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Global Existence of Periodic Solutions in a Four-Neuron Network Model with Delays*

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Abstract. In this paper, we consider a delayed differential system that models a network of four neurons with memory. Using the global Hopf bifurcation theorem for FDE due to Wu and a Bendixson's criterion for high-dimensional ODE due to Li and Muldowney, we obtain a group of conditions which guarantee the model to have multiple periodic solutions when the sum of delays is sufficiently large.

AMS Subject Classifications: 34K13, 34K18,92B20

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1. Introduction

Since Hopfield [12] proposed a simplified neural network model, in which each neuron is represented by a linear circuit consisting of a resistor and a capacitor, connected to other neurons via nonlinear sigmoidal activation functions, called transfer functions, there has been great interest in studying the dynamical properties of neural networks. Based on the Hopfield neural network model, Marcus and Weatervelt [15] argued that the nonlinear sigmoidal activation functions connecting neurons may include delays due to finite propagation time or due to delays in electronic components in hardware

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realization and hence they proposed the following delayed differential equations

$$C_i\dot{u}_i(t) = -\frac{1}{R_i}u_i(t) + \sum_{j=1}^{n} T_{ij} f_j(u_j(t-\tau_j)), \quad i = 1, 2, \cdots, n. \tag{1.1}$$

In general case the dynamics of the system (1.1) can be very sophisticated and its analytical investigation is not possible. To approach the problem Baldi and Atiya began investigation from the simplest architecture class capable to sustain oscillations, i.e. a ring of neurons connected cyclically [1,2]

$$\dot{u}_i(t) = -\frac{u_i}{T_i} + J_{ii-1} f_{i-1}(u_{i-1}(t-\tau_{i-1})), \quad i = 1, 2, \cdots, n. \tag{1.2}$$

Campbell [3] generalized Baldi and Atiya's model (1.2) to a network that consists of a ring of neurons where the jth element receives two time-delayed inputs: one from itself and another from the previous element. Campbell obtained sufficient conditions for local stability and bifurcations. Several papers are devoted to the existence and stability of periodic solutions of delayed neural network models with two or three neurons, see [3-7, 9, 11, 14, 22, 23, 25]. Recently, we [14] studied the stability and Hopf bifurcation for the Baldi and Atiya's model (1.2) with $n = 4$

$$\begin{aligned} \dot{u}_1(t) &= -\mu_1 u_1(t) + a_1 f_1(u_4(t-\tau_4)), \\ \dot{u}_2(t) &= -\mu_2 u_2(t) + a_2 f_2(u_1(t-\tau_1)), \\ \dot{u}_3(t) &= -\mu_3 u_3(t) + a_3 f_3(u_2(t-\tau_2)), \\ \dot{u}_4(t) &= -\mu_4 u_4(t) + a_4 f_4(u_3(t-\tau_3)). \end{aligned} \tag{1.3}$$

A group of sufficient conditions for the asymptotic stability of the equilibrium and for the existence of Hopf bifurcations is obtained, and a formula for determining the direction and stability of the Hopf bifurcation is derived.

The purpose of this paper is to investigate the global existence of multiple periodic solutions for (1.3). The method of showing the existence of non-constant periodic solutions is the S^1-equivariant degree. More precisely, we shall use the global Hopf bifurcation theory due to Wu [30] for functional differential equations, which was established using a purely topological argument. Meanwhile, the Bendixson's criterion on higher dimensional ordinary differential equations due to Li and Muldowney [13] shall be used to rule out the existence of nonconstant periodic solution for zero delays.

We would like to mention that there are several articles on the global existence of periodic solutions in delayed differential equations based on the global Hopf bifurcation theory due to Wu [30], for example, see Ruan and Wei [17], Song and Wei [18], Song, Wei and Han [19], Song, Wei and Yuan [20], Sun and Han [21], Wei and Li [23], Wei and Yuan [26], Wei and Zou [27], Wen and Wang [28] and Wu [30].

The rest of the paper is organized as follows: In Section 2, we present the local Hopf bifurcation results of [14], and the higher dimensional Bendixson's criterion of Li and Muldowney [13]. The global existence of multiple periodic solutions is discussed in Section 3. As an example, the model

$$\begin{aligned}
\dot{u}_1(t) &= -\mu u_1(t) + a_1 \tanh u_4(t-\tau_4),\\
\dot{u}_2(t) &= -\mu u_2(t) + a_2 \tanh u_1(t-\tau_1),\\
\dot{u}_3(t) &= -\mu u_3(t) + a_3 \tanh u_2(t-\tau_2),\\
\dot{u}_4(t) &= -\mu u_4(t) + a_4 \tanh u_3(t-\tau_3),
\end{aligned} \tag{1.4}$$

is analyzed and some numerical simulations are presented in Section 4.

2. Preliminary Results

We present some preliminary results to be used in the next section to establish global existence of non-constant periodic solutions.

Let $x_1(t) = u_1(t-\tau_1-\tau_2-\tau_3),\ x_2(t) = u_2(t-\tau_2-\tau_3),\ x_3(t) = u_3(t-\tau_3),\ x_4(t) = u_4(t)$ and $\tau = \tau_1+\tau_2+\tau_3+\tau_4$, Then *Eq.*(1.3) becomes

$$\begin{aligned}
\dot{x}_1(t) &= -\mu_1 x_1(t) + a_1 f_1(x_4(t-\tau)),\\
\dot{x}_2(t) &= -\mu_2 x_2(t) + a_2 f_2(x_1(t)),\\
\dot{x}_3(t) &= -\mu_3 x_3(t) + a_3 f_3(x_2(t)),\\
\dot{x}_4(t) &= -\mu_4 x_4(t) + a_4 f_4(x_3(t)).
\end{aligned} \tag{2.1}$$

We make the following assumption:

(H1) For $i = 1,2,3,4$, constant $\mu_i > 0,\ f_i \in C^2,\ f_i(0) = 0$, and there exists $L > 0$ such that $|f_i(x)| \le L$ for $x \in R$. The origin $(0,0,0,0)$ is the unique equilibrium of (2.1).

Under the hypothesis (H_1), the linearization of system (2.1) at the origin $(0,0,0,0)$ is given by

$$\begin{aligned}
\dot{x}_1(t) &= -\mu_1 x_1(t) + a_1 f_1'(0) x_4(t-\tau),\\
\dot{x}_2(t) &= -\mu_2 x_2(t) + a_2 f_2'(0) x_1(t),\\
\dot{x}_3(t) &= -\mu_3 x_3(t) + a_3 f_3'(0) x_2(t),\\
\dot{x}_4(t) &= -\mu_4 x_4(t) + a_4 f_4'(0) x_3(t).
\end{aligned} \tag{2.2}$$

The characteristic equation associated with (2.2) is

$$\lambda^4 + a\lambda^3 + b\lambda^2 + c\lambda + d + re^{-\lambda\tau} = 0, \tag{2.3}$$

where

$$a = \sum_{i=1}^{4} \mu_i,\ b = \sum_{1\le i<j\le 4} \mu_i\mu_j,\ c = \sum_{1\le i<j<k\le 4} \mu_i\mu_j\mu_k,\ d = \prod_{i=1}^{4} \mu_i, r = -\prod_{i=1}^{4} a_i f_i'(0).$$

Denote

$$p = a^2 - 2b = \sum_{i=1}^{4} \mu_i^2 > 0,$$

$$q = b^2 + 2d - 2ac = \sum_{1 \le i < j \le 4} \mu_i^2 \mu_j^2 > 0,$$

$$u = c^2 - 2bd = \sum_{1 \le i < j < k \le 4} \mu_i^2 \mu_j^2 \mu_k^2 > 0,$$

and

$$v = d^2 - r^2 = \prod_{i=1}^{4} \mu_i^2 - \prod_{i=1}^{4} a_i^2 f_i'^2(0).$$

Hence, ω solves

$$\omega^8 + p\omega^6 + q\omega^4 + u\omega^2 + v = 0 \tag{2.4}$$

when $i\omega$ is a imaginary root of Eq.(2.3). Denote

$$h(z) = z^4 + pz^3 + qz^2 + uz + v.$$

From the fact that $p > 0, q > 0$ and $u > 0$ it follows that $h(z)$ has an unique positive zero if and only if $v > 0$. Let ω_0 be the unique positive root of Eq.(2.4) when $h(z)$ has an unique zero. Denote

$$\bar{\tau}_k = \frac{1}{\omega_0}[\arcsin\left(\frac{c\omega_0 - a\omega_0^3}{r}\right) + 2k\pi], \quad k = 0, 1, 2, \cdots. \tag{2.5}$$

We make the following hypothesis at first.

(H2) $$\prod_{i=1}^{4} \mu_i + \prod_{i=1}^{4} a_i f_i'(0) < 0, \quad \prod_{i=1}^{4} \mu_i - \prod_{i=1}^{4} a_i f_i'(0) > 0 \quad \text{and}$$
$$\Big(\sum_{1 \le i < j < k \le 4} \mu_i \mu_j \mu_k\Big)\Big(\sum_{i \ne j, 1 \le i, j \le 4} \mu_i^2 \mu_j + 2 \sum_{1 \le i < j < k \le 4} \mu_i \mu_j \mu_k\Big)$$
$$-\Big(\sum_{i=1}^{4} \mu_i\Big)^2\Big(\prod_{i=1}^{4} \mu_i - \prod_{i=1}^{4} a_i f_i'(0)\Big) > 0.$$

One can shows that (H_2) implies that $v < 0$. Hence Eq.(2.4) has an unique positive root denoted by ω_0. This implies that (2.5) make sense. From Theorem 2.5 of Li and Wei [14], we have the following result.

Lemma 2.1. *Suppose (H_2) is satisfied. Then the equilibrium (0,0,0,0) of (2.1) is asymptotically stable when $\tau \in [0, \bar{\tau}_0)$, and unstable when $\tau > \bar{\tau}_0$. Moreover, at $\tau = \bar{\tau}_j, j = 0, 1, 2, \cdots, \pm i\omega_0$ is a pair of simple imaginary roots of (2.3).*

Let $\lambda(\tau) = \alpha(\tau) + i\omega(\tau)$ be the root of (2.3) satisfying

$$\alpha(\bar{\tau}_k) = 0, \quad \omega(\bar{\tau}_k) = \omega_0.$$

Then, from Lemma 2.3 of Li and Wei [14], we have the following transversality condition.

Lemma 2.2. *If (H_2) is satisfied,then*

$$\alpha'(\bar{\tau}_k) > 0. \tag{2.6}$$

Moreover, system (2.1) undergoes a Hopf bifurcation at the origin when $\tau = \bar{\tau}_k,\ k = 0, 1, 2, \cdots$.

When applying the global Hopf bifurcation theorem of Wu [30] to show the global existence of non-constant periodic solutions, we need to prove that (2.1) with $\tau = 0$ has no non-constant periodic solutions. This will be done by using a high-dimensional Bendixson's criterion of Li and Muldowney [13].

Li and Muldowney [13] generalized the classical Bindixson's criterion to n-dimensional ordinary differential equations

$$\dot{x} = f(x), \quad x \in R^n, \quad f \in C^1 \tag{2.7}$$

for any finite n. As shown in [13], to derive a high-dimensional Bendixson's criterion, it is sufficient to show that the second compound equation

$$z'(t) = \frac{\partial f}{\partial x}^{[2]}(x(t, x_0))z(t), \tag{2.8}$$

with respect to a solution $x(t, x_0) \in D$ to (2.7) is equi-uniformly asymptotically stable, namely, for each $x_0 \in D$, system (2.8) is uniformly asymptotically stable, and the exponential decay rate is uniform for x_0 in each compact subset of D, where $D \subset R^n$ is an open connected set. Here $\frac{\partial f}{\partial x}^{[2]}$ is the second additive compound matrix of the Jacobian matrix $\frac{\partial f}{\partial x}$. It is an $\begin{pmatrix} n \\ 2 \end{pmatrix} \times \begin{pmatrix} n \\ 2 \end{pmatrix}$ matrix, and thus (2.8) is a linear system of dimension $\begin{pmatrix} n \\ 2 \end{pmatrix}$ (see Fiedler [10] and Muldowney [16]). The equi-uniform asymptotic stability of (2.8) implies the exponential decay of the surface area of any compact two-dimensional surface in D. If D is simply connected, this precludes the existence of any invariant simple closed rectifiable curve in D, including periodic orbits. In particular, the following result is proved in Li and Muldowney [13].

Lemma 2.3. *Let* $D \subset R^n$ *be a simply connected region. Assume that the family of linear systems*

$$z'(t) = \frac{\partial f}{\partial x}^{[2]}(x(t, x_0))z(t), \quad x_0 \in D$$

is equi-uniformly asymptotically stable. Then

(a) D *contains no simple closed invariant curves including periodic orbits, homoclinic orbits, heteroclinic cycles;*

(b) each semi-orbit in D *converges to a single equilibrium.*

In particular, if D *is positively invariant and contains an unique equilibrium* $\bar{x}$*, then* $\bar{x}$ *is globally asymptotically stable in* D.

The required uniform asymptotic stability of the family of linear systems (2.8) can be proved by constructing a suitable Lyapunov function. For instance, (2.8) is equi-uniformly asymptotically stable if there exists a positive definite function $V(z)$, such that, $\frac{dV(z)}{dt}|_{(2.8)}$ is negative definite, and V and $\frac{dV}{dt}|_{(2.8)}$ are both independent of x_0.

For a 4×4 matrix

$$A = \begin{pmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \\ a_{41} & a_{42} & a_{43} & a_{44} \end{pmatrix},$$

its second additive compound matrix $A^{[2]}$ is, see Li and Muldowney [13],

$$A^{[2]} = \begin{pmatrix} a_{11}+a_{22} & a_{23} & a_{24} & -a_{13} & -a_{14} & 0 \\ a_{32} & a_{11}+a_{33} & a_{34} & a_{12} & 0 & -a_{14} \\ a_{42} & a_{43} & a_{11}+a_{44} & 0 & a_{12} & a_{13} \\ -a_{31} & a_{21} & 0 & a_{22}+a_{33} & a_{34} & -a_{24} \\ -a_{41} & 0 & a_{21} & a_{43} & a_{22}+a_{44} & a_{23} \\ 0 & -a_{41} & a_{31} & -a_{42} & a_{32} & a_{33}+a_{44} \end{pmatrix}. \tag{2.9}$$

Consider system (2.1) with $\tau = 0$,

$$\begin{aligned} \dot{x}_1 &= -\mu_1 x_1 + a_1 f_1(x_4), \\ \dot{x}_2 &= -\mu_2 x_2 + a_2 f_2(x_1), \\ \dot{x}_3 &= -\mu_3 x_3 + a_3 f_3(x_2), \\ \dot{x}_4 &= -\mu_4 x_4 + a_4 f_4(x_3). \end{aligned} \tag{2.10}$$

We make the following assumption.

(H3) There exist $\alpha,\ \beta,\ \gamma,\ \xi,\ \eta > 0$ such that

$$\begin{aligned} \sup_{x\in R}\Big\{ &-(\mu_1+\mu_2) + \frac{1}{\eta}|a_1 f_1'(x)|,\ -(\mu_1+\mu_3) + \beta|a_1 f_1'(x)| + \frac{\beta}{\alpha}|a_3 f_3'(x)|, \\ &-(\mu_1+\mu_4) + \frac{\gamma}{\beta}|a_4 f_4'(x)|,\ -(\mu_2+\mu_3) + \frac{\xi}{\beta}|a_2 f_2'(x)|, \\ &-(\mu_2+\mu_4) + \frac{\eta}{\gamma}|a_2 f_2'(x)| + \frac{\eta}{\xi}|a_4 f_4'(x)|,\ -(\mu_3+\mu_4) + \frac{1}{\eta}|a_3 f_3'(x)| \Big\} < 0. \end{aligned} \tag{2.11}$$

Lemma 2.4. *If the hypotheses* (H_1) *and* (H_3) *are satisfied, then the system (2.10) has no non-constant periodic solutions. Furthermore, the unique equilibrium* $(0,0,0,0)$ *is globally asymptotically stable in* R^4.

Proof. First of all, we prove that the solutions of (2.10) are bounded. Let

$$V(x_1,x_2,x_3,x_4) = \frac{1}{2}\sum_{i=1}^{4} x_i^2.$$

Then the derivative of V with respect to t along a solution of (2.10) is

$$\frac{dV}{dt}|_{(2.10)} = -\sum_{i=1}^{4}\mu_i x_i^2 + a_1 x_1 f_1(x_4) + a_2 x_2 f_2(x_1) + a_3 x_3 f_3(x_2) + a_4 x_4 f_4(x_3).$$

Using assumption (H_1) we get

$$\frac{dV}{dt}|_{(2.10)} \leq -\sum_{i=1}^{4} \mu_i x_i^2 + L\sum_{i=1}^{4} |a_i x_i|.$$

Let $\mu = \min_i\{\mu_i\}$ *and* $M \geq \max\{1, \frac{L}{\mu}\sum_{i=1}^{4}|a_i|\}$. Thus, when $(\sum_{i=1}^{4} x_i^2)^{\frac{1}{2}} = A \geq M$, we have

$$\begin{aligned} \tfrac{dV}{dt}|_{(2.10)} &\leq -\mu A^2 + AL\sum_{i=1}^{4}|a_i| \\ &= A(-\mu A + L\sum_{i=1}^{4}|a_i|) \\ &< 0. \end{aligned}$$

This shows that solutions of (2.10) are uniformly ultimately bounded.

Denote $x = (x_1, x_2, x_3, x_4)^T$ and

$$\begin{aligned} f(x_1, x_2, x_3, x_4) = \quad & (-\mu_1 x_1 + a_1 f_1(x_4), -\mu_2 x_2 + a_2 f_2(x_1), \\ & -\mu_3 x_3 + a_3 f_3(x_2), -\mu_4 x_4 + a_4 f_4(x_3))^T. \end{aligned}$$

We have

$$\frac{\partial f}{\partial x} = \begin{pmatrix} -\mu_1 & 0 & 0 & a_1 f_1'(x_4) \\ a_2 f_2'(x_1) & -\mu_2 & 0 & 0 \\ 0 & a_3 f_3'(x_2) & -\mu_3 & 0 \\ 0 & 0 & a_4 f_4'(x_3) & -\mu_4 \end{pmatrix}$$

and, by (2.9)

$$\frac{\partial f}{\partial x}^{[2]} =$$

$$\begin{pmatrix} -(\mu_1+\mu_2) & 0 & 0 & 0 & -a_1 f_1'(x_4) & 0 \\ a_3 f_3'(x_2) & -(\mu_1+\mu_3) & 0 & 0 & 0 & -a_1 f_1'(x_4) \\ 0 & a_4 f_4'(x_3) & -(\mu_1+\mu_4) & 0 & 0 & 0 \\ 0 & a_2 f_2'(x_1) & 0 & -(\mu_2+\mu_3) & 0 & 0 \\ 0 & 0 & a_2 f_2'(x_1) & a_4 f_4'(x_3) & -(\mu_2+\mu_4) & 0 \\ 0 & 0 & 0 & 0 & a_3 f_3'(x_2) & -(\mu_3+\mu_4) \end{pmatrix}.$$

The second compound system

$$\begin{pmatrix} \dot z_1 \\ \dot z_2 \\ \dot z_3 \\ \dot z_4 \end{pmatrix} = \frac{\partial f}{\partial x}^{[2]} \begin{pmatrix} z_1 \\ z_2 \\ z_3 \\ z_4 \end{pmatrix}$$

is

$$\begin{aligned} \dot z_1(t) &= -(\mu_1+\mu_2)z_1 - a_1 f_1'(x_4(t))z_5, \\ \dot z_2(t) &= -(\mu_1+\mu_3)z_2 + a_3 f_3'(x_2(t))z_1 - a_1 f_1'(x_4(t))z_6, \\ \dot z_3(t) &= -(\mu_1+\mu_4)z_3 + a_4 f_4'(x_3(t))z_2, \\ \dot z_4(t) &= -(\mu_2+\mu_3)z_4 + a_2 f_2'(x_1(t))z_2, \\ \dot z_5(t) &= -(\mu_2+\mu_4)z_5 + a_2 f_2'(x_1(t))z_3 + a_4 f_4'(x_3(t))z_4, \\ \dot z_6(t) &= -(\mu_3+\mu_4)z_6 + a_3 f_3'(x_2(t))z_5, \end{aligned} \tag{2.12}$$

where $x(t) = (x_1(t), x_2(t), x_3(t), x_4(t))^T$ is a solution of the system (2.10) with $x(0) = x_0 \in R^4$. Set

$$W(z) = \max\{\alpha|z_1|, \beta|z_2|, \gamma|z_3|, \xi|z_4|, \eta|z_5|, |z_6|\}, \tag{2.13}$$

where $\alpha, \beta, \gamma, \xi, \eta > 0$ are constants. Then direct calculation leads to the following inequalities

$$\begin{aligned}
&\tfrac{d^+}{dt}\alpha|z_1(t)| \leq -(\mu_1+\mu_2)\alpha|z_1| + \tfrac{1}{\eta}|a_1 f_1'(x_4(t))| \cdot \eta|z_5|,\\
&\tfrac{d^+}{dt}\beta|z_2(t)| \leq -(\mu_1+\mu_3)\beta|z_2| + \tfrac{\beta}{\alpha}|a_3 f_3'(x_2(t))| \cdot \alpha|z_1| + \beta|a_1 f_1'(x_4(t))| \cdot |z_6|,\\
&\tfrac{d^+}{dt}\gamma|z_3(t)| \leq -(\mu_1+\mu_4)\gamma|z_3| + \tfrac{\gamma}{\beta}|a_4 f_4'(x_3(t))| \cdot \beta|z_2|,\\
&\tfrac{d^+}{dt}\xi|z_4(t)| \leq -(\mu_2+\mu_3)\xi|z_4| + \tfrac{\xi}{\beta}|a_2 f_2'(x_1(t))| \cdot \beta|z_2|,\\
&\tfrac{d^+}{dt}\eta|z_5(t)| \leq -(\mu_2+\mu_4)\eta|z_5| + \tfrac{\eta}{\gamma}|a_2 f_2'(x_1(t))| \cdot \gamma|z_3| + \tfrac{\eta}{\xi}|a_4 f_4'(x_3(t))| \cdot \xi|z_4|,\\
&\tfrac{d^+}{dt}|z_6(t)| \leq -(\mu_3+\mu_4)|z_6| + \tfrac{1}{\eta}|a_3 f_3'(x_2(t))| \cdot \eta|z_5|,
\end{aligned}$$

where $\frac{d^+}{dt}$ denotes the right-hand derivative. Therefore,

$$\frac{d^+}{dt}W(z(t)) \leq \zeta(t) \cdot W(z(t)),$$

with

$$\begin{aligned}
\zeta(t) = \max\{&-(\mu_1+\mu_2) + \frac{1}{\eta}|a_1 f_1'(x_4(t))|, -(\mu_1+\mu_3) + \beta|a_1 f_1'(x_4(t))| + \frac{\beta}{\alpha}|a_3 f_3'(x_2(t))|,\\
&-(\mu_1+\mu_4) + \frac{\gamma}{\beta}|a_4 f_4'(x_3(t))|, -(\mu_2+\mu_3) + \frac{\xi}{\beta}|a_2 f_2'(x_1(t))|, -(\mu_2+\mu_4)\\
&+ \frac{\eta}{\gamma}|a_2 f_2'(x_1(t))| + \frac{\eta}{\xi}|a_4 f_4'(x_3(t))|, -(\mu_3+\mu_4) + \frac{1}{\eta}|a_3 f_3'(x_2(t))|\}.
\end{aligned}$$

Thus, under the assumption (H_3), and by the boundedness of solution to (2.10), there exists a $\delta > 0$ such that $\zeta(t) \leq -\delta < 0$, and thus

$$W(z(t)) \leq W(z(s)) \cdot e^{-\delta(t-s)}, \quad t \geq s > 0.$$

This establishes the equi-uniform asymptotic stability of the second compound system (2.12), and hence the conclusion of the lemma follows from the result of Li and Muldowney [12]. □

3. Global Existence of Periodic Solutions

In this section, we shall regard the sum of delays $\tau = \sum_{i=1}^{4} \tau_i$ as parameter and employ a S^1-degree theory developed in Erbe et al. [8] to investigate the global existence of non-constant periodic solutions. More specifically, we shall apply a global Hopf bifurcation theorem, Theorem 3.3 of Wu [30], which was developed using the S^1-degree theory.

Theorem 3.1. *Suppose that the hypotheses $(H_1), (H_2)$ and (H_3) are satisfied. Then the system (2.1) has at least k non-constant periodic solutions when $\tau > \bar{\tau}_k, k \geq 1$, where $\bar{\tau}_k$ is defined by (2.4).*

Proof. We regard (τ, p) as parameters and apply Theorem 3.3 in Wu [30]. By (H_1) we know that $0 = (0,0,0,0)$ as the unique equilibrium of (2.1). Hence, the stationary solution of (2.1) is the form $(0, \tau, p)$, and the corresponding characteristic function is

$$\begin{aligned} q(\lambda) = \ & \lambda^4 + (\sum_{i=1}^{4} \mu_i)\lambda^3 + (\sum_{1\leq i<j\leq 4} \mu_i\mu_j)\lambda^2 + (\sum_{1\leq i<j<k\leq 4} \mu_i\mu_j\mu_k)\lambda \\ & + \prod_{i=1}^{4} \mu_i - (\prod_{i=1}^{4} a_i f_i'(0))e^{-\lambda\tau}. \end{aligned} \tag{3.1}$$

Clearly, $q(\lambda)$ is continuous in $(\tau, p, \lambda) \in R_+ \times R_+ \times C$.

To locate centers, we consider

$$\begin{aligned} q(i\tfrac{2m\pi}{p}) = \ & (\tfrac{2m\pi}{p})^4 - i(\sum_{i=1}^{4} \mu_i)(\tfrac{2m\pi}{p})^3 - (\sum_{1\leq i<j\leq 4} \mu_i\mu_j)(\tfrac{2m\pi}{p})^2 \\ & + i(\sum_{1\leq i<j<k\leq 4} \mu_i\mu_j\mu_k)\tfrac{2m\pi}{p} + \prod_{i=1}^{4} \mu_i - (\prod_{i=1}^{4} a_i f_i'(0))e^{-i\frac{2m\pi}{p}\tau}. \end{aligned}$$

Using Lemma 2.1 we know that $(0, \tau, p)$ is a center if and only if $m = 1, \tau = \bar{\tau}_k$ and $p = \frac{2\pi}{\omega_0}$. In particular, $(0, \bar{\tau}_k, \frac{2\pi}{\omega_0})$ is a center and all centers are isolated. In fact, the set of centers is countable and can be expresses as

$$\{(0, \bar{\tau}_k, \frac{2\pi}{\omega_0}); k = 0, 1, 2, \cdots\},$$

where $\bar{\tau}_k$ is defined by (2.5).

Consider $q(\lambda)$ with $m = 1$. By Lemmas 2.1 and 2.2, for fixed k, there exist $\varepsilon > 0, \delta > 0$ and a smooth curve $\lambda : (\bar{\tau}_k - \delta, \bar{\tau}_k + \delta) \to C$ such that $q(\lambda(\tau)) = 0, |\lambda(\tau) - i\omega_0| < \varepsilon$ for all $\tau \in (\bar{\tau}_k - \delta, \bar{\tau}_k + \delta)$, and

$$\lambda(\bar{\tau}_k) = i\omega_0, \quad \frac{d}{d\tau} Re\lambda(\tau)|_{\tau=\bar{\tau}_k} > 0.$$

Let

$$\Omega_\varepsilon = \{(u, p) : 0 < u < \varepsilon, |p - \frac{2\pi}{\omega_0}| < \varepsilon\}.$$

Clearly, if $|\tau - \bar{\tau}_k| < \delta$ and $(u, p) \in \partial\Omega_\varepsilon$ such that $q(u + i\frac{2\pi}{p}) = 0$, then $\tau = \bar{\tau}_k, u = 0, p = \frac{2\pi}{\omega_0}$. This verifies hypothesis (A4) for $m = 1$ in Theorem 3.3 of Wu [12]. Moreover, if we put

$$H_m^{\pm}(0, \bar{\tau}_k, \frac{2\pi}{\omega_0})(u, p) = \Delta_{(0, \bar{\tau}_k \pm \delta, p)}(u + im\frac{2\pi}{p}),$$

then, at $m = 1$, we have

$$\begin{aligned} \gamma_m(0, \bar{\tau}_k, \tfrac{2\pi}{\omega_0}) & = deg_B(H_m^-(0, \bar{\tau}_k, \tfrac{2\pi}{\omega_0}), \Omega_\varepsilon) - deg_B(H_m^+(0, \bar{\tau}_k, \tfrac{2\pi}{\omega_0}), \Omega_\varepsilon) \\ & = -1. \end{aligned}$$

By Theorem 3.2 of Wu [30], we conclude that the connected component $C(0, \bar{\tau}_k, \frac{2\pi}{\omega_0})$ through $(0, \bar{\tau}_k, \frac{2\pi}{\omega_0})$ in Σ is nonempty, where

$$\Sigma = cl\{(x, \tau, p) : x \text{ is a } p-\text{periodic solution of (2.1)}\}.$$

Obviously, the first crossing number of each center is always -1. Therefore, we conclude that $C(0, \bar{\tau}_k, \frac{2\pi}{\omega_0})$ is unbounded by Theorem 3.3 of Wu [30].

Next, we prove that the periodic solutions of (2.1) are uniformly bounded. Let $r(t) = [\sum_{i=1}^{4} x_i^2(t)]^{\frac{1}{2}}$, differentiating $r(t)$ along a solution of (2.1) we have

$$\begin{aligned}\dot{r}(t) &= \tfrac{1}{r(t)} \sum_{i=1}^{4} x_i(t)\dot{x}_i(t) \\ &= \tfrac{1}{r(t)}[-\sum_{i=1}^{4} \mu_i x_i^2(t) + a_1 x_1(t) f_1(x_4(t-\tau)) \\ &\quad + a_2 x_2(t) f_2(x_1(t)) + a_3 x_3(t) f_3(x_2(t)) + a_4 x_4(t) f_4(x_3(t))] \\ &\leq \tfrac{1}{r(t)}[-\mu \sum_{i=1}^{4} x_i^2(t) + L \sum_{i=1}^{4} |a_i x_i(t)|],\end{aligned}$$

where μ is defined by Lemma 2.4. Similar to the first part of proof of Lemma 2.4, we know that if there exists $t_0 > 0$ such that $r(t_0) \geq M$, then

$$\dot{r}(t_0) < 0.$$

It follows that if $x(t) = (x_1(t), x_2(t), x_3(t), x_4(t))^T$ is a periodic solution of (2.1), then $r(t) < M$ for all t. This shows that the periodic solutions of (2.1) are uniformly bounded.

Next, we explain that the system (2.1) has no non-constant τ-periodic solution. In fact, if $x(t) = (x_1(t), x_2(t), x_3(t), x_4(t))^T$ is a τ-periodic solution of (2.1), then $x(t)$ is τ-periodic solution of the ordinary differential equation (2.10). Applying Lemma 2.4 we know that, under the hypothesis (H_3), equation (2.10) has no non-constant periodic solution.

By the definition of $\bar{\tau}_k$ (see (2.5)), we know that

$$\omega_0 \bar{\tau}_k > 2\pi, \quad k = 1, 2, \cdots,$$

and hence

$$\frac{2\pi}{\omega_0} < \bar{\tau}_k, \quad k = 1, 2, \cdots.$$

From Lemma 2.1, we know that $\bar{\tau}_0 > 0$. Hence for $\tau > \bar{\tau}_k$, there exists an integer m such that $\frac{\tau}{m} < \frac{2\pi}{\omega_0} < \tau$. Since system (2.1) has no τ-periodic solution, it has no $\frac{\tau}{n}$-periodic solution for any integer n. This implies that the period p of a periodic solution on the connected component $C(0, \bar{\tau}_k, \frac{2\pi}{\omega_0})$ satisfies $\frac{\tau}{m} < p < \tau$. So we know that the periods of the periodic solutions of the system (2.1) on $C(0, \bar{\tau}_k, \frac{2\pi}{\omega_0})$ are uniformly bounded.

From the discussion above, we have shown that the projection of $C(0, \bar{\tau}_k, \frac{2\pi}{\omega_0})$ onto the τ-space must be unbounded. Otherwise, there exists τ^* such that projection of

$C(0, \bar{\tau}_k, \frac{2\pi}{\omega_0})$ onto the τ-space is included on a interval $(0, \tau^*)$. Then, from $C(0, \bar{\tau}_k, \frac{2\pi}{\omega_0})$ is unbounded, it follows that the projection of $C(0, \bar{\tau}_k, \frac{2\pi}{\omega_0})$ onto the x-space or onto the p-space is unbounded. There arises a contradiction.

Applying Lemma 2.4 again, we know that system (2.1) has no non-constant periodic solution when $\tau = 0$. Thus, the projection of $C(0, \bar{\tau}_k, \frac{2\pi}{\omega_0})$ onto the τ-space must be an interval $[T, \infty)$ with $0 < T \leq \bar{\tau}_k$. This shows that, for each $\tau > \bar{\tau}_k (k \geq 1)$, system (2.1) has a non-constant periodic solution on $C(0, \bar{\tau}_k, \frac{2\pi}{\omega_0})$. Therefore, if $\tau > \bar{\tau}_k \geq \bar{\tau}_1$, system (2.1) has at least k periodic solutions. This completes the proof of Theorem 3.1. □

Remark 3.2. From Lemma 2.4 we know that, under the hypothesis (H_1) and (H_3), the unique equilibrium $(0, 0, 0, 0)$ of system (2.1) with $\tau = 0$ is globally asymptotically stable in R^4. However, under the hypothesis $(H_1), (H_2)$ and (H_3), system (2.1) has at least k non-constant periodic solutions when $\tau > \bar{\tau}_k (k \geq 1)$. This demonstrates how time delays influence the dynamics of system (2.1).

4. An Example

Consider the neural networks with four neurons and delays

$$\begin{aligned}
\dot{u}_1(t) &= -\mu u_1(t) + a_1 f(u_4(t - \tau_4)), \\
\dot{u}_2(t) &= -\mu u_2(t) + a_2 f(u_1(t - \tau_1)), \\
\dot{u}_3(t) &= -\mu u_3(t) + a_3 f(u_2(t - \tau_2)), \\
\dot{u}_4(t) &= -\mu u_4(t) + a_4 f(u_3(t - \tau_3)).
\end{aligned} \tag{4.1}$$

Applying the results of Sections 2 and 3, we establish the global existence of periodic solutions for the system (4.1).

We make the following assumptions on $f(x)$.

(P_1). $\mu > 0,\ f \in C^2,\ xf(x) > 0$ *for* $x \neq 0$, *and origin* $(0, 0, 0, 0)$

is the unique equilibrium of (4.1).

(P_2). *There exists* $L > 0$ *such that* $|f(x)| \leq L$ *for* $x \in R$, *and*

$$-4\mu^4 < \prod_{i=1}^{4} a_i f'^4(0) < -\mu^4. \tag{4.2}$$

(P_3). $|Af'(x)| < 2\mu$ *for* $x \in R$, *where* $A = max\{a_1, a_2, a_3, a_4, a_1 + a_3, a_2 + a_4\}$.

The inequality (4.2) ensures that

$$\omega^8 + 4\mu^2\omega^6 + 6\mu^4\omega^4 + 4\mu^6\omega^2 + \mu^8 - (\prod_{i=1}^{4} a_i^2) f'^8(0) = 0$$

has an unique root denoted by ω_0. Let

$$\bar{\tau}_k = \frac{1}{\omega_0}[\arcsin\left(\frac{4\mu\omega_0^3 - 4\mu^3\omega_0}{(\prod_{i=1}^{4} a_i)f'^4(0)}\right) + 2k\pi], \quad k = 0, 1, 2, \cdots. \tag{4.3}$$

Set $\tau = \sum_{i=1}^{4} \tau_i$, by Theorem 3.1, we have the following result.

Theorem 4.1. *Suppose that $(P_1), (P_2)$ and (P_3) are satisfied. Then the system (4.1) has at least k non-constant periodic solutions when $\tau > \bar{\tau}_k, k \geq 1$.*

Applying Theorem 4.1 to $f(x) = tanh(x)$ we have the following corollary.

Corollary 4.2. *For the four-neuron network with four delays:*

$$\begin{aligned} \dot{u}_1(t) &= -\mu u_1(t) + a_1 \tanh u_4(t - \tau_4), \\ \dot{u}_2(t) &= -\mu u_2(t) + a_2 \tanh u_1(t - \tau_1), \\ \dot{u}_3(t) &= -\mu u_3(t) + a_3 \tanh u_2(t - \tau_2), \\ \dot{u}_4(t) &= -\mu u_4(t) + a_4 \tanh u_3(t - \tau_3), \end{aligned} \tag{4.4}$$

where $\mu > 0$ and $a_i (i = 1, 2, 3, 4)$ are constants. If

$$-4\mu^4 < \prod_{i=1}^{4} a_i < -\mu^4$$

is satisfied, then the system (4.4) has at least k non-constant periodic solutions when $\tau > \bar{\tau}_k$ and $k \geq 1$, where $\tau = \sum_{i=1}^{4} \tau_i$, and $\bar{\tau}_k$ is defined in (4.3) with $f'(0) = 1$.

As an example, consider system (4.4) with $\mu = 1, a_1 = 0.95, a_2 = 1.6, a_3 = 0.95, a_4 = -1.56$, such that a_i satisfy the condition in Corollary 4.2. In this case it can be calculated that $\omega_0 = 0.707$, and for $k = 0, 1, 2, \cdots$,

$$\bar{\tau}_k = 0.96, 9.85, 18.74, 27.62, 36.5, 45.4, 54.28, \cdots.$$

The delays are chosen as $\tau_1 = 7, \tau_2 = 5, \tau_3 = 6, \tau_4 = 5$ so that $\tau = \sum_{i=1}^{4} \tau_i = 23$ is between the two Hopf bifurcation values $\bar{\tau}_2 = 18.74$ and $\bar{\tau}_3 = 27.62$. Periodic solutions are shown in Figure 1.

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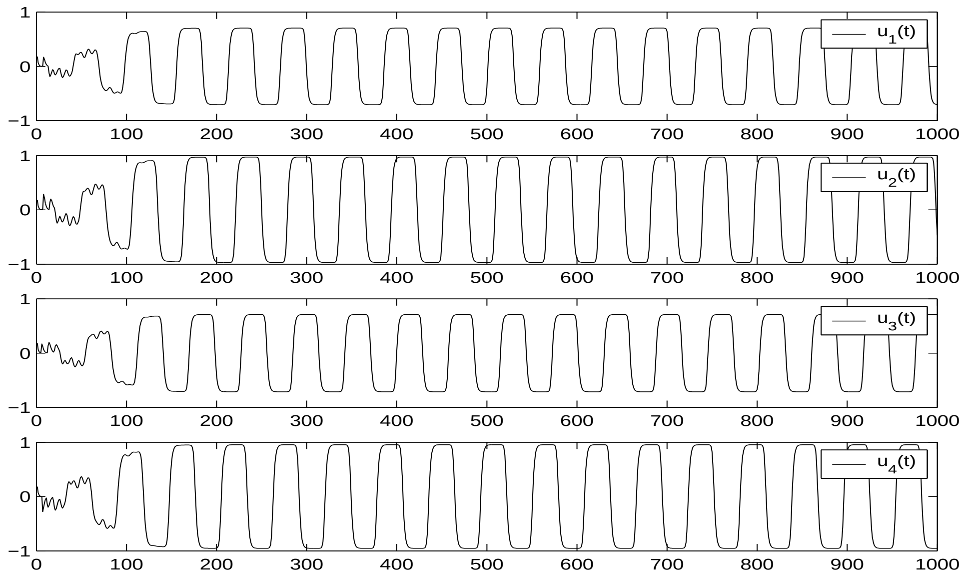

Figure 1: $\tau = \sum_{i=1}^{4} \tau_i = 23$, a periodic solution occurs.

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