\rho_m-1}\eta(t)\right]^T Q\left[\sum_{s=k-\rho_M}^{k-\rho_m-1}\eta(t)\right],\\
\sum_{s=-\rho_M}^{-\rho_m-1}\sum_{s=k+j}^{k-1} \eta_S^T Q \eta_S &\leq \frac{-2}{(\rho_M-\rho_m)(\rho_M+\rho_m+1)} \times \\
&\quad \left[\sum_{s=-\rho_M}^{-\rho_m-1}\sum_{s=k+j}^{k-1}\eta(t)\right]^T Q\left[\sum_{s=-\rho_M}^{-\rho_m-1}\sum_{s=k+j}^{k-1}\eta(t)\right].
\end{aligned}$$

Lemma 2.4. [3] *Given constant matrices $\Omega_1,\ \Omega_2,\ \Omega_3$, where $\Omega_1 = \Omega_1^T > 0$ and $\Omega_2 = \Omega_2^T > 0$. Then $\Omega_1 + \Omega_3^T\Omega_2^{-1}\Omega_3 < 0$ if and only if* $\begin{bmatrix} \Omega_1 & \Omega_3^T \\ \Omega_3 & -\Omega_2 \end{bmatrix} < 0.$

Lemma 2.5. [3] *Let $D,\ N$ and $F(k)$ be the real matrices of appropriate dimensions with $F(k)$ satisfying $F^T(k)F(k) \leq I$. Then we have the following inequalities holds:*

(i) *for* $\epsilon > 0,\ DF(k)N + N^TF^T(k)D^T \leq \epsilon^{-1}DD^T + \epsilon N^TN$
(ii) *for* $\epsilon > 0$ *and* $P - \epsilon DD^T > 0$,

$$(A+DF(k)N)P(A+DF(k)N)^T \leq A^T(P^{-1}-\epsilon DD^T)^{-1}A + \epsilon^{-1}N^TN.$$

3. Non-Fragile known fault-tolerant control design

In this section, we study the asymptotical stabilization of discrete time NCS without uncertainty, when the actuator fault matrix G is exactly known and in the absence and the presence of non-fragile control. For this consideration, the nominal form of the NCS (2.10) is as follows

$$x_{k+1} = Ax_k + \delta_0 BG\widehat{K}Hx_{\rho,1} + \overline{\delta_0}BG\widehat{K}Hx_{\rho,2} \tag{3.1}$$

Theorem 3.1. *The discrete time NCS* (3.1) *is asymptotically stable with known actuator failure parameter matrix G, the output feedback reliable control and $\Delta(k) = 0$ if there exist symmetric matrices $X > 0$, $\widehat{R}_n > 0$, $\widehat{S}_n > 0$, $\widehat{Q}_m = \begin{bmatrix} \widehat{Q}_{m1} & \widehat{Q}_{m2} \\ * & \widehat{Q}_{m3} \end{bmatrix} > 0$, any matrices M_{mn}, $(m = 1, \cdots, 6, n = 1, 2, 3, 4)$ and matrix Y with appropriate dimensions, such that the following LMI holds,*

$$\hat{\Omega} = \begin{bmatrix} \hat{\Omega}_{16,16} & \sqrt{\rho_{11}}\widehat{M}_1 & \sqrt{\rho_{11}}\widehat{M}_2 & \sqrt{\rho_0}\widehat{M}_3 & \sqrt{\rho_{21}}\widehat{M}_4 & \sqrt{\rho_{21}}\widehat{M}_5 & \sqrt{\rho_M}\widehat{M}_6 \\ * & -(\widehat{R}_1 + \widehat{R}_2) & 0 & 0 & 0 & 0 & 0 \\ * & * & -\widehat{R}_1 & 0 & 0 & 0 & 0 \\ * & * & * & -\widehat{R}_2 & 0 & 0 & 0 \\ * & * & * & * & -(\widehat{R}_3 + \widehat{R}_4) & 0 & 0 \\ * & * & * & * & * & -\widehat{R}_3 & 0 \\ * & * & * & * & * & * & -\widehat{R}_4 \end{bmatrix} < 0, \tag{3.2}$$

$$\begin{aligned}
\hat{\Omega}_{1,1} &= \widehat{Q}_{11} + \widehat{Q}_{21} + \widehat{Q}_{31} + \widehat{Q}_{41} + \widehat{Q}_{51} + \widehat{Q}_{61} + \rho_{11}\widehat{Q}_{11} + \rho_{21}\widehat{Q}_{41} - 2\frac{\rho_{11}^2}{\rho_{12}}\widehat{S}_1 - 2\frac{\rho_0^2}{\rho_{13}}\widehat{S}_2 \\
&\quad -2\frac{\rho_{21}^2}{\rho_{22}}\widehat{S}_3 - 2\frac{\rho_M^2}{\rho_{23}}\widehat{S}_4 + 2\lambda_1 AX - 2\lambda_1 X + 2\widehat{M}_{31} + 2\widehat{M}_{61},\ \hat{\Omega}_{1,2} = 2\widehat{M}_{21}, \\
\hat{\Omega}_{1,3} &= 2\lambda_1\delta_0 BGY + 2XA^T\lambda_2 - 2\lambda_2 X + 2\widehat{M}_{11} - 2\widehat{M}_{21} + 2\widehat{M}_{32}^T - 2\widehat{M}_{31} + 2\widehat{M}_{62}^T, \\
\hat{\Omega}_{1,4} &= 2\lambda_1\overline{\delta_0}BGY + 2XA^T\lambda_3 - 2X\lambda_3 + 2\widehat{M}_{33}^T + 2\widehat{M}_{41} - 2\widehat{M}_{51}^T + 2\widehat{M}_{63}^T - 2\widehat{M}_{61}, \\
\hat{\Omega}_{1,5} &= -2\widehat{M}_{11} + 2\widehat{M}_{51},\ \hat{\Omega}_{1,6} = -2\widehat{M}_{41},\ \hat{\Omega}_{1,7} = 2X + 2\widehat{Q}_{12} + 2\widehat{Q}_{22} + 2\widehat{Q}_{32} + 2\widehat{Q}_{42} \\
&\quad +2\widehat{Q}_{52} + 2\widehat{Q}_{62} + 2\rho_{11}\widehat{Q}_{12} + 2\rho_{21}\widehat{Q}_{42} - 2X\lambda_1 + 2XA^T\lambda_4 - 2\lambda_4 X + 2\widehat{M}_{34}^T + 2\widehat{M}_{64}^T, \\
\hat{\Omega}_{1,13} &= 4\frac{\rho_{11}}{\rho_{12}}\widehat{S}_1,\ \hat{\Omega}_{1,14} = 4\frac{\rho_0}{\rho_{13}}\widehat{S}_2,\ \hat{\Omega}_{1,15} = 4\frac{\rho_{21}}{\rho_{22}}\widehat{S}_3,\ \hat{\Omega}_{1,16} = 4\frac{\rho_M}{\rho_{23}}\widehat{S}_4,\ \hat{\Omega}_{2,2} = -\widehat{Q}_{21}, \\
\hat{\Omega}_{2,3} &= 2\widehat{M}_{22}^T,\ \hat{\Omega}_{2,4} = 2\widehat{M}_{23}^T,\ \hat{\Omega}_{2,7} = 2\widehat{M}_{24}^T,\ \hat{\Omega}_{2,8} = -2\widehat{Q}_{22},\ \hat{\Omega}_{3,3} = -\widehat{Q}_{11} + 2\lambda_2\delta_0 BGY \\
&\quad +2\widehat{M}_{12} - 2\widehat{M}_{22} - 2\widehat{M}_{32},\ \hat{\Omega}_{3,4} = 2\lambda_2\overline{\delta_0}BGY + (\lambda_3\delta_0 BGY)^T + 2\widehat{M}_{13}^T \\
&\quad -2\widehat{M}_{23}^T - 2\widehat{M}_{33}^T + 2\widehat{M}_{42} - 2\widehat{M}_{52} - 2\widehat{M}_{62},\ \hat{\Omega}_{3,5} = -2\widehat{M}_{12} + 2\widehat{M}_{52},\ \hat{\Omega}_{3,6} = -2\widehat{M}_{42}, \\
\hat{\Omega}_{3,7} &= -2\lambda_2 X + 2(\lambda_4\delta_0 BGY)^T + 2\widehat{M}_{14}^T - 2\widehat{M}_{24}^T - 2\widehat{M}_{34}^T,\ \hat{\Omega}_{3,9} = -2\widehat{Q}_{12}, \\
\hat{\Omega}_{4,4} &= -\widehat{Q}_{41} + 2\lambda_3\overline{\delta_0}BGY + 2\widehat{M}_{43} - 2\widehat{M}_{53} - 2\widehat{M}_{63},\ \hat{\Omega}_{4,5} = -2\widehat{M}_{13} + 2\widehat{M}_{53}, \\
\hat{\Omega}_{4,6} &= -2\widehat{M}_{43},\ \hat{\Omega}_{4,7} = -2\lambda_3 X + 2(\lambda_4\overline{\delta_0}BGY)^T + 2\widehat{M}_{44}^T - 2\widehat{M}_{54}^T - \widehat{M}_{64}^T, \\
\hat{\Omega}_{4,10} &= -2\widehat{Q}_{42},\ \hat{\Omega}_{5,5} = -\widehat{Q}_{31} - \widehat{Q}_{51},\ \hat{\Omega}_{5,7} = -2\widehat{M}_{14}^T + 2\widehat{M}_{54}^T,\ \hat{\Omega}_{5,11} = -2\widehat{Q}_{32} - 2\widehat{Q}_{52},
\end{aligned}$$

$$\begin{aligned}
\hat{\Omega}_{6,6} &= -\widehat{Q}_{61},\ \hat{\Omega}_{6,7} = -2\widehat{M}_{44}^T,\ \hat{\Omega}_{6,12} = -2\widehat{Q}_{62},\ \hat{\Omega}_{7,7} = \widehat{Q}_{13} + \widehat{Q}_{23} + \widehat{Q}_{33} + \widehat{Q}_{43}\\
&\quad + \widehat{Q}_{53} + \widehat{Q}_{63} + \rho_{11}\widehat{Q}_{13} + \rho_{21}\widehat{Q}_{43} + \rho_{11}\widehat{R}_1 + \rho_0\widehat{R}_2 + \rho_{21}\widehat{R}_3 + \rho_M\widehat{R}_4 + \frac{\rho_{12}}{2}\widehat{S}_1\\
&\quad + \frac{\rho_{13}}{2}\widehat{S}_2 + 2\frac{\rho_{22}}{2}\widehat{S}_3 + 2\frac{\rho_{23}}{2}\widehat{S}_4 + X - 2\lambda_4 X,\ \hat{\Omega}_{8,8} = -\widehat{Q}_{23},\ \hat{\Omega}_{9,9} = -\widehat{Q}_{13},\\
\hat{\Omega}_{10,10} &= -\widehat{Q}_{43},\ \hat{\Omega}_{11,11} = -\widehat{Q}_{33} - \widehat{Q}_{53},\ \hat{\Omega}_{12,12} = -\widehat{Q}_{63},\ \hat{\Omega}_{13,13} = -\frac{2}{\rho_{12}}\widehat{S}_1,\\
\hat{\Omega}_{14,14} &= -\frac{2}{\rho_{13}}\widehat{S}_2,\ \hat{\Omega}_{15,15} = -\frac{2}{\rho_{22}}\widehat{S}_3,\ \hat{\Omega}_{16,16} = -\frac{2}{\rho_{23}}\widehat{S}_4,\\
\hat{M}_i &= \begin{bmatrix} \hat{M}_{i1} & 0 & \hat{M}_{i2} & \hat{M}_{i3} & 0_{2n} & \hat{M}_{i4} & 0_{9n} \end{bmatrix},\ i = 1, \cdots, 6,\ \rho_{11} = \rho_0 - \rho_m,\\
\rho_{12} &= \rho_{11}(\rho_0 + \rho_m + 1),\ \rho_{13} = \rho_0(\rho_0 + 1),\ \rho_{21} = \rho_M - \rho_0,\ \rho_{22} = \rho_{21}(\rho_M + \rho_0 + 1),\\
\rho_{23} &= \rho_M(\rho_M + 1),
\end{aligned}$$

and the other parameters are zero. In this case, the output feedback reliable controller gain is given by $K = YX^{-1}H^{-1}$.

Proof. In order to obtain the asymptotically stable result for (3.1), we choose piece-wise Lyapunov-Krasovskii functional candidate $V(x_k, k)$ as,

$$V(x_k, k) = \sum_{n=1}^{8} V_n(x_k, k), \tag{3.3}$$

where,

$$\begin{aligned}
V_1(x_k, k) &= x_k^T P x_k,\\
V_2(x_k, k) &= \sum_{s=k-\rho_{k,1}}^{k-1} \lambda_s^T Q_1 \lambda_s + \sum_{s=k-\rho_m}^{k-1} \lambda_s^T Q_2 \lambda_s + \sum_{s=k-\rho_0}^{k-1} \lambda_s^T Q_3 \lambda_s,\\
V_3(x_k, k) &= \sum_{s=k-\rho_{k,2}}^{k-1} \lambda_s^T Q_4 \lambda_s + \sum_{s=k-\rho_0}^{k-1} \lambda_s^T Q_5 \lambda_s + \sum_{s=k-\rho_M}^{k-1} \lambda_s^T Q_6 \lambda_s,\\
V_4(x_k, k) &= \sum_{s=-\rho_0+1}^{-\rho_m} \sum_{j=k+s}^{k-1} \lambda_j^T Q_1 \lambda_j + \sum_{s=-\rho_M+1}^{-\rho_0} \sum_{j=k+s}^{k-1} \lambda_j^T Q_4 \lambda_j,\\
V_5(x_k, k) &= \sum_{s=-\rho_0}^{-\rho_m-1} \sum_{j=k+s}^{k-1} \eta_j^T R_1 \eta_j + \sum_{s=-\rho_0}^{-1} \sum_{j=k+s}^{k-1} \eta_j^T R_2 \eta_j,\\
V_6(x_k, k) &= \sum_{s=-\rho_M}^{-\rho_0-1} \sum_{j=k+s}^{k-1} \eta_j^T R_3 \eta_j + \sum_{s=-\rho_M}^{-1} \sum_{j=k+s}^{k-1} \eta_j^T R_4 \eta_j,\\
V_7(x_k, k) &= \sum_{l=-\rho_0}^{-\rho_m-1} \sum_{j=l}^{-1} \sum_{s=k+j}^{k-1} \eta_s^T S_1 \eta_s + \sum_{l=-\rho_0}^{-1} \sum_{j=l}^{-1} \sum_{s=k+j}^{k-1} \eta_s^T S_2 \eta_s,\\
V_8(x_k, k) &= \sum_{l=-\rho_M}^{-\rho_0-1} \sum_{j=l}^{-1} \sum_{s=k+j}^{k-1} \eta_s^T S_3 \eta_s + \sum_{l=-\rho_M}^{-1} \sum_{j=l}^{-1} \sum_{s=k+j}^{k-1} \eta_s^T S_4 \eta_s,
\end{aligned}$$

with $\lambda_k^T = \begin{bmatrix} x_k^T & \eta_k^T \end{bmatrix}$ and $\eta_k = x_{k+1} - x_k$. Let us define the forward difference of $V_n(x_k, k)$ as $\Delta V_n(x_k, k) = V_n(x_{k+1}, k+1) - V_n(x_k, k)$. Then we have,

$$\Delta V_1(x_k,k) = x_{k+1}^T P x_{k+1} - x_k^T P x_k \tag{3.4}$$

$$\begin{aligned}
\Delta V_2(x_k,k) &= \sum_{s=k+1-\rho_{(k+1),1}}^{k} \lambda_s^T Q_1 \lambda_s - \sum_{s=k-\rho_{k,1}}^{k-1} \lambda_s^T Q_1 \lambda_s + \sum_{s=k+1-\rho_m}^{k} \lambda_s^T Q_2 \lambda_s \\
&\quad - \sum_{s=k-\rho_m}^{k-1} \lambda_s^T Q_2 \lambda_s + \sum_{s=k+1-\rho_0}^{k} \lambda_s^T Q_3 \lambda_s - \sum_{s=k-\rho_0}^{k-1} \lambda_s^T Q_3 \lambda_s, \\
&= \lambda_k^T \Big(Q_1 + Q_2 + Q_3 \Big) \lambda_k - \lambda_{\rho,1} Q_1 \lambda_{\rho,1} - \lambda_{\rho,m} Q_2 \lambda_{\rho,m} - \lambda_{\rho,0} Q_3 \lambda_{\rho,0} \\
&\quad + \sum_{s=k+1-\rho_{(k+1),1}}^{k-\rho_m} \lambda_S^T Q_1 \lambda_S, \qquad (3.5)
\end{aligned}$$

$$\begin{aligned}
\Delta V_3(x_k,k) &= \sum_{s=k+1-\rho_{(k+1),2}}^{k} \lambda_s^T Q_4 \lambda_s - \sum_{s=k-\rho_{k,2}}^{k-1} \lambda_s^T Q_4 \lambda_s + \sum_{s=k+1-\rho_0}^{k} \lambda_s^T Q_5 \lambda_s \\
&\quad - \sum_{s=k-\rho_0}^{k-1} \lambda_s^T Q_5 \lambda_s + \sum_{s=k+1-\rho_M}^{k} \lambda_s^T Q_6 \lambda_s - \sum_{s=k-\rho_M}^{t-1} \lambda_s^T Q_6 \lambda_s, \\
&= \lambda_k^T \Big(Q_4 + Q_5 + Q_6 \Big) \lambda_k - \lambda_{\rho,2} Q_4 \lambda_{\rho,2} - \lambda_{\rho,0} Q_5 \lambda_{\rho,0} - \lambda_{\rho,M} Q_6 \lambda_{\rho,M} \\
&\quad + \sum_{s=k+1-\rho_{(k+1),2}}^{k-\rho_0} \lambda_S^T Q_4 \lambda_S, \qquad (3.6)
\end{aligned}$$

$$\begin{aligned}
\Delta V_4(x_k,k) &= \sum_{s=-\rho_0+1}^{-\rho_m} \Bigg[\sum_{j=k+1+s}^{k} \lambda_j^T Q_1 \lambda_j - \sum_{j=k+s}^{k-1} \lambda_j^T Q_1 \lambda_j \Bigg] \\
&\quad + \sum_{s=-\rho_M+1}^{-\rho_0} \Bigg[\sum_{j=k+1+s}^{k} \lambda_j^T Q_4 \lambda_j - \sum_{j=k+s}^{k-1} \lambda_j^T Q_4 \lambda_j \Bigg], \\
&= \sum_{s=-\rho_0+1}^{-\rho_m} \Big[\lambda_k^T Q_1 \lambda_k - \lambda_{k+s}^T Q_1 \lambda_{k+s} \Big] + \sum_{s=-\rho_M+1}^{-\rho_0} \Big[\lambda_k^T Q_4 \lambda_k - \lambda_{k+s}^T Q_4 \lambda_{k+s} \Big], \\
&= \lambda_k^T \big(\rho_{11} Q_1 + \rho_{21} Q_4 \big) \lambda_k - \sum_{s=k-\rho_0+1}^{k-\rho_m} \lambda_s^T Q_1 \lambda_s - \sum_{s=k-\rho_M+1}^{k-\rho_0} \lambda_s^T Q_4 \lambda_s, \qquad (3.7)
\end{aligned}$$

$$\begin{aligned}
\Delta V_5(x_k,k) &= \sum_{s=-\rho_0}^{-\rho_m-1} \Bigg[\sum_{j=k+1+s}^{k} \eta_j^T R_1 \eta_j - \sum_{j=k+s}^{k-1} \eta_j^T R_1 \eta_j \Bigg] \\
&\quad + \sum_{s=-\rho_0}^{-1} \Bigg[\sum_{j=k+1+s}^{k} \eta_j^T R_2 \eta_j - \sum_{j=k+s}^{k-1} \eta_j^T R_2 \eta_j \Bigg], \\
&= \sum_{s=-\rho_0}^{-\rho_m-1} \Big[\eta_k^T R_1 \eta_k - \eta_{k+s}^T R_1 \eta_{k+s} \Big] + \sum_{s=-\rho_0}^{-1} \Big[\eta_k^T R_2 \eta_k - \eta_{k+s}^T R_2 \eta_{k+s} \Big], \\
&= \eta_k^T \big(\rho_{11} R_1 + \rho_0 R_2 \big) \eta_k - \sum_{s=k-\rho_0}^{k-\rho_m-1} \eta_s^T R_1 \eta_s - \sum_{s=k-\rho_0}^{k-1} \eta_s^T R_2 \eta_s, \qquad (3.8)
\end{aligned}$$

$$\begin{aligned}
\Delta V_6(x_k,k) &= \sum_{s=-\rho_M}^{-\rho_0-1}\Big[\sum_{j=k+1+s}^{k}\eta_j^T R_3\eta_j - \sum_{j=k+s}^{k-1}\eta_j^T R_3\eta_j\Big] \\
&\quad + \sum_{s=-\rho_M}^{-1}\Big[\sum_{j=k+1+s}^{k}\eta_j^T R_4\eta_j - \sum_{j=k+s}^{k-1}\eta_j^T R_4\eta_j\Big], \\
&= \sum_{s=-\rho_M}^{-\rho_0-1}\Big[\eta_k^T R_3\eta_k - \eta_{k+s}^T R_3\eta_{k+s}\Big] + \sum_{s=-\rho_M}^{-1}\Big[\eta_k^T R_4\eta_k - \eta_{k+s}^T R_4\eta_{k+s}\Big], \\
&= \eta_k^T\big(\rho_{21}R_3+\rho_M R_4\big)\eta_k - \sum_{s=k-\rho_M}^{k-\rho_0-1}\eta_s^T R_3\eta_s + \sum_{s=k-\rho_M}^{k-1}\eta_s^T R_4\eta_s, \qquad (3.9) \\
\Delta V_7(x_k,k) &= \sum_{l=-\rho_0}^{-\rho_m-1}\sum_{j=l}^{-1}\Big[\sum_{s=k+1+j}^{k}\eta_s^T S_1\eta_s - \sum_{s=k+j}^{k-1}\eta_s^T S_1\eta_s\Big] \\
&\quad + \sum_{l=-\rho_0}^{-1}\sum_{j=l}^{-1}\Big[\sum_{s=k+1+j}^{k}\eta_s^T S_2\eta_s - \sum_{s=k+j}^{k-1}\eta_s^T S_2\eta_s\Big] \\
&= \sum_{l=-\rho_0}^{-\rho_m-1}\Big[(-l)\eta_k^T S_1\eta_k - \sum_{j=l}^{-1}\eta_{k+j}^T S_1\eta_{k+j}\Big] \\
&\quad + \sum_{s=-\rho_0}^{-1}\Big[(-l)\eta_k^T S_2\eta_k - \sum_{j=l}^{-1}\eta_{k+j}^T S_2\eta_{k+j}\Big] \\
&= \eta_k^T\big(\frac{\rho_{11}}{2}S_1 + \frac{\rho_{13}}{2}S_2\big)\eta_k - \sum_{l=-\rho_0}^{-\rho_m-1}\sum_{j=k+l}^{k-1}\eta_j^T S_1\eta_j \\
&\quad - \sum_{l=-\rho_0}^{-1}\sum_{j=k+l}^{k-1}\eta_j^T S_2\eta_j, \qquad (3.10) \\
\Delta V_8(x_k,k) &= \sum_{l=-\rho_M}^{-\rho_0-1}\sum_{j=l}^{-1}\Big[\sum_{s=k+1+j}^{k}\eta_s^T S_3\eta_s - \sum_{s=k+j}^{k-1}\eta_s^T S_3\eta_s\Big] \\
&\quad + \sum_{l=-\rho_M}^{-1}\sum_{j=l}^{-1}\Big[\sum_{s=k+1+j}^{k}\eta_s^T S_4\eta_s - \sum_{s=k+j}^{k-1}\eta_s^T S_4\eta_s\Big] \\
&= \sum_{l=-\rho_M}^{-\rho_0-1}\Big[(-l)\eta_k^T S_3\eta_k - \sum_{j=l}^{-1}\eta_{k+j}^T S_3\eta_{k+j}\Big] \\
&\quad + \sum_{l=-\rho_M}^{-1}\Big[(-l)\eta_k^T S_4\eta_k - \sum_{j=l}^{-1}\eta_{k+j}^T S_4\eta_{k+j}\Big] \\
&= \eta_k^T\big(\frac{\rho_{21}}{2}S_3 + \frac{\rho_{23}}{2}S_4\big)\eta_k - \sum_{l=-\rho_M}^{-\rho_0-1}\sum_{j=k+l}^{k-1}\eta_j^T S_3\eta_j - \sum_{l=-\rho_M}^{-1}\sum_{j=k+l}^{k-1}\eta_j^T S_4\eta_j \qquad (3.11)
\end{aligned}$$

Coimbing (3.4) - (3.11), we get,

$$\begin{aligned}\Delta V(x_k,k) &= 2x_k^T P\eta_k+\lambda_k^T\Big(Q_1+Q_2+Q_3+Q_4+Q_5+Q_6+\rho_{11}Q_1+\rho_{21}Q_4\Big)\lambda_k\\&-\lambda_{\rho,1}Q_1\lambda_{\rho,1}-\lambda_{\rho,m}Q_2\lambda_{\rho,m}-\lambda_{\rho,0}(Q_3+Q_5)\lambda_{\rho,0}-\lambda_{\rho,2}Q_4\lambda_{\rho,2}\\&-\lambda_{\rho,M}Q_6\lambda_{\rho,M}+\eta_k^T\Big(P+\rho_{11}R_1+\rho_0R_2+\rho_{21}R_3+\rho_MR_4\\&+\frac{1}{2}\rho_{11}S_1+\frac{1}{2}\rho_{13}S_2+\frac{1}{2}\rho_{21}S_3+\frac{1}{2}\rho_{23}S_2\Big)\eta_k-\sum_{s=k-\rho_0}^{k-\rho_m-1}\eta_S^TR_1\eta_S\\&-\sum_{s=k-\rho_0}^{k-1}\eta_S^TR_2\eta_S-\sum_{s=k-\rho_M}^{k-\rho_0-1}\eta_S^TR_3\eta_S-\sum_{s=k-\rho_M}^{k-1}\eta_S^TR_4\eta_S\\&-\sum_{l=-\rho_0}^{-\rho_m-1}\sum_{j=k+l}^{k-1}\eta_j^TS_1\eta_j-\sum_{l=-\rho_0}^{-1}\sum_{j=k+l}^{k-1}\eta_j^TS_2\eta_j\\&-\sum_{l=-\rho_M}^{-\rho_0-1}\sum_{j=k+l}^{k-1}\eta_j^TS_3\eta_j-\sum_{l=-\rho_M}^{-1}\sum_{j=k+l}^{k-1}\eta_j^TS_4\eta_j \qquad (3.12)\end{aligned}$$

By using Lemma 2.3, we obtain

$$\begin{aligned}-\sum_{l=-\rho_0}^{-\rho_m-1}\sum_{j=k+l}^{k-1}\eta_j^TS_1\eta_j &\le -\frac{2}{\rho_{12}}\Big[\sum_{l=-\rho_0}^{-\rho_m-1}\sum_{j=k+l}^{k-1}\eta_j\Big]^TS_1\Big[\sum_{l=-\rho_0}^{-\rho_m-1}\sum_{j=k+l}^{k-1}\eta_j\Big],\\&\le -\frac{2}{\rho_{12}}\Big[\rho_{11}x_k+\sum_{l=k-\rho_0}^{k-\rho_m-1}x_k\Big]^TS_1\Big[\rho_{11}x_k+\sum_{l=k-\rho_0}^{k-\rho_m-1}x_k\Big], \qquad (3.13)\\-\sum_{l=-\rho_0}^{-1}\sum_{j=k+l}^{k-1}\eta_j^TS_2\eta_j &\le -\frac{2}{\rho_{13}}\Big[\sum_{l=-\rho_0}^{-1}\sum_{j=k+l}^{k-1}\eta_j\Big]^TS_2\Big[\sum_{l=-\rho_0}^{-1}\sum_{j=k+l}^{k-1}\eta_j\Big],\\&\le -\frac{2}{\rho_{13}}\Big[\rho_0x_k+\sum_{l=k-\rho_0}^{k-1}x_k\Big]^TS_2\Big[\rho_0x_k+\sum_{l=k-\rho_0}^{k-1}x_k\Big], \qquad (3.14)\\-\sum_{s=-\rho_M}^{-\rho_0-1}\sum_{j=k+l}^{k-1}\eta_j^TS_3\eta_j &\le -\frac{2}{\rho_{22}}\Big[\sum_{l=-\rho_M}^{-\rho_0-1}\sum_{j=k+l}^{k-1}\eta_j\Big]^TS_3\Big[\sum_{l=-\rho_M}^{-\rho_0-1}\sum_{j=k+l}^{k-1}\eta_j\Big],\\&\le -\frac{2}{\rho_{22}}\Big[\rho_{21}x_k+\sum_{l=k-\rho_M}^{k-\rho_0-1}x_k\Big]^TS_3\Big[\rho_{21}x_k+\sum_{l=k-\rho_M}^{k-\rho_0-1}x_k\Big], \qquad (3.15)\\-\sum_{l=-\rho_M}^{-1}\sum_{j=k+l}^{k-1}\eta_j^TS_4\eta_j &\le -\frac{2}{\rho_{23}}\Big[\sum_{l=-\rho_M}^{-1}\sum_{j=k+l}^{k-1}\eta_j\Big]^TS_4\Big[\sum_{l=-\rho_M}^{-1}\sum_{j=k+l}^{k-1}\eta_j\Big],\\&\le -\frac{2}{\rho_{23}}\Big[\rho_Mx_k+\sum_{l=k-\rho_M}^{k-1}x_k\Big]^TS_4\Big[\rho_Mx_k+\sum_{l=k-\rho_M}^{k-1}x_k\Big] \qquad (3.16)\end{aligned}$$

On the other hand, for any appropriately dimensioned matrices $M_n,\ n = 1, \dots, 6,$ the following inequalities hold;

$$2\zeta_k^T M_1\Big[x_{\rho,1} - x_{\rho_0} - \sum_{s=k-\rho_0}^{k-\rho_{k,1}-1} \eta_S\Big] = 0, \tag{3.17}$$

$$2\zeta_k^T M_2\Big[x_{\rho,m} - x_{\rho,1} - \sum_{s=k-\rho,1}^{k-\rho_m-1} \eta_S\Big] = 0, \tag{3.18}$$

$$2\zeta_k^T M_3\Big[x_k - x_{\rho,1} - \sum_{s=k-\rho,1}^{k-1} \eta_S\Big] = 0, \tag{3.19}$$

$$2\zeta_k^T M_4\Big[x_{\rho_{k,2}} - x_{\rho_M} - \sum_{s=k-\rho_M}^{k-\rho_{k,2}-1} \eta_S\Big] = 0, \tag{3.20}$$

$$2\zeta_k^T M_5\Big[x_{\rho,0} - x_{\rho,2} - \sum_{s=k-\rho,2}^{k-\rho_0-1} \eta_S\Big] = 0, \tag{3.21}$$

$$2\zeta_k^T M_6\Big[x_k - x_{\rho,2} - \sum_{s=k-\rho,2}^{k-1} \eta_S\Big] = 0, \tag{3.22}$$

$$(\rho_0 - \rho_{k,1})\zeta_k^T M_1(R_1 + R_2)^{-1} M_1^T \zeta_k - \sum_{s=k-\rho_0}^{k-\rho_1(k)-1} \zeta_k^T M_1(R_1 + R_2)^{-1} M_1^T \zeta_k = 0, \tag{3.23}$$

$$(\rho_{k,1} - \rho_m)\zeta_k^T M_2 R_1^{-1} M_2^T \zeta_k - \sum_{s=k-\rho_{k,1}}^{t-\rho_m-1} \zeta_k^T M_2 R_1^{-1} M_2^T \zeta_k = 0, \tag{3.24}$$

$$\rho_{k,1}\zeta_k^T M_3 R_2^{-1} M_3^T \zeta_k - \sum_{s=k-\rho_{k,1}}^{k-1} \zeta_k^T M_3 R_2^{-1} M_3^T \zeta_k = 0, \tag{3.25}$$

$$(\rho_M - \rho_{k,2})\zeta_k^T M_4(R_3 + R_4)^{-1} M_4^T \zeta_k - \sum_{s=k-\rho_M}^{k-\rho_{k,2}-1} \zeta_k^T M_4(R_3 + R_4)^{-1} M_4^T \zeta_k = 0, \tag{3.26}$$

$$(\rho_{k,2} - \rho_0)\zeta_k^T M_5 R_3^{-1} M_5^T \zeta_k - \sum_{s=k-\rho_{k,2}}^{k-\rho_0-1} \zeta_k^T M_5 R_3^{-1} M_5^T \zeta_k = 0, \tag{3.27}$$

$$\rho_{k,2}\zeta_k^T M_6 R_4^{-1} M_6^T \zeta_k - \sum_{s=k-\rho_{k,2}}^{k-1} \zeta_k^T M_6 R_4^{-1} M_6^T \zeta_k = 0. \tag{3.28}$$

where $\zeta_k^T = \Big[x_k^T\ x_{\rho,1}^T\ x_{\rho,2}^T\ \eta_k^T\Big]^T$. Also, for any matrix N of appropriate dimensions the following inequality holds:

$$2\zeta_k^T N\Big[(A - I)x_k + \delta_0 BGKHx_{\rho,1} + \overline{\delta_0} BGKHx_{\rho,2} - \eta_k\Big] = 0. \tag{3.29}$$

Combining (3.12) - (3.29) we obtain,

$$\Delta V(x_k,k) \quad \leq \quad \xi_k^T\Omega_1\xi_k \tag{3.30}$$

where

$$\Omega_1 = \begin{bmatrix} \Omega_{16,16} & \sqrt{\rho_{11}}M_1 & \sqrt{\rho_{11}}M_2 & \sqrt{\rho_0}M_3 & \sqrt{\rho_{21}}M_4 & \sqrt{\rho_{21}}M_5 & \sqrt{\rho_M}M_6 \\ * & -(R_1+R_2) & 0 & 0 & 0 & 0 & 0 \\ * & * & -R_1 & 0 & 0 & 0 & 0 \\ * & * & * & -R_2 & 0 & 0 & 0 \\ * & * & * & * & -(R_3+R_4) & 0 & 0 \\ * & * & * & * & * & -R_3 & 0 \\ * & * & * & * & * & * & -R_4 \end{bmatrix}, \tag{3.31}$$

$$\xi_k^T = \Big[x_k^T \; x_{\rho_m}^T \; x_{\rho,1}^T \; x_{\rho,2}^T \; x_{\rho_0}^T \; x_{\rho_M}^T \; \eta_k^T \; \eta_{\rho_m}^T \; \eta_{\rho,1}^T \; \eta_{\rho,2}^T \; \eta_{\rho_0}^T \; \eta_{\rho_M}^T \; \sum_{s=k-\rho_0}^{k-\rho_m-1} x_S^T \; \sum_{s=k-\rho_0}^{k-1} x_S^T$$
$$\sum_{s=k-\rho_M}^{k-\rho_0-1} x_S^T \; \sum_{s=k-\rho_M}^{k-1} x_S^T\Big]^T,$$

$$\begin{aligned}
\Omega_{1,1} &= Q_{11}+Q_{21}+Q_{31}+Q_{41}+Q_{51}+Q_{61}+\rho_{11}Q_{11}+\rho_{21}Q_{41}-2\frac{\rho_{11}^2}{\rho_{12}}S_1 \\
&\quad -2\frac{\rho_0^2}{\rho_{13}}S_2-2\frac{\rho_{21}^2}{\rho_{22}}S_3-2\frac{\rho_M^2}{\rho_{23}}S_4+2N_1A-2N_1+2M_{31}+2M_{61},\ \Omega_{1,2}=2M_{21}, \\
\Omega_{1,3} &= 2N_1\delta_0BGKH+2A^TN_2^T-2N_2^T+2M_{11}-2M_{21}+2M_{32}^T-2M_{31}+2M_{62}^T, \\
\Omega_{1,4} &= 2N_1\overline{\delta_0}BGKH+2A^TN_3^T-2N_3^T+2M_{33}^T+2M_{41}-2M_{51}^T+2M_{63}^T-2M_{61}, \\
\Omega_{1,5} &= -2M_{11}+2M_{51},\ \Omega_{1,6}=-2M_{41},\ \Omega_{1,7}=2P+2Q_{12}+2Q_{22}+2Q_{32}+2Q_{42} \\
&\quad +2Q_{52}+2Q_{62}+2\rho_{11}Q_{12}+2\rho_{21}Q_{42}-2N_1+2A^TN_4^T-2N_4^T+2M_{34}^T+2M_{64}^T, \\
\Omega_{1,13} &= 4\frac{\rho_{11}}{\rho_{12}}S_1,\ \Omega_{1,14}=4\frac{\rho_0}{\rho_{13}}S_2,\ \Omega_{1,15}=4\frac{\rho_{21}}{\rho_{22}}S_3,\ \Omega_{1,16}=4\frac{\rho_M}{\rho_{23}}S_4,\ \Omega_{2,2}=-Q_{21}, \\
\Omega_{2,3} &= 2M_{22}^T,\ \Omega_{2,4}=2M_{23}^T,\ \Omega_{2,7}=2M_{24}^T,\ \Omega_{2,8}=-2Q_{22},\ \Omega_{3,3}=+2N_2\delta_0BGKH \\
&\quad -Q_{11}+2M_{12}-2M_{22}-2M_{32},\ \Omega_{3,4}=2N_2\overline{\delta_0}BGKH+(N_3\delta_0BGKH)^T+2M_{13}^T \\
&\quad -2M_{23}^T-2M_{33}^T+2M_{42}-2M_{52}-2M_{62},\ \Omega_{3,5}=-2M_{12}+2M_{52},\ \Omega_{3,6}=-2M_{42}, \\
\Omega_{3,7} &= -2N_2+2(N_4\delta_0BGKH)^T+2M_{14}^T-2M_{24}^T-2M_{34}^T,\ \Omega_{3,9}=-2Q_{12}, \\
\Omega_{4,4} &= -Q_{41}+2N_3\overline{\delta_0}BGKH+2M_{43}-2M_{53}-2M_{63},\ \Omega_{4,5}=-2M_{13}+2M_{53}, \\
\Omega_{4,6} &= -2M_{43},\ \Omega_{4,7}=-2N_3+2(N_4\overline{\delta_0}BGKH)^T+2M_{44}^T-2M_{54}^T-2M_{64}^T, \\
\Omega_{4,10} &= -2Q_{42},\ \Omega_{5,5}=-Q_{31}-Q_{51},\ \Omega_{5,7}=-2M_{14}^T+2M_{54}^T,\ \Omega_{5,11}=-2Q_{32}-2Q_{52}, \\
\Omega_{6,6} &= -Q_{61},\ \Omega_{6,7}=-2M_{44}^T,\ \Omega_{6,12}=-2Q_{62},\ \Omega_{7,7}=P+Q_{13}+Q_{23}+Q_{33}+Q_{43} \\
&\quad +Q_{53}+Q_{63}+\rho_{11}Q_{13}+\rho_{21}Q_{43}+\rho_{11}R_1+\rho_0R_2+\rho_{21}R_3+\rho_MR_4+\frac{\rho_{11}}{2}S_1 \\
&\quad +\frac{\rho_{13}}{2}S_2+\frac{\rho_{21}}{2}S_3+\frac{\rho_{23}}{2}S_4-2N_4,\ \Omega_{8,8}=-Q_{23},\ \Omega_{9,9}=-Q_{13},\ \Omega_{10,10}=-Q_{43}, \\
\Omega_{11,11} &= -Q_{33}-Q_{53},\ \Omega_{12,12}=-Q_{63},\ \Omega_{13,13}=-\frac{2}{\rho_{12}}S_1,\ \Omega_{14,14}=-\frac{2}{\rho_{13}}S_2, \\
\Omega_{15,15} &= -\frac{2}{\rho_{22}}S_3,\ \Omega_{16,16}=-\frac{2}{\rho_{23}}S_4, \\
M_i &= \begin{bmatrix} M_{i1} & 0 & M_{i2} & M_{i3} & 0_{2n} & M_{i4} & 0_{9n} \end{bmatrix},\ i=1,\cdots,6,
\end{aligned}$$

and other parameters are zero.

In order to obtain the output feedback controller gain matrices, let us define $N_i = \lambda_i P$ $(i = 1,2,3,4)$, where λ_i is the design parameter. Pre- and post-multiplying (3.31) by $\text{diag}\{X, \cdots, X\} \in \mathbb{R}^{22\times 22}$, letting $\widehat{R}_n = XR_nX$, $\widehat{S}_n = XS_nX$, $\widehat{Q}_m = \begin{bmatrix} \widehat{Q}_{m1} & \widehat{Q}_{m2} \\ * & \widehat{Q}_{m3} \end{bmatrix} > 0, \widehat{Q}_{m1} = XQ_{m1}X, \widehat{Q}_{m2} = XQ_{m2}X, \widehat{Q}_{m3} = XQ_{m3}X, \widehat{M}_{mn} = XM_{mn}X$ with $X = P^{-1}$, $m = 1, \ldots, 6$, $n = 1, 2, 3, 4,$. we obtain LMI (3.2). Thus we conclude by the Lyapunov stability theory that the reliable NCS (3.1) without uncertainties is asymptotically stable, which completes the proof.

In the following theorem, we extend the results obtained in the previous theorem to design the non-fragile controller $\widehat{K} = K + \Delta K(k)$ for the discrete time NCS (3.1) with known actuator failure G.

Theorem 3.2. *The non-fragile discrete time NCS* (3.1) *is asymptotically stable with known actuator failure parameter matrix G and the output feedback reliable non-fragile control if there exist symmetric matrices $X > 0$, $\widehat{R}_n > 0$, $\widehat{S}_n > 0$, $\widehat{Q}_m = \begin{bmatrix} \widehat{Q}_{m1} & \widehat{Q}_{m2} \\ * & \widehat{Q}_{m3} \end{bmatrix} > 0$, any matrices M_{mn}, $(m = 1, \cdots, 6, n = 1, 2, 3, 4)$, matrix Y with appropriate dimensions and positive scalars ϵ_i, $(i = 1, \cdots, 4)$, such that the following LMI holds,*

$$\widetilde{\Omega} = \begin{bmatrix} \hat{\Omega} & \widetilde{\mathbb{M}} \\ * & -\widetilde{\epsilon} \end{bmatrix} < 0, \tag{3.32}$$

$$\widetilde{\mathbb{M}} = \begin{bmatrix} \epsilon_1\widetilde{\mathbb{M}}_1 & \widetilde{\mathbb{N}}_1 & \epsilon_2\widetilde{\mathbb{M}}_2 & \widetilde{\mathbb{N}}_2 & \epsilon_3\widetilde{\mathbb{M}}_3 & \widetilde{\mathbb{N}}_3 & \epsilon_4\widetilde{\mathbb{M}}_4 & \widetilde{\mathbb{N}}_4 \end{bmatrix},$$

$$\widetilde{\epsilon} = \begin{bmatrix} \epsilon_1 & \epsilon_1 & \epsilon_2 & \epsilon_2 & \epsilon_3 & \epsilon_3 & \epsilon_4 & \epsilon_4 \end{bmatrix}^T,$$
$$\widetilde{\mathbb{M}}_1 = \begin{bmatrix} 0_{2n} & \epsilon_1\lambda_1\sqrt{\delta_0}M^TG^TB^T & \epsilon_1\lambda_1\sqrt{\overline{\delta}_0}M^TG^TB^T & 0_{12n} \end{bmatrix}^T, \ \widetilde{\mathbb{N}}_1 = \begin{bmatrix} NHX & 0_{15n} \end{bmatrix}^T,$$
$$\widetilde{\mathbb{M}}_2 = \begin{bmatrix} 0_{2n} & \epsilon_2\lambda_2\sqrt{\delta_0}M^TG^TB^T & \epsilon_2\lambda_2\sqrt{\overline{\delta}_0}M^TG^TB^T & 0_{12n} \end{bmatrix}^T, \ \widetilde{\mathbb{N}}_2 = \begin{bmatrix} 0_{2n} & NHX & 0_{13n} \end{bmatrix}^T,$$
$$\widetilde{\mathbb{M}}_3 = \begin{bmatrix} 0_{2n} & \epsilon_3\lambda_3\sqrt{\delta_0}M^TG^TB^T & \epsilon_3\lambda_3\sqrt{\overline{\delta}_0}M^TG^TB^T & 0_{12n} \end{bmatrix}^T, \ \widetilde{\mathbb{N}}_3 = \begin{bmatrix} 0_{3n} & NHX & 0_{12n} \end{bmatrix}^T,$$
$$\widetilde{\mathbb{M}}_4 = \begin{bmatrix} 0_{2n} & \epsilon_4\lambda_4\sqrt{\delta_0}M^TG^TB^T & \epsilon_4\lambda_4\sqrt{\overline{\delta}_0}M^TG^TB^T & 0_{12n} \end{bmatrix}^T, \ \widetilde{\mathbb{N}}_4 = \begin{bmatrix} 0_{6n} & NHX & 0_{9n} \end{bmatrix}^T,$$

other parameters are defined as in Theorem 3.1. In this case, the output feedback non-fragile reliable controller gain is given by $K = YX^{-1}H^{-1}$.

Proof: In this theorem, by including the non-fragile control $\widehat{K} = K + \Delta K(k)$, LMI (3.31) in Theorem 3.1 can be written as

$$\begin{aligned} \Omega = \ & \Omega_1 + \mathbb{M}_1F(k)\mathbb{N}_1 + \mathbb{N}_1^TF^T(k)\mathbb{M}_1^T + \mathbb{M}_2F(k)\mathbb{N}_2 + \mathbb{N}_2^TF^T(k)\mathbb{M}_2^T \\ & + \mathbb{M}_3F(k)\mathbb{N}_3 + \mathbb{N}_3^TF^T(k)\mathbb{M}_3^T + \mathbb{M}_4F(k)\mathbb{N}_4 + \mathbb{N}_4^TF^T(k)\mathbb{M}_4^T \end{aligned} \tag{3.33}$$

where

$$\mathbb{M}1 = \begin{bmatrix} 0_{2n} & N_1\sqrt{\delta_0}M^TG^TB^T & N_1\sqrt{\overline{\delta}_0}M^TG^TB^T & 0_{12n} \end{bmatrix}^T, \ \mathbb{N}_1 = \begin{bmatrix} NH & 0_{15n} \end{bmatrix}^T,$$
$$\mathbb{M}_2 = \begin{bmatrix} 0_{2n} & N_2\sqrt{\delta_0}M^TG^TB^T & N_2\sqrt{\overline{\delta}_0}M^TG^TB^T & 0_{12n} \end{bmatrix}^T, \ \mathbb{N}_2 = \begin{bmatrix} 0_{2n} & NH & 0_{13n} \end{bmatrix}^T,$$
$$\mathbb{M}_3 = \begin{bmatrix} 0_{2n} & N_3\sqrt{\delta_0}M^TG^TB^T & N_3\sqrt{\overline{\delta}_0}M^TG^TB^T & 0_{12n} \end{bmatrix}^T, \ \mathbb{N}_3 = \begin{bmatrix} 0_{3n} & NH & 0_{12n} \end{bmatrix}^T,$$
$$\mathbb{M}_4 = \begin{bmatrix} 0_{2n} & N_4\sqrt{\delta_0}M^TG^TB^T & N_4\sqrt{\overline{\delta}_0}M^TG^TB^T & 0_{12n} \end{bmatrix}^T, \ \mathbb{N}_4 = \begin{bmatrix} 0_{6n} & NH & 0_{9n} \end{bmatrix}^T.$$

Further, it follows from (3.33) and Lemma 2.5 that there exist scalars ϵ_i, $(i = 1, \cdots, 4)$ such that

$$\begin{aligned}\Omega &= \Omega_1 + \epsilon_1 \mathbb{M}_1 \mathbb{M}_1^T + \epsilon_1^{-1} \mathbb{N}_1 \mathbb{N}_1^T + \epsilon_2 \mathbb{M}_2 \mathbb{M}_2^T + \epsilon_2^{-1} \mathbb{N}_1 \mathbb{N}_2^T + \epsilon_3 \mathbb{M}_3 \mathbb{M}_3^T \\ &\quad + \epsilon_3^{-1} \mathbb{N}_1 \mathbb{N}_3^T + \epsilon_4 \mathbb{M}_4 \mathbb{M}_4^T + \epsilon_4^{-1} \mathbb{N}_1 \mathbb{N}_4^T \end{aligned} \tag{3.34}$$

Then by using Lemma 2.4, it is easy to get

$$\Omega = \begin{bmatrix} \Omega_1 & \mathbb{M} \\ * & -\widetilde{\epsilon} \end{bmatrix} \tag{3.35}$$

where $\mathbb{M} = \begin{bmatrix} \epsilon_1 \mathbb{M}_1 & \mathbb{N}_1 & \epsilon_2 \mathbb{M}_2 & \mathbb{N}_2 & \epsilon_3 \mathbb{M}_3 & \mathbb{N}_3 & \epsilon_4 \mathbb{M}_4 & \mathbb{N}_4 \end{bmatrix}$, Pre- and post-multiplying (3.35) by $\mathrm{diag}\{\underbrace{X, \cdots, X}_{22}, \underbrace{I, \cdots, I}_{8}\} \in \mathbb{R}^{30\times 30}$, we obtain LMI (3.32) which completes the proof.

4. Robust non-fragile unknown fault-tolerant control design

In the previous section, we developed criteria for reliable stabilization of discrete-time random delays NCS (3.1) with and without non-fragile control. This section extended the above results to study the robust unknown reliable control problem with and without non-fragile control. First, we present the following lemma.

Lemma 4.1. [6, 24] *Let B, D, G, X and Y be matrices of appropriate dimensions, F be an uncertain matrix such that $F^T F \leq I$ and Δ be a diagonal uncertain matrix satisfying $\Delta^T \Delta \leq I$. Then there exist a positive definite diagonal matrix U and a positive scalar ϵ such that*

$$U - \epsilon G G^T > 0 \tag{4.1}$$

and

$$\begin{aligned} B\Delta Y + Y^T \Delta^T B^T + DFX + X^T F^T D^T + B\Delta G F X + X^T F^T G^T \Delta^T B^T &\leq \\ \epsilon D D^T + \epsilon^{-1} X^T X + BUB^T + (Y + \epsilon G D^T)^T (U - \epsilon G G^T)^{-1} (Y + \epsilon G D^T) & \end{aligned} \tag{4.2}$$

Proof: It follows from Lemma 2.5 that

$$\begin{aligned} & B\Delta Y + Y^T \Delta^T B^T + DFX + X^T F^T D^T + B\Delta G F X + X^T F^T G^T \Delta^T B^T \\ = \ & B\Delta Y + Y^T \Delta^T B^T + (D + B\Delta G) F X + X^T F^T (D + B\Delta G)^T \\ \leq \ & B\Delta Y + Y^T \Delta^T B^T + \epsilon (D + B\Delta G)(D + B\Delta G)^T + \epsilon^{-1} X^T X \\ \leq \ & \epsilon D D^T + \epsilon^{-1} X^T X + B\Delta (Y + \epsilon G D^T) + (Y + \epsilon G D^T)^T \Delta^T B^T \\ & + \epsilon B \Delta G G^T \Delta^T B^T \end{aligned} \tag{4.3}$$

Obviously, there exists a positive definite diagonal matrix U satisfying the matrix inequality (4.1). Therefore,

$$\begin{aligned} & B\Delta (Y + \epsilon G D^T) + (Y + \epsilon G D^T)^T \Delta^T B^T + \epsilon B \Delta G G^T \Delta^T B^T \\ \leq \ & (Y + \epsilon G D^T)^T (U - \epsilon G G^T)^{-1} (Y + \epsilon G D^T) + B \Delta U \Delta^T B^T \\ \leq \ & (Y + \epsilon G D^T)^T (U - \epsilon G G^T)^{-1} (Y + \epsilon G D^T) + B U^{\frac{1}{2}} \Delta \Delta^T U^{\frac{1}{2}} B^T \\ \leq \ & (Y + \epsilon G D^T)^T (U - \epsilon G G^T)^{-1} (Y + \epsilon G D^T) + B U B^T. \end{aligned} \tag{4.4}$$

It is seen that, (4.3) and (4.4) leads to (4.2), which completes the proof.

Moreover, it should be noted that the results obtained in Theorems 3.1 and 3.2 are applicable for the actuator failure matrix G, which is exactly known and fixed. However, it should be pointed out that the actuator failures may not always be fixed ones. It may occur within a range of intervals. So in this section, we assume that the actuator failure G occurs in a range of interval and satisfies the following assumption:

Assumption 2: G is the actuator fault matrix defined as follows

$$G = \text{diag}\{g_1, g_2, \dots, g_m\}, \quad 0 \leq \underline{g}_i \leq g_i \leq \overline{g}_i, \quad \overline{g}_i \geq 1 \tag{4.5}$$

where $\underline{g}_i$ and $\overline{g}_i$, $i = 1, 2, \dots, m$ are given constants. If $g_i = 0$, i^{th} actuator completely fails whereas i^{th} actuator is normal if $g_i = 1$. Define

$$G_0 = \text{diag}\{g_{10}, g_{20}, \dots g_{m0}\}, \quad g_{i0} = \frac{\overline{g}_i + \underline{g}_i}{2}, \tag{4.6}$$

$$G_1 = \text{diag}\{g_{11}, g_{21}, \dots g_{m1}\}, \quad g_{i1} = \frac{\overline{g}_i - \underline{g}_i}{2}. \tag{4.7}$$

Then the matrix G can be written as

$$G = G_0 + G_1\Sigma, \quad \Sigma = \text{diag}\{\theta_1, \dots, \theta_p\}, \ |\theta_i| \leq g_{i1}, \ (i = 1, \cdots, p). \tag{4.8}$$

Now, we design the robust controller for unknown actuator failure matrix G, which satisfies the constraints (4.5) - (4.8). The following theorem designs the reliable state feedback controller using the conditions obtained in Theorem 3.1.

Theorem 4.2. *The uncertain discrete time NCS* (2.10) *is robustly asymptotically stable with unknown actuator failure parameter matrix G, the output feedback reliable control and $\Delta K(k) = 0$ if there exist symmetric matrices $X > 0$, $\widehat{R}_n > 0$, $\widehat{S}_n > 0$, $\widehat{Q}_m = \begin{bmatrix} \widehat{Q}_{m1} & \widehat{Q}_{m2} \\ * & \widehat{Q}_{m3} \end{bmatrix} > 0$, any matrices M_{mn}, $(m = 1, \cdots, 6, n = 1, 2, 3, 4)$, matrix Y with appropriate dimensions and positive scalars ϵ_i $(i = 1, 2)$, such that the following LMI holds:*

$$\hat{\Theta} = \begin{bmatrix} \breve{\Omega} & \hat{\Theta}_1 & \hat{\Theta}_2 & \hat{\Theta}_3 & \hat{\Theta}_4 \\ * & -\epsilon_1 & 0 & 0 & 0 \\ * & * & -\epsilon_1 & 0 & 0 \\ * & * & * & -\epsilon_2 & 0 \\ * & * & * & * & -\epsilon_2 \end{bmatrix} < 0, \tag{4.9}$$

$$\hat{\Theta}_1 = \begin{bmatrix} \epsilon_1\lambda_1 G_1^T & 0 & \epsilon_1\lambda_2 G_1^T & \epsilon_1\lambda_3 G_1^T & 0_{2n} & \epsilon_1\lambda_4 G_1^T & 0_{15n} \end{bmatrix},$$
$$\hat{\Theta}_2 = \begin{bmatrix} 0_{2n} & \delta_0 BY & \overline{\delta_0} BY & 0_{18n} \end{bmatrix}, \ \hat{\Theta}_4 = \begin{bmatrix} N_a X & 0_{21n} \end{bmatrix},$$
$$\hat{\Theta}_3 = \begin{bmatrix} \epsilon_2\lambda_1 W^T & 0 & \epsilon_2\lambda_2 W & \epsilon_2\lambda_3 W & 0_{2n} & \epsilon_2\lambda_4 W & 0_{15n} \end{bmatrix},$$

and the other parameters are defined in Theorem 3.1. In this case the output feedback reliable controller gain, is given by $K = YX^{-1}H^{-1}$.

Proof. By replacing the matrix A by $A + W\wp(k)N_a$ and G by $G_0 + G_1\Sigma$ in Theorem 3.1, we see thet

$$\hat{\Theta} = \check{\Omega} + \hat{\Theta}_1^T \wp(k)\hat{\Theta}_2 + \hat{\Theta}_2^T \wp(k)\hat{\Theta}_1 + \hat{\Theta}_3^T \Sigma\hat{\Theta}_4 + \hat{\Theta}_4^T \Sigma\hat{\Theta}_3 \tag{4.10}$$

where $\check{\Omega}$ is obtained by replacing G by G_0 in $\hat{\Omega}$. Further it follows from Lemma 2.5 and (4.10) that

$$\hat{\Theta} = \check{\Omega} + \epsilon_1^{-1}\hat{\Theta}_1^T\hat{\Theta}_1 + \epsilon_1\hat{\Theta}_2^T\hat{\Theta}_2 + \epsilon_2^{-1}\hat{\Theta}_3^T\hat{\Theta}_3 + \epsilon_2\hat{\Theta}_4^T\hat{\Theta}_4 \tag{4.11}$$

Then, it is easy to see that (4.11) is equivalent to LMI (4.9) by Lemma 2.4. Hence completes the proof.

In the following theorem, we extend the results obtained in the previous theorem to design the non-fragile controller $\hat{K} = K + \Delta K(k)$ for the uncertain discrete time NCS (2.10) with random delays.

Theorem 4.3. *The non-fragile uncertain discrete time NCS* (2.10) *is robustly asymptotically stable with unknown actuator failure parameter matrix G and the output feedback reliable non-fragile control if there exist symmetric matrices $X > 0$, $\widehat{R}_n > 0$, $\widehat{S}_n > 0$, $\widehat{Q}_m = \begin{bmatrix} \widehat{Q}_{m1} & \widehat{Q}_{m2} \\ * & \widehat{Q}_{m3} \end{bmatrix} > 0$, any matrices M_{mn} $(m = 1, \cdots, 6, n = 1, 2, 3, 4)$, matrix Y with appropriate dimensions any positive definite diagonal matrix U and positive scalars ϵ_i $(i = 1, 2)$, such that the following LMI holds,*

$$\check{\Theta} = \begin{bmatrix} \check{\Omega} & \check{\Theta}_1 & \check{\Theta}_2 & \check{\Theta}_3 & \check{\Theta}_4 & \check{\Theta}_5 & \check{\Theta}_6 & \check{\Theta}_7 & \check{\Theta}_8 & \check{\Theta}_9 \\ * & -\epsilon_1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ * & * & -\epsilon_1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ * & * & * & -\epsilon_1 & 0 & 0 & 0 & 0 & 0 & 0 \\ * & * & * & * & \check{\Theta}_{4,4} & 0 & 0 & 0 & 0 & 0 \\ * & * & * & * & * & \check{\Theta}_{5,5} & 0 & 0 & 0 & 0 \\ * & * & * & * & * & * & -U & 0 & 0 & 0 \\ * & * & * & * & * & * & * & -U & 0 & 0 \\ * & * & * & * & * & * & * & * & -\epsilon_2 & 0 \\ * & * & * & * & * & * & * & * & * & -\epsilon_2 \end{bmatrix} < 0, \tag{4.12}$$

$$\check{\Theta}_2 = \begin{bmatrix} \epsilon_1(\lambda_1\delta_0 BG_0M) \\ 0 \\ \epsilon_1(\lambda_2\delta_0 BG_0M) \\ \epsilon_1(\lambda_3\delta_0 BG_0M) \\ 0_{2n} \\ \epsilon_1(\lambda_4\delta_0 BG_0M) \\ 0_{15n} \end{bmatrix}, \ \check{\Theta}_3 = \begin{bmatrix} \epsilon_1(\lambda_1\overline{\delta_0} BG_0M) \\ 0 \\ \epsilon_1(\lambda_2\overline{\delta_0} BG_0M) \\ \epsilon_1(\lambda_3\overline{\delta_0} BG_0M) \\ 0_{2n} \\ \epsilon_1(\lambda_4\overline{\delta_0} BG_0M) \\ 0_{15n} \end{bmatrix},$$

$$\check{\Theta}_5 = \begin{bmatrix} (Y^TG_1^T + \epsilon_1\lambda_1\overline{\delta_0}BG_0MM^TG_1) \\ 0 \\ (Y^TG_1^T + \epsilon_1\lambda_1\overline{\delta_0}BG_0MM^TG_1) \\ (Y^TG_1^T + \epsilon_1\lambda_1\overline{\delta_0}BG_0MM^TG_1) \\ 0_{2n} \\ (Y^TG_1^T + \epsilon_1\lambda_1\overline{\delta_0}BG_0MM^TG_1) \\ 0_{15n} \end{bmatrix}, \ \check{\Theta}_6 = \begin{bmatrix} \lambda_1\delta_0BU \\ 0 \\ \lambda_1\delta_0BU \\ \lambda_1\delta_0BU \\ 0_{2n} \\ \lambda_1\delta_0BU \\ 0_{15n} \end{bmatrix},$$

$$\check{\Theta}_7 = \begin{bmatrix} \lambda_1\overline{\delta_0}BU \\ 0 \\ \lambda_1\overline{\delta_0}BU \\ \lambda_1\overline{\delta_0}BU \\ 0_{2n} \\ \lambda_1\overline{\delta_0}BU \\ 0_{15n} \end{bmatrix}, \ \check{\Theta}_4 = \begin{bmatrix} (Y^TG_1^T + \epsilon_1\lambda_1\delta_0BG_0MM^TG_1) \\ 0 \\ (Y^TG_1^T + \epsilon_1\lambda_1\delta_0BG_0MM^TG_1) \\ (Y^TG_1^T + \epsilon_1\lambda_1\delta_0BG_0MM^TG_1) \\ 0_{2n} \\ (Y^TG_1^T + \epsilon_1\lambda_1\delta_0BG_0MM^TG_1) \\ 0_{15n} \end{bmatrix},$$

$$\hat{\Theta}_8 = \begin{bmatrix} \epsilon_2\lambda_1W^T & 0 & \epsilon_2\lambda_2W & \epsilon_2\lambda_3W & 0_{2n} & \epsilon_2\lambda_4W & 0_{15n} \end{bmatrix},$$
$$\hat{\Theta}_9 = \begin{bmatrix} N_aX & 0_{21n} \end{bmatrix}, \ \check{\Theta}_1 = \begin{bmatrix} 0_{2n} & N & N & 0_{18n} \end{bmatrix},$$
$$\check{\Theta}_{4,4} = \check{\Theta}_{5,5} = -(U - \epsilon_1MM^T)$$

and the other parameters are defined in Theorem 3.1. In this case the output feedback controller gain is given by $K = YX^{-1}H^{-1}$.

Proof: The proof immediately follows by applying the non-fragile controller $\hat{K} = K + \Delta K(k)$, unknown actuator failure matrix G and uncertain parameters in Theorem 4.2. Further applying Lemma 4.1, we obtain (4.12). Thus we conclude by Lyapunov stability theory that the non-fragile uncertain discrete time NCS (2.10) with unknown reliable control is robustly asymptotically stable. The proof is completed.

Remark 4.4. In the absence of non-fragile, reliable controls and random delay, the uncertain discrete time NCS (2.10) is as follows:

$$\begin{aligned} x_{k+1} &= \bar{A}(k)x_k + \bar{B}(k)u_k \\ y_k &= Hx_k \end{aligned} \tag{4.13}$$

Choosing the control input delay as $u_k = KHx_{k,\rho}$, the NCS (4.13) is written as,

$$x_{k+1} = (A + \wp A(k))x_k + (B + \wp B(k))KHx_{k,\rho} \tag{4.14}$$

where the parameter uncertainties are defined as
$\begin{bmatrix} \wp A(k) & \wp B(k) \end{bmatrix} = W\wp(k) \begin{bmatrix} N_1 & N_2 \end{bmatrix}$.

First we consider the case of matrices A and B being fixed, i.e., $\wp A(k) = 0$ and $\wp B(k) = 0$. Then the nominal form of the NCS (4.14) can be written as

$$x_{k+1} = Ax_k + BKHx_{k,\rho} \tag{4.15}$$

Corollary 4.5. *The discrete time NCS* (4.15) *is asymptotically stable with the output feedback control* $u_k = KHx_{k,\rho}$ *if there exist symmetric positive matrices* $X > 0$, $\widehat{R}_n > 0$, $\widehat{S}_n > 0$, $n = 1, 2$, $\widehat{Q}_m = \begin{bmatrix} \widehat{Q}_{m1} & \widehat{Q}_{m2} \\ * & \widehat{Q}_{m3} \end{bmatrix} > 0$, *any matrices* M_{mp}, $(m, p = 1, 2, 3)$ *and matrix* Y *with appropriate dimensions, such that the following LMI holds:*

$$\Xi_1 = \begin{bmatrix} \Xi_{10,10} & \sqrt{\rho_{11}}\widehat{M_1} & \sqrt{\rho_{11}}\widehat{M_2} & \sqrt{\rho_M}\widehat{M_3} \\ * & -(\widehat{R}_1 + \widehat{R}_2) & 0 & 0 \\ * & * & -\widehat{R}_1 & 0 \\ * & * & * & -\widehat{R}_2 \end{bmatrix} < 0, \tag{4.16}$$

where,

$$\begin{aligned}
\Xi_{1,1} &= \widehat{Q}_{11} + \widehat{Q}_{21} + \widehat{Q}_{31} + \rho_{11}\widehat{Q}_{11} - 2\frac{\rho_{11}^2}{\rho_{12}}\widehat{S}_1 - 2\frac{\rho_M^2}{\rho_{13}}\widehat{S}_2 + 2\lambda_1 AX - 2\lambda_1 X + 2\widehat{M}_{31}, \\
\Xi_{1,2} &= 2\widehat{M}_{21},\ \Xi_{1,3} = 2\lambda_1 BY + 2\lambda_2 XA^T - 2\lambda_2 X + 2\widehat{M}_{11}^T - 2\widehat{M}_{21} + 2\widehat{M}_{32}^T - 2\widehat{M}_{31}, \\
\Xi_{1,4} &= 2\widehat{M}_{11}^T,\ \Xi_{1,5} = X + \widehat{Q}_{12} + \widehat{Q}_{22} + \widehat{Q}_{32} + \rho_{11}\widehat{Q}_{12} + 2\widehat{M}_{33}^T - 2\lambda_1 X + 2\lambda_3 XA^T - 2\lambda_3 X, \\
\Xi_{1,9} &= 4\frac{\rho_1 1}{\rho_{12}}\widehat{S}_1,\ \Xi_{1,10} = 4\frac{\rho_M}{\rho_{13}}\widehat{S}_2,\ \Xi_{2,2} = -\widehat{Q}_{21},\ \Xi_{2,3} = 2\widehat{M}_{22}^T,\ \Xi_{2,5} = 2\widehat{M}_{23}^T,\ \Xi_{2,6} = -\widehat{Q}_{22}, \\
\Xi_{3,3} &= -\widehat{Q}_{11} + 2\lambda_2 BY - 2\widehat{M}_{12} - 2\widehat{M}_{22} - 2\widehat{M}_{32},\ \Xi_{3,4} = -2\widehat{M}_{12},\ \Xi_{3,5} = 2\widehat{M}_{13}^T - 2\widehat{M}_{23}^T \\
&\quad -2\widehat{M}_{33}^T - 2\lambda_2 X + 2Y^T B^T \lambda_3,\ \Xi_{3,7} = -\widehat{Q}_{12},\ \Xi_{4,4} = -\widehat{Q}_{31} - 2\widehat{M}_{13}^T,\ \Xi_{4,8} = -\widehat{Q}_{32}, \\
\Xi_{5,5} &= X + \widehat{Q}_{13} + \widehat{Q}_{23} + \widehat{Q}_{33} + \rho_{11}\widehat{Q}_{13} + \rho_{11}\widehat{R}_1 + \rho_M\widehat{R}_2 + \frac{1}{2}\rho_{11}\widehat{S}_1 + \frac{1}{2}\rho_M\widehat{S}_2 - 2\lambda_3 X, \\
\Xi_{6,6} &= -\widehat{Q}_{23},\ \Xi_{7,7} = -\widehat{Q}_{13},\ \Xi_{8,8} = -\widehat{Q}_{33},\ \Xi_{9,9} = -\frac{1}{\rho_{12}}\widehat{S}_1,\ \Xi_{10,10} = -\frac{1}{\rho_{13}}\widehat{S}_2, \\
\hat{M}_i &= \begin{bmatrix} \hat{M}_{i1} & 0 & \hat{M}_{i2} & 0 & \hat{M}_{i3} & 0_{5n} \end{bmatrix},\ i = 1, \cdots, 3,\ \rho_{11} = \rho_M - \rho_m, \\
\rho_{12} &= \rho_{11}(\rho_M + \rho_m + 1),\ \rho_{13} = \rho_M(\rho_M + 1).
\end{aligned}$$

and the remaining position parameters are zero. In this case the output feedback controller gain is given by $K = YX^{-1}H^{-1}$.

Proof: Consider the Lyapunov-Krasovskii functional candidate $V(x_k, k)$ as

$$V(x_k, k) = \sum_{n=1}^{5} V_n(x_k, k), \tag{4.17}$$

where

$$\begin{aligned}
V_1(x_k,k) &= x_k^T P x_k,\\
V_2(x_k,k) &= \sum_{s=k-\rho_{k,1}}^{k-1} \lambda_s^T Q_1 \lambda_s + \sum_{s=k-\rho_m}^{k-1} \lambda_s^T Q_2 \lambda_s + \sum_{s=k-\rho_M}^{k-1} \lambda_s^T Q_3 \lambda_s,\\
V_3(x_k,k) &= \sum_{s=-\rho_M+1}^{-\rho_m} \sum_{j=k+s}^{k-1} \lambda_j^T Q_1 \lambda_j,\\
V_4(x_k,k) &= \sum_{j=-\rho_M}^{-\rho_m-1} \sum_{s=k+j}^{k-1} \eta_s^T R_1 \eta_s + \sum_{j=-\rho_M}^{-1} \sum_{s=k+j}^{k-1} \eta_s^T R_2 \eta_s,\\
V_5(x_k,k) &= \sum_{l=-\rho_M}^{-\rho_m-1} \sum_{j=l}^{-1} \sum_{s=k+j}^{k-1} \eta_s^T S_1 \eta_s + \sum_{l=-\rho_M}^{-1} \sum_{j=l}^{-1} \sum_{s=k+j}^{k-1} \eta_s^T S_2 \eta_s,
\end{aligned}$$

The proof of this corollary is similar to Theorem 3.1 and hence omitted.
Now, we extend the results of Corollary 4.5 to uncertain NCS (4.14), which yields the following corollary.

Corollary 4.6. *The uncertain NCS* (4.14) *is robustly asymptotically stable with the output feedback control* $u_k = KHx_{k,\rho}$ *if there exist symmetric positive matrices* $X > 0,\ \widehat{R}_n > 0,\ \widehat{S}_n > 0,\ n = 1, 2,\ \widehat{Q}_m = \begin{bmatrix} \widehat{Q}_{m1} & \widehat{Q}_{m2} \\ * & \widehat{Q}_{m3} \end{bmatrix} > 0,$ *any matrices* $M_{mp},\ (m, p = 1, 2, 3)$, *appropriate dimensions matrix* Y *and positive scalar* ϵ, *such that the following LMI holds:*

$$\hat{\Xi} = \begin{bmatrix} \Xi_1 & \Xi_2 & \Xi_3 \\ * & -\epsilon & 0 \\ * & * & -\epsilon \end{bmatrix} < 0, \tag{4.18}$$

$\Xi_2 = \begin{bmatrix} \epsilon\lambda_1 W & 0 & \epsilon\lambda_2 W & 0 & \epsilon\lambda_3 W & 0_{5n} \end{bmatrix}^T,\ \Xi_3 = \begin{bmatrix} N_1 X & 0 & N_2 Y & 0_{7n} \end{bmatrix}^T$

and the remaining parameters are share the same expressions as those in (4.16).

5. Numerical simulations

This section provides four numerical examples along with simulation results to illustrate the effectiveness and less conservative of the developed theoretical results. More precisely, in Example 5.1 we consider four cases to demonstrate the obtained result. Case I deals with the reliable control design for the nominal form NCS given in (3.1) with $\Delta K(k) = 0$, and Case II investigates the non-fragile reliable control design for NCS (3.1). In both, the cases actuator fault matrix is known, whereas Case III and IV discuss the unknown actuator fault matrix for uncertain discrete-time NCS without and with non-fragile control, respectively.

Example 5.1. Consider the closed loop reliable control for discrete-time NCS (3.1) with the following parameters:

$$A = \begin{bmatrix} 0.08 & 0 \\ 0 & 0.09 \end{bmatrix},\ B = \begin{bmatrix} 1.2536 \\ -0.8226 \end{bmatrix},\ H = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix},$$

Case :I ($\Delta K(k) = 0$ with known actuator fault matrix:)

By setting the uncertain free case in the control gain matrix, i.e., $\Delta K(k) = 0$, and choosing the designing parameters $\lambda_1 = 0.0049,\ \lambda_2 = 0.0006,\ \lambda_3 = 0.0053,\ \lambda_4 = 1.0988$ and the remaining parameters as $\rho_m = 1,\ \rho_0 = 2,\ \rho_M = 8,\ G = 0.7,\ \delta_0 = 0.4$in MATLAB LMI toolbox the LMI constraint obtained in Theorem 3.1 is solved. It can be easily found that the obtained constraints are solvable and feasible, which are not provided here due to page length. Based on the above parameter values, the considered discrete-time NCS (3.1) is asymptotically stable with a known actuator fault matrix.

Case :II ($\Delta K(k) \neq 0$ with known actuator fault matrix:)

Consider a non-fragile controller such that the resulting closed loop discrete time NCS (3.1) is asymptotically stable with known actuator failure parameter matrix $G = 0.7$. Further, the non-fragile uncertain parameters are give as follows:

$$M = \begin{bmatrix} 0.0127 & 0.0414 \end{bmatrix},\ N = \begin{bmatrix} 0.1012 & 0 \\ 0 & 0.0063 \end{bmatrix}.$$

Solving the LMI in Theorem 3.2, with the above parameters together with $\rho_m = 1,\ \rho_0 = 2,\ \rho_M = 7,\ G = 0.7,\ \delta_0 = 0.4$ and the same designing parameters as same in Case I, we get a feasible solution that guarantees the closed loop form of the considered NCS (3.1) is asymptotically stable with non-fragile controller. For both the cases, the corresponding output feedback control gain matrices and the maximum delay bound of ρ_M are listed in Table 1 for Theorem 3.1 & 3.2 respectively. It is observed from Table 1 that when the non-fragile controller appears in the NCS (3.1) the upper bound ρ_M decreases compare to normal control.

Table 1. Comparison of the maximum delay bound ρ_M for the cases I and II

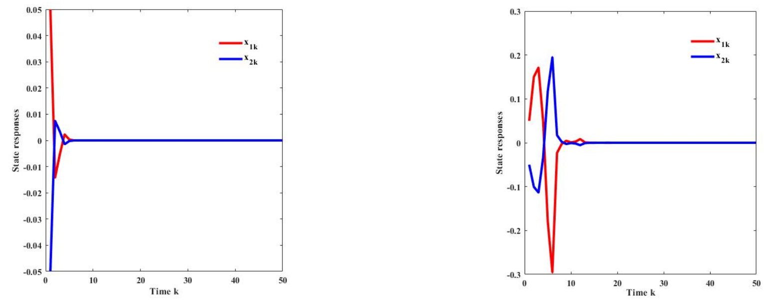
Cases	$\widehat{K} = K + \Delta K(k)$ when $\Delta K(k) = 0$	$\widehat{K} = K + \Delta K(k)$ when $\Delta K(k) \neq 0$
Gain Matrix $\widehat{K}$	$\begin{bmatrix} 0.0037 & -0.0024 \end{bmatrix}$	$\begin{bmatrix} 0.0026 & 0.0001 \end{bmatrix}$
Maximum upper bound ρ_M	8	7

Moreover, in order to reflect the effectiveness of the developed design scheme, simulation results are presented in Figures 2- 4. For this, the initial condition of the discrete time NCS (3.1) is chosen as $x(0) = \begin{bmatrix} 0.05 & -0.05 \end{bmatrix}$ and the unknown time varying uncertain matrix is given by,

$$F(k) = \begin{cases} 0.01\sin(k), & 0 \leq k \leq 50, \\ 0, & otherwise. \end{cases}$$

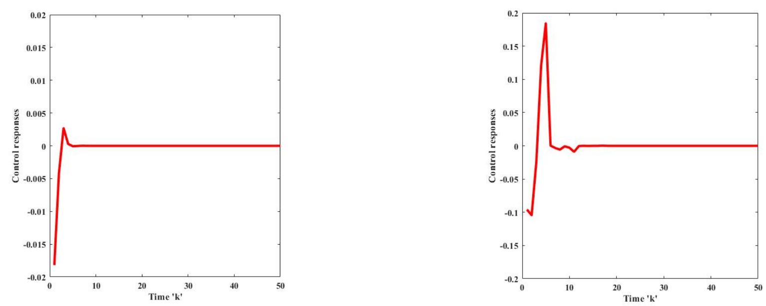
For cases I & II, the state responses of the considered NCS (3.1) are presented in Figures 2 (a) and (b). Figure 2 (a) represents the time response of the state vector x_k without non-fragile. Figure 2 (b) represents the time response of the state vector x_k of the non-fragile NCS. Time histories of the reliable control forces $u^f(k)$ with and without non-fragile control acting on the NCS (3.1) are given in Figure 3 (a) and (b) respectively. Further, Figure 4(a) describes the Bernoulli random variable δ_k and Figure 4(b) represents the variation of time-varying random delay $\rho_{k,1}, \rho_{k,2}$. The simulation results reveal that the considered non-fragile reliable discrete time

NCS with random delay is stabilizable via the proposed output feedback control law.



when $\Delta K(k) = 0$ when $\Delta K(k) \neq 0$

FIGURE 2. State responses of the Closed loop NCS (3.1)



when $\Delta K(k) = 0$ when $\Delta K(k) \neq 0$

FIGURE 3. Control forces of controllers of nominal (3.1)

Case :III($\Delta K(k) = 0$ with robust unknown actuator fault matrix:)
In the following case, we consider the problem of robust unknown reliable controller design for a uncertain discrete time NCS (2.10) without non-fragile control. In additionally, we choose the uncertain parameters as,

$$W = \begin{bmatrix} \overline{\alpha} & 0 \end{bmatrix}^T, \ N_a = \begin{bmatrix} 0.3231 & 0 \\ 0 & 0.3381 \end{bmatrix}, \ \wp(k) = \alpha(k)/\overline{\alpha}$$

where $|\alpha(k)| \leq \overline{\alpha}$.

When the actuator fault matrix is not exactly known and assumed to occur in the interval $0.2 \leq G \leq 0.9$, then the reliable controller can be designed by solving the LMI conditions in (4.9). For the above fault matrix inequality G together with parameters in Case.2 with $\rho_M = 6$, the feasible solutions are obtained without non-fragile controller.

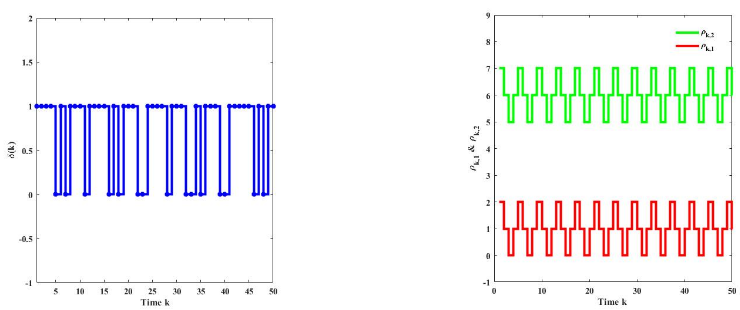


FIGURE 4. a) Simulation of Bernoulli random variables, b) Simulation of Time varying delay

More precisely, we assume that both the lower and upper bounds of the delay ρ_k are known. Our purpose is to determine the maximum value of $\overline{\alpha}$ such that the uncertain NCS (2.10) is robustly asymptotically stable. The calculated maximum value of $\overline{\alpha}$ for different time varying interval ρ_k is given in Table 2. It is clear from Table 2 that $\overline{\alpha}$ decreases as time varying interval increases.

TABLE 2. Calculated upper values of $\overline{\alpha}$ for Case III

	$1 \leq \rho(k) \leq 3$	$1 \leq \rho(k) \leq 4$	$1 \leq \rho(k) \leq 5$	$1 \leq \rho(k) \leq 6$	$1 \leq \rho(k) \leq 7$
Case III	0.048	0.038	0.028	0.017	0.007

Case :IV($\Delta K(k) \neq 0$ with robust unknown actuator fault matrix:)
Next, we consider an output feedback non-fragile controller such that, for all admissible uncertainties as well as unknown actuator failures occurring in the NCS model, the resulting closed-loop system is robustly asymptotically stable. The sensor fault matrix G is assumed to satisfy $0.2 \leq G \leq 0.9$. Then it follows from (4.8) that $G_0 = 0.2$ and $G_1 = 0.9$ solving the LMI in Theorem 4.3 under the same parameters considered in the previous cases, we obtain the feasible solutions for the values of $\rho_m = 1,\ \rho_0 = 2,\ \rho_M = 6,\ \delta_0 = 0.4$. The calculated maximum value of $\overline{\alpha}$ for different time-varying interval ρ_k is given in Table 3. Whereas the robustness indices of output feedback control gain matrices and the maximum delay bound of ρ_M for the cases III and IV respectively is given Table 4.

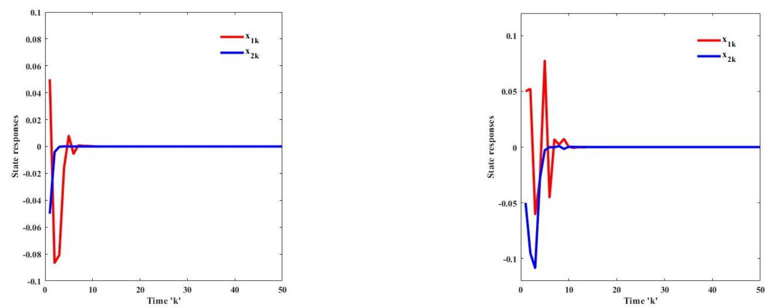
TABLE 3. Calculated upper values of $\overline{\alpha}$ for Case IV

	$1 \leq \rho(k) \leq 3$	$1 \leq \rho(k) \leq 4$	$1 \leq \rho(k) \leq 5$	$1 \leq \rho(k) \leq 6$
Case III	0.0193	0.0104	0.0007	0.0006

The corresponding simulation results are plotted in Figures 5 - 7 for both $\Delta K(k) = 0$ and $\Delta K(k) \neq 0$. The state responses of the uncertain closed-loop

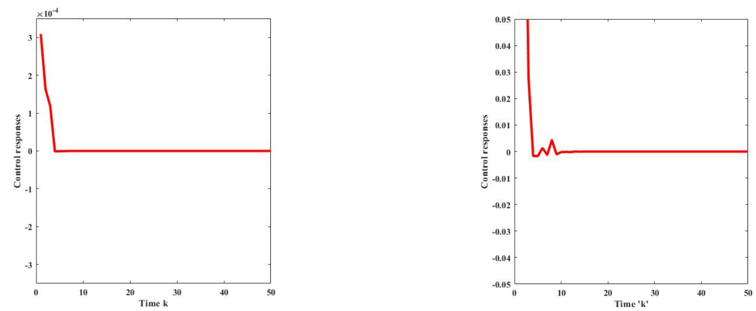
TABLE 4. Comparison of the maximum delay bound ρ_M for the cases III and IV

Cases	$\widehat{K} = K + \Delta K(k)$ when $\Delta K(k) = 0$	$\widehat{K} = K + \Delta K(k)$ when $\Delta K(k) \neq 0$
Gain Matrix $\widehat{K}$	$\begin{bmatrix} -0.3707 & -0.0323 \end{bmatrix}$	$\begin{bmatrix} 0.4706 & -0.0759 \end{bmatrix}$
Maximum upper bound ρ_M	6	4



when $\Delta K(k) = 0$ when $\Delta K(k) \neq 0$

FIGURE 5. State responses of the Closed loop NCS system (3.1)



when $\Delta K(k) = 0$ when $\Delta K(k) \neq 0$

FIGURE 6. Control forces of controllers of nominal system (3.1)

discrete time NCS system (2.10) in the presence and the absence of non-fragile control terms are exhibited in Figures 5(a) and 5(b), respectively. Even though the state responses of the closed-loop NCS (2.10) reach the equilibrium point quickly compare to the presence of a non-fragile controller. So we conclude that the considered uncertain discrete-time NCS in Example 5.1 is robustly asymptotically stable through the obtained controller gain. Further, the simulated random variables δ_k and the variation of time-varying delays $\rho_{k,1}, \rho_{k,2}$ are demonstrated in Figures 7(a) and (b), respectively.

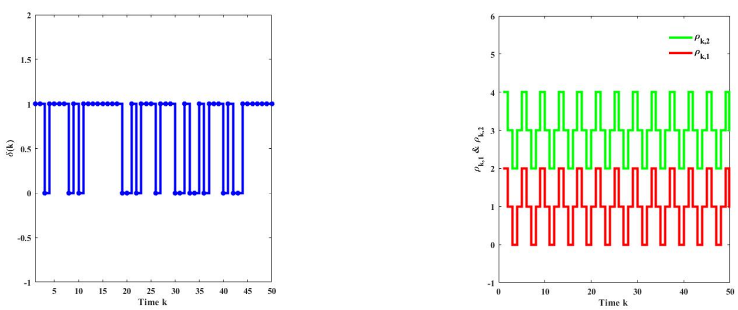


FIGURE 7. a) Simulation of Bernoulli random variables, b) Simulation of Time varying delay

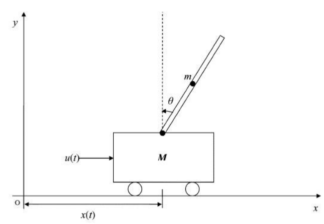


FIGURE 8. Inverted pendulum system

It is observed that, the state vectors of discrete-time NCS for both $\Delta K(k) = 0$ and $\Delta K(k) \neq 0$ cases are shown in Figures 2 and 5 respectively, and the corresponding controller performances are shown in Figures 3, 6, respectively for both nominal and uncertain NCS. We can say that the obtained controller designs compared to the non-fragile controller make the state trajectories converge well and quickly to an equilibrium point. Thus, simulation results reveal that the designed controller can stabilize the uncertain closed-loop NCS (2.10) effectively in the presence and absence of uncertainties.

Example 5.2. In this example, we consider an inverted pendulum system with a delayed control input. The inverted pendulum on a cart is depicted in Figure 8. In this system, a pendulum is attached to the side of a cart by means of a pivot which allows the pendulum to swing in the xy-plane. A force u^f is applied to the cart in the x-direction to keep the pendulum balanced upright. x_k is the displacement of the center of mass of the cart from the origin O. θ is the angle of the pendulum from the top vertical. M and m are the masses of the cart and the pendulum, respectively; l is the half-length of the pendulum (i.e., the distance from the pivot to the center of mass of the pendulum). It is assumed that the pendulum is modeled as a thin rod and the surface to be friction-free. Then, by applying Newton's second law, we arrive at the equations of motion for the system [4, 13].

$$(M+m)\ddot{x} + ml\ddot{\theta}\cos\theta - ml\dot{\theta}^2\sin\theta = u^f \quad (5.1)$$

$$ml\ddot{x}\cos\theta + \frac{4}{3}ml^2\ddot{\theta} - mgl\sin\theta = 0 \quad (5.2)$$

where g is the acceleration due to gravity. Now, by selecting state variables

$$z = \begin{bmatrix} z_1 \\ z_2 \end{bmatrix} = \begin{bmatrix} \theta \\ \dot{\theta} \end{bmatrix}$$

and by linearizing the above model at the equilibrium point $z = 0$, we obtain the following statespace model:

$$\dot{z}_t = \begin{bmatrix} 0 & 1 \\ \frac{3(M+m)g}{l(4M+m)} & 0 \end{bmatrix} z_t + \begin{bmatrix} 0 \\ -\frac{3}{l(4M+m)} \end{bmatrix} u_t^f \quad (5.3)$$

Here the parameters are selected as $M = 8.0$kg, $m = 2.0$kg, $l = 0.5$m, $g = 9.8\text{m/s}^2$. By assuming the sampling time to be $T_s = 30$m/s, the discretized model for the above pendulum system in (5.3) is given by,

$$x_{k+1} = \begin{bmatrix} 1.0087 & 0.0301 \\ 0.5202 & 1.0078 \end{bmatrix} x_k + \begin{bmatrix} -0.0001 \\ 0.0053 \end{bmatrix} u_k^f \quad (5.4)$$

Assume the lower delay bound $\rho_m = 1$(i.e., $d_m h = 30ms$). In comparison, by solving the LMI in Corollary 4.5 of this correspondence paper, we have easily found feasible solution, that the discrete time NCS is asymptotically stable for any delay less than $\rho_M = 5$ (i.e., $d_M h = 150ms$) with the design parameters as $\lambda_1 = 0.7232$, $\lambda_2 = 0.3474$ and $\lambda_3 = 8.6547$. Moreover, it is clearly seen from Table 5 that the maximum upper bound obtained in this paper is bigger than the value in [4, 5], which concludes that the proposed controller has less conservative and better performance. From the obtained solutions, the state feedback control gain matrices are calculated as

$$K = \begin{bmatrix} -159.4809 - 38.5120 \end{bmatrix} \quad (5.5)$$

TABLE 5. Calculated upper bound ρ_M for different values of ρ_m

Method	[5]	Theorem 1 in [4]	Theorem 3 in [4]	Corollary 4.6
ρ_M	1	2	3	4

For simulation purpose, we take the initial condition $x(0) = \begin{bmatrix} 0.05 & -0.05 \end{bmatrix}^T$. The simulation result of the open and closed loop form of discrete time NCS is given in Figure 9 to show the effectiveness of controller gain (5.5). The unstability of the state responses of open loop form of NCS is revealed in Figure 9(a) whereas state responses of closed loop form of NCS converges to equilibrium point in Figure 9(b).

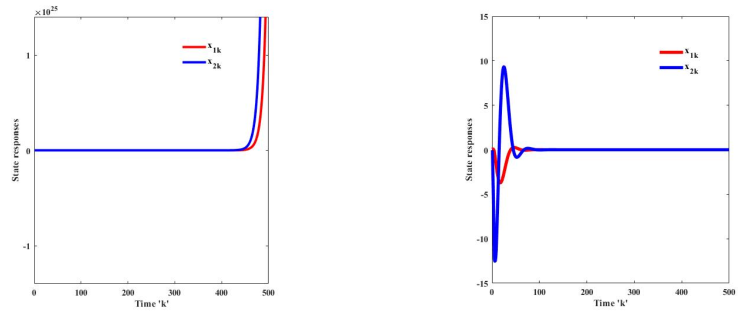


FIGURE 9. State responses of the open and closed loop NCS (5.4)

Example 5.3. Consider the uncertain closed loop NCS (4.14) with the norm bounded parameter uncertainties with the following parameters:

$$A = \begin{bmatrix} 0.8 & 0 \\ 0 & 0.9 \end{bmatrix}, \ B = \begin{bmatrix} -0.1 & 0 \\ -0.1 & -0.1 \end{bmatrix}, \ H = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \ W = \begin{bmatrix} \bar{\alpha} \\ 0 \end{bmatrix},$$
$$N_1 = \begin{bmatrix} 1 & 0 \end{bmatrix}, \ N_2 = \begin{bmatrix} 0 & 0 \end{bmatrix}, \ \wp(k) = \alpha(k)/\bar{\alpha}$$

where $|\alpha(k)| \leq \overline{\alpha}$.

Assume that both lower and upper bounds of the delay $\rho(k)$ are known. From Corollary 4.6 the calculated maximum value of $\bar{\alpha}$ is presented in Table 6 with the designing parameter values $\lambda_1 = 0.7947$, $\lambda_2 = 0.5774$ and $\lambda_3 = 8.6547$ such that the system in (4.14) with the uncertainties is robustly asymptotically stable. For comparison, the method from [5] is also simulated under the same conditions and the results are listed in Table 6. It is observed that our results are much less conservative than the previous ones.

TABLE 6. Calculated upper values of $\bar{\alpha}$ for different cases

	$1 \leq \rho(k) \leq 2$	$3 \leq \rho(k) \leq 5$	$5 \leq \rho(k) \leq 7$	$2 \leq \rho(k) \leq 7$	$2 \leq \rho(k) \leq 8$
[5]	-	0.1615	0.1300	0.0830	Infeasible
Corollary 4.6	0.2277	0.1914	0.1696	0.1696	0.1572

Example 5.4. Consider the following discrete-time system with a time-varying state delay [5]

$$x(k+1) = \begin{bmatrix} 0.8 & 0 \\ 0.05 & 0.9 \end{bmatrix} x_k + \begin{bmatrix} -0.1 & 0 \\ -0.2 & -0.1 \end{bmatrix} x_{k,\rho}$$

By solving the LMI in Corollary 4.5 is feasible with the designing parameter values $\lambda_1 = 0.8909$, $\lambda_2 = 0.3342$ and $\lambda_3 = 4.7204$, then the calculated maximum upper bound ρ_M for different values ρ_m is presented in Table 7. However, the upper bound of time delay obtained in [4, 5] are smaller than that of our paper. Thus,

TABLE 7. Calculated upper bound ρ_M for different values ρ_m

	$\rho_m = 2$	$\rho_m = 4$	$\rho_m = 6$	$\rho_m = 10$	$\rho_m = 12$
[5]	7	8	9	12	13
Theorem 1 in [4]	13	13	14	15	16
Theorem 3 in [4]	13	13	14	15	16
Corollary 4.5	20	20	22	26	28

we can conclude that the our proposed controller yields better performance than the work in [4, 5].

6. Conclusion

The non-fragile reliable output feedback control problem for a class of discrete-time NCS with data packet dropout and transmission delays induced by network channels via randomly occurring time-varying delays has been investigated. The non-fragile reliable output feedback control has been designed for the proposed closed-loop NCS. By implementing the Lyapunov technique together with the LMI approach and free weighting matrix, delay-dependent sufficient conditions are obtained in terms of LMIs for the existence of a non-fragile reliable controller, which ensures the robust asymptotic stability of the NCS. Finally, four numerical examples show the less conservativeness of the obtained results and demonstrate the effectiveness of the proposed non-fragile reliable control law.

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A. Arunkumar: Karupa Foundation, Mettupalayam 641301, Tamil Nadu, India
E-mail address: arunapm@yahoo.com

Hugo Leiva: Universidad de los Andes, Facultad de Ciencias, Departamento de Matematica, Merida 5101-Venezuela
E-mail address: hleivatoka@gmail.com

V. Dhanya: Karupa Foundation, Mettupalayam 641301, Tamil Nadu, India
E-mail address: velmurugan.dhanya@gmail.com

K. P. Sridhar: Karpagam Academy of Higher Education, Coimbatore 641021, Tamil Nadu, India
E-mail address: capsridhar@gmail.com

A. Tridane: Mathematical Sciences Department, United Arab Emirates University, Al Ain, UAE.
E-mail address: a-tridane@uaeu.ac.ae